Topics in Geometric Mechanics: Week 14

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Recall

We left off with:

- momentum maps;
- coadjoint orbits.

Recall - Momentum map

Definition

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ a Lie algebra of smooth hamiltonians on $(M, \{,\})$. An homomorphism

$$\Psi : \mathfrak{g} \to \mathfrak{h} \qquad \xi \mapsto h_\xi$$

induces

$$\psi : M \to \mathfrak{g}^* \qquad \langle \psi(x), \xi \rangle = h_\xi(x).$$

We call ψ a momentum map.

Theorem

If ψ is a momentum map, then it is a Poisson map.

Convexity

Assumption: (M^{2n}, ω) is a compact symplectic manifold.

Theorem

(Atiyah-Guillemin-Sternberg) Let $T = \mathbf{T}^n$ have an effective Hamiltonian action on M with momentum map $\psi : M \to \mathfrak{t}^*$. Then $\psi(M)$ is a compact, convex subset of $\mathfrak{t}^*$.

Theorem

(Kirwan) If G is a compact Lie group with an effective Hamiltonian action on M with momentum map ψ, and $T \lhd G$ is a maximal torus, then $\psi(M) \cap \mathfrak{t}^*_+$ is a compact, convex polytope.

Theorem

(Delzant) There is a distinguished family D of convex polytopes which are the image of a torus momentum map. Given such a polytope, one can reconstruct (M, ω), the T-action and ψ.

Lemma

Let $x \in M$ *have the stabilizer subgroup* $T_x \lhd T$ *of dimension* $k \leq n$. *There are symplectic coordinates* $(x_1, y_1, \ldots, x_k, y_k, \ldots, x_n, y_n)$ *on a neighbourhood of* x *and a basis* $t_1, \ldots, t_n$ *of* $\mathfrak{t}^*$ *such that*

$$\psi(p) - \psi(x) = \sum_{a=1}^{k} \frac{1}{2}(x_a^2 + y_a^2) t_a + \sum_{b=k+1}^{n} x_b t_b.$$

Example

Let (M, ω) be $\mathbf{S}^2$ with its unit area form. Let $T = \mathbf{T}^1$ act by rotation about the vertical z-axis. The momentum map is $\psi = z$.

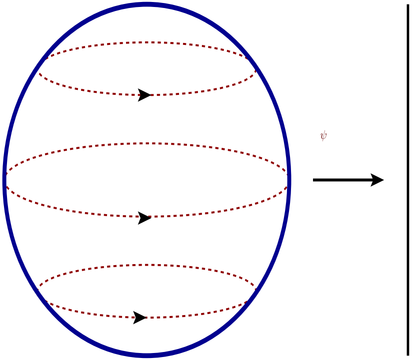

Figure: The momentum map.

Example

Let $M = \mathbf{S}^2 \times \mathbf{S}^2$ with ω the sum of the unit area forms, and $T = \mathbf{T}^2$ acting by rotating about each vertical axis.

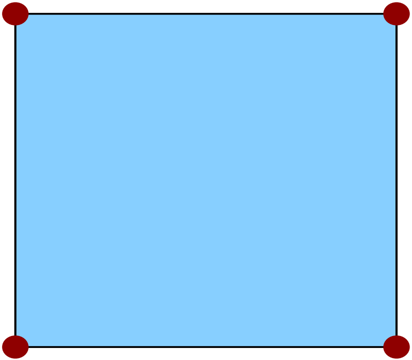

Figure: The image of the momentum map.

Liouville-Arnold Theorem

Let (M^{2n}, ω) be a symplectic manifold, $f_1, \ldots, f_n$ be Poisson commuting, independent functions.

Theorem

Let T be a connected, compact component of a regular level set $F_c = \{f_1 = c_1, \ldots, f_n = c_n\}$. Then $T \cong \mathbf{T}^n$ and there is a neighbourhood $U \supset T$ and coordinates $(\theta, I) : U \to \mathbf{T}^n \times \mathbf{R}^n$ such that

1. $$\omega | U = \sum_{i=1}^{n} \mathrm{d}\theta_i \wedge \mathrm{d}I_i \,,$$
2. $$f_i | U = F_i(I),$$
3. $$X_{f_i} = \sum_{j=1}^{n} \frac{\partial F_i}{\partial I_j} \frac{\partial}{\partial \theta_j}.$$

Simple Pendulum

Take

$$H = \frac{1}{2}p^2 + \cos(x).$$

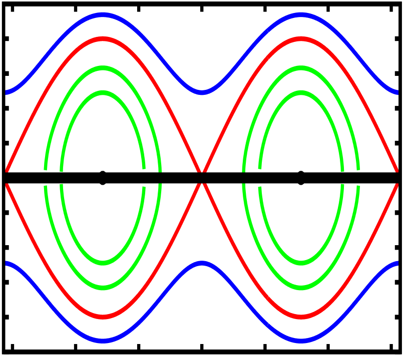

Figure: Pendulum.