Topics in Geometric Mechanics: Week 3

Leo Butler

February 1, 2012

The Spherical Pendulum - Homework 2

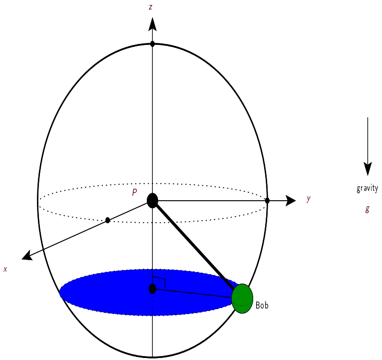

Figure: Spherical Pendulum.

The Spherical Pendulum - Homework 2 - correction

We let $A : V \to V$ be a linear transformation, $x, y \in V$.

$$\exp(tA) = I + tA + \frac{1}{2}t^2A^2 + \cdots$$

$$\exp(tA) \cdot (x \otimes y) := (\exp(tA)x) \otimes (\exp(tA)y)$$

Week 3

Recall:

- we looked at classical Newtonian mechanics;
- we looked at some tensor algebra;
- let's continue with some tensor algebra.

Standing Notational Conventions

- V, W are finite-dimensional vector spaces; V^*, W^* are their duals.
- $\mathsf{T}^*(V)$ is the tensor algebra over V with tensor product $\otimes$.
- $\mathsf{S}^*(V)$ is the symmetric tensor algebra over V with tensor product $\cdot$.
- $\Lambda^*(V)$ is the alternating tensor algebra over V with wedge product $\wedge$.
- $\mathrm{Hom}(V; W)$ is the vector space of linear transformations $V \to W$.
- Given $v \in V, \phi \in V^*$, define

$$\langle \phi, v \rangle = \phi(v).$$

Adjoints and Transposes

- A linear transformation

$$\begin{array}{ccc} V & \xrightarrow{A} & W \\ & \text{induces} & \\ V^* & \xleftarrow{A^*} & W^* \end{array}$$

- Defined by

$$\langle A^*\phi, v\rangle := \langle \phi, Av\rangle \qquad \phi \in W^*, v \in V.$$

-

$$(\mathrm{Im}\, A)^\perp = \mathrm{Ker}\, A^* \qquad \mathrm{Ker}\, A = (\mathrm{Im}\, A^*)^\perp$$

Traces

- For $x \otimes y \in V \otimes V^*$ define
$$\mathrm{Tr}\,(x \otimes y) := \langle y, x \rangle.$$
- The Tr form extends to a linear function $\mathrm{Hom}(V; V) \cong V \otimes V^* \to F$.
- For $A \in \mathrm{Hom}(V; V)$
$$\mathrm{Tr}\,(A) = \sum_{i=1}^{n} \langle \phi_i, Ae_i \rangle = \sum_{i=1}^{n} \langle A^* \phi_i, e_i \rangle = \mathrm{Tr}\,(A^*)$$

Traces

- The map

$$\begin{array}{ccc} A & \mapsto & \mathrm{Tr}_A \\ x \otimes y & \mapsto & \langle y, Ax \rangle =: \mathrm{Tr}_A(x \otimes y) \end{array}$$

  is an isomorphism

$$\mathrm{Hom}(V; V)^* \cong \mathrm{Hom}(V; V).$$

Volume forms and determinants

- V is an n-dimensional vector space
- $\Lambda^n(V)$ is 1-dimensional;
- $\Lambda^k(V)$ is $\binom{n}{k}$-dimensional for $0 \leq k \leq n$.

Duality

- There is a natural dual V^* and there are often *unnatural* duals, too.

Inner products

- V is a vector space with basis $e_1, \dots, e_n$
- Define $\Omega = e_1 \wedge \cdots \wedge e_n$
- Let $\eta \in \Lambda^k(V), \rho \in \Lambda^{n-k}(V)$. Define

$$\langle \eta, \rho \rangle := \eta \wedge \rho / \Omega.$$