Topics in Geometric Mechanics: Week 6

Leo Butler

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Week 6

Last week we looked at Lie groups:
A Lie group is a group G that satisfies

- G is a smooth manifold
- Multiplication $\mu : G \times G \to G$ is a smooth map

$$\mu(g_1, g_2) = g_1 \cdot g_2.$$

- Inversion $\iota : G \to G$ is a smooth map

$$\iota(g) = g^{-1}.$$

Lie Group Actions

Definition

An action of a Lie group G on a manifold M is a smooth map $\alpha : G \times M \to M$ s.t.

$$\alpha(h, \alpha(g, x)) = \alpha(hg, x)$$

for all $h, g \in G$, $x \in M$. One often writes $g \cdot x$ for $\alpha(g, x)$.

Lie Group Actions - Examples

- G acting on $M = G$ by left translations
- G acting on $M = G$ by $g \cdot x = xg^{-1}$
- $G = H \times H$ acting on $M = H$ by $(h_0, h_1) \cdot x = h_0 x h_1^{-1}$
- $G = \mathrm{GL}(\mathbf{R}^n)$ acting on $M = \mathbf{R}^n$ by its standard linear representation
- $G = \mathrm{GL}(V)$ acting on $\mathsf{T}^k(V)$ by the induced transformations
- $G = \mathrm{O}(\mathbf{R}^n)$ acting on $M = \mathbf{S}^{n-1} \subset \mathbf{R}^n$ by linear transformations
- G acting on $\mathfrak{g}$ (its Lie algebra) by the adjoint action

The action on the sphere

Let us see that $G = \mathrm{O}(\mathbf{R}^n)$ acting on $\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n \,\mathrm{s.t.}\, |x| = 1\}$ is a Lie group action.

- The map $m : \mathrm{Mat}_{n\times n}(\mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}^n$ given by

  $$m(A, x) = Ax$$

  is smooth and satisfies $m(A, m(B, x)) = m(BA, x)$.
- The inclusion map $\iota : \mathrm{GL}(n; \mathbf{R}) \hookrightarrow \mathrm{Mat}_{n\times n}(\mathbf{R})$ is smooth.
- The composition $\mu = m \circ (\iota \times id) : \mathrm{GL}(n; \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}^n$ is smooth and satisfies $\mu(h, \mu(g, x)) = \mu(hg, x)$.
- The inclusion $\zeta : \mathrm{O}(\mathbf{R}^n) \hookrightarrow \mathrm{GL}(n; \mathbf{R})$ is a Lie group homomorphism.
- The map $\eta = \mu \circ (\zeta \times id) : \mathrm{O}(\mathbf{R}^n) \times \mathbf{R}^n \to \mathbf{R}^n$ is a Lie group action.

Facts about Actions

Let $\alpha : G \times M \to M$ be a Lie group action.

- If $H < G$ is a closed subgroup, then the restriction $\alpha_H : H \times M \to M$ is a Lie group action.

Facts about Actions

- If $N \subset M$ is a smooth submanifold invariant under G (i.e. $g \cdot n \in N$ for all $n \in N, g \in G$), then the restriction $\alpha_N : G \times N \to N$ is a Lie group action.

Facts about Actions

- Let $m \in M$. Let $\mathrm{stab}_G(m) = \{g \in G \,\mathrm{s.t.}\, g \cdot m = m\}$ be the stabiliser subgroup.

Facts about Actions

- Let $G \cdot m = \{g \cdot m \, \text{s.t.} \, g \in G\}$ be the orbit of the point m.

Facts about Actions

- For each $x = g \cdot m \in G \cdot m$, the stabiliser $\mathrm{stab}_G(x) = g\mathrm{stab}_G(m)g^{-1}$.
- If G is compact, then $G \cdot m$ is a smooth submanifold with a transitive Lie group action of G. It is diffeomorphic to G/H where $H = \mathrm{stab}_G(m)$.

Facts about Actions

- For each subgroup $H \leq G$, let

$$M_H = \{m \in M \,\mathrm{s.t.}\, \mathrm{stab}_G(m) \text{ is conjugate to } H\}.$$

- M is a disjoint union

$$M = \coprod_{H \leq G} M_H.$$

- If G is compact, then M_H is a smooth submanifold with a smooth Lie group action of G on it for all H.

An Example

Let $M = \mathbf{S}^2 \subset \mathbf{R}^3$ and let

$$G = \mathrm{O}(\mathbf{R}^1) \times \mathrm{O}(\mathbf{R}^2).$$

- $\mathrm{stab}_G(N) = \mathrm{stab}_G(-N) = 1 \times \mathrm{O}(\mathbf{R}^2)$.
- $\mathrm{stab}_G(E) = \mathrm{O}(\mathbf{R}^1) \times \mathrm{O}(\mathbf{R}^1) \times 1$.
- $\mathrm{stab}_G(X) = 1 \times \mathrm{O}(\mathbf{R}^1) \times 1$.

An Example

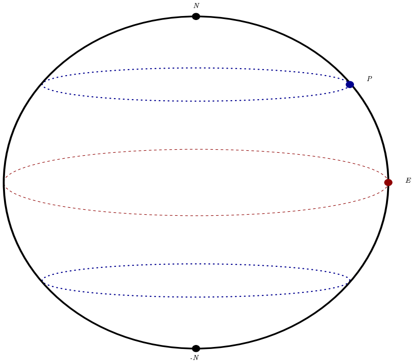

Figure: The orbits and stabilizers of points on the 2-sphere.

A Second Example

Let $G = \mathrm{GL}(\mathbf{C}^3)$ act on $M = \mathrm{Mat}_{3\times 3}(\mathbf{C})$ by

$$g \cdot x = gxg^{-1}.$$

- By the Jordan canonical form theorem, every $x \in \mathrm{Mat}_{3\times 3}(\mathbf{R})$ is conjugate to one of ($a, b, c \in \mathbf{C}^\times$ are distinct)
  - Semisimples:

$$D = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \qquad E = \begin{bmatrix} a & & \\ & a & \\ & & c \end{bmatrix} \qquad F = aI$$

  - Non-semisimples:

$$U = \begin{bmatrix} a & 1 & \\ & a & \\ & & b \end{bmatrix} \qquad V = \begin{bmatrix} a & 1 & \\ & a & 1 \\ & & a \end{bmatrix}$$

A Second Example

The stabilisers of each element type:

- $\mathrm{stab}_G(D) = A = \{\text{all diagonal elements}\}$. ("complex torus")
- $\mathrm{stab}_G(E) = \mathrm{GL}(\mathbf{C}^2) \times \mathrm{GL}(\mathbf{C}^1) =$
  $\left\{ x = \begin{bmatrix} g & \\ & c \end{bmatrix} \text{ s.t. } g \in \mathrm{GL}(\mathbf{C}^2), c \in \mathbf{C}^\times = \mathrm{GL}(\mathbf{C}^1) \right\}$.
- $\mathrm{stab}_G(F) = G$.
- $\mathrm{stab}_G(U) = N_1 \times \mathrm{GL}(\mathbf{C}^1) = \left\{ x = \begin{bmatrix} s & t & \\ & s & \\ & & c \end{bmatrix} \text{ s.t. } s, c \neq 0 \right\}$.
- $\mathrm{stab}_G(U) = N_2 = \left\{ x = \begin{bmatrix} s & t & \\ & s & t \\ & & s \end{bmatrix} \text{ s.t. } s \neq 0 \right\}$.

A Third Example

Let $M = \mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ be the 2-dimensional torus. Let $G = \mathbf{R}v/\mathbf{Z}^2$ be the 1-parameter subgroup of $\mathbf{T}^2$ generated by $v = (1, \sqrt{2})$.

- G acts freely on M.
- $G \cdot x$ is dense in M; it is never an embedded submanifold.

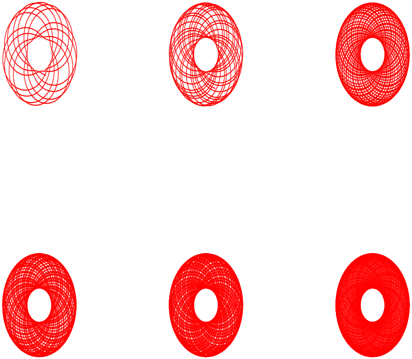

Figure: Intervals of increasing length in the 1-parameter subgroup.