Topics in Geometric Mechanics: Week 7

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Week 7

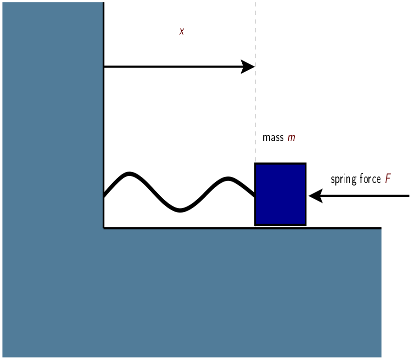
This week we will look at Lagrangian mechanics.

- Lagrangian mechanics uses geometry and the calculus of variations
- In most cases, the equations are those of Newtonian mechanics
- Energy, rather than Forces, play the principal role

An example

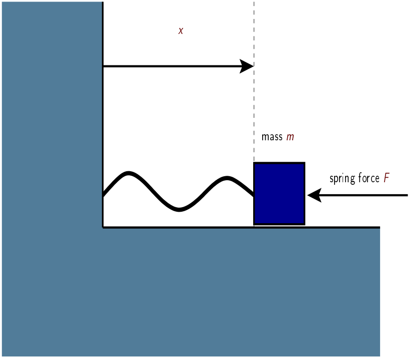
- Hooke's Law: $F_{\text{spring}} = -kx$.
- Newton's Equations: $-kx = m\ddot{x}$
- Conservation: $-kx\dot{x} = m\ddot{x}\dot{x}$ implies

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$



An example

- Hooke's Law: $E_{\mathrm{spring}} = -\int_0^x F \, \mathrm{d}x = \frac{1}{2}kx^2.$
- Kinetic Energy: $E_{\mathrm{kinetic}} = \frac{1}{2}m\dot{x}^2$
- Newton's Equations: $\frac{d}{dt}\left(\frac{\partial E_{\mathrm{kinetic}}}{\partial \dot{x}}\right) + \frac{\partial E_{\mathrm{spring}}}{\partial x} = 0$



From Newton to Lagrange

Theorem

Let a mechanical system have a configuration space M, an open subset of $\mathbf{R}^n$. Let the forces acting on the system, F, be conservative. Then Newton's equations for this mechanical system are equivalent to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x},$$

where L is the difference of kinetic and potential energy.

Proof.

Since F is conservative, the work done is path-independent. Define a function

$$V(x) = -\int_0^x \langle F(c(t)), \mathrm{d}c \rangle = \int_0^1 \langle F(tx), x \rangle \,\mathrm{d}t$$

where $c(t) = tx$. The fundamental theorem of calculus shows that the gradient of V at x is $-F(x)$.

Define

$$L(x, \dot{x}) = \frac{1}{2} m |\dot{x}|^2 - V(x).$$

Then

$$m\ddot{x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \qquad \text{and} \qquad F(x) = -\nabla V(x) = \frac{\partial L}{\partial x}.$$

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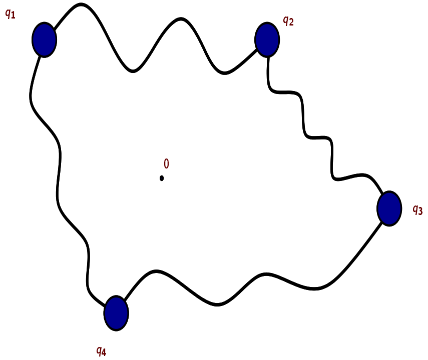
Example - springs and masses

$$F(\mathbf{q}) = -A\mathbf{q} \qquad V(q) = \int_0^1 \langle A(t\mathbf{q}), \mathbf{q}\rangle \,\mathrm{d}t = \frac{1}{2}\langle A\mathbf{q}, \mathbf{q}\rangle.$$

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \sum \frac{1}{2} m_i |\dot{\mathbf{q}}_i|^2 - V(\mathbf{q}).$$



Constrained Systems

Definition

Let a mechanical system have a configuration manifold $M \subset \mathbf{R}^n$. We say the system is constrained when M is a closed submanifold with $\dim M < n$.

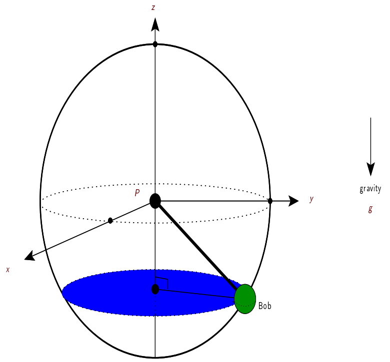


Figure: A constrained mechanical system.

Acceleration in a Constrained System

Definition

Let a mechanical system have a configuration manifold $M \subset \mathbf{R}^n$. A smooth curve $c : I \subset \mathbf{R} \to M$ has the derivative

$$\mathrm{d}_t\, c = \dot{c}(t)$$

and acceleration

$$\frac{D\dot{c}(t)}{dt} = \Pi_{c(t)}(\ddot{c}(t)),$$

where $\Pi_x : T_x\mathbf{R}^n \to T_x M$ is the orthogonal projection.

Kinetic Energy and Acceleration

Definition

Let (M, g) be a Riemannian manifold, $v \in T_x M$. The kinetic energy of v is

$$T(x, v) = \frac{1}{2} g_x(v, v) = \frac{1}{2} |v|_x^2.$$

Definition

Let (M, g) be a Riemannian manifold, and $c : I \subset \mathbf{R} \to M$ be a C^2 curve. We define the acceleration of c to be

$$\nabla_{\dot{c}} \dot{c} \equiv \frac{D\dot{c}}{dt}.$$