Topics in Geometric Mechanics: Week 12

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Recall

We left off with:

- Poisson manifolds;
- Poisson maps;

Momentum map

Definition

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ a Lie algebra of smooth hamiltonians on $(M, \{,\})$. An homomorphism

$$\Psi : \mathfrak{g} \to \mathfrak{h} \qquad \xi \mapsto h_\xi$$

induces

$$\psi : M \to \mathfrak{g}^* \qquad \langle \psi(x), \xi \rangle = h_\xi(x).$$

We call ψ a momentum map.

Examples

1. Let $h \in C^\infty(M)$ and $\mathfrak{h} = \mathbf{R} \cdot h$, $\mathfrak{g} = \mathbf{R}$, and $\Psi(1) = h$. Then $\psi(x) = h(x)$. This is the tautological momentum map.

2. For $\mathfrak{g} = \mathfrak{sp}(\mathbf{R}^{2n})$, $\mathfrak{h} = S^2(\mathbf{R}^{2n})$ from above,

$$\psi(z) = \frac{1}{2} zz'J \qquad \psi : \mathbf{R}^{2n} \to \mathfrak{sp}(\mathbf{R}^{2n})^* \equiv \mathfrak{sp}(\mathbf{R}^{2n}).$$

3. For $\mathfrak{g} = \mathfrak{so}(\mathbf{R}^n)$ acting on $\mathbf{R}^{2n} = T^*\mathbf{R}^n$,

$$\psi(z) = \frac{1}{2}(px' - xp') \qquad \psi : \mathbf{R}^{2n} \to \mathfrak{so}(\mathbf{R}^n)^* \equiv \mathfrak{so}(\mathbf{R}^n).$$

Theorem

Let $\psi : M \to \mathfrak{g}^*$ be a momentum map. Then ψ is a Poisson map.

Momentum maps and symplectic manifolds

Let (M, Ω) be a symplectic manifold, G a Lie group that acts by Poisson diffeomorphisms. For $\xi \in \mathfrak{g}$, define the vector field on M

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot x \qquad \forall x \in M.$$

Theorem

If $H^1(M)$ and $H^2(\mathfrak{g})$ vanishes, then there is a momentum map $\psi : M \to \mathfrak{g}^*$ such that the hamiltonian of ξ_M is $\psi^* \xi$ for all $\xi \in \mathfrak{g}$.

Proof.

If $H^1(M)$ vanishes, then all closed 1-forms are exact. The locally hamiltonian vector field ξ_M therefore has a hamiltonian $h = h_\xi$. We get a linear map $\Psi : \mathfrak{g} \to C^\infty(M)$, $\xi \mapsto h_\xi$. We know that $h_{[\xi,\eta]}$ is a hamiltonian of $[\xi, \eta]_M$ and that $\{h_\xi, h_\eta\}$ is a hamiltonian, too. Therefore,

$$c(\xi, \eta) = h_{[\xi,\eta]}(x) - \{h_\xi, h_\eta\}(x)$$

is independent of $x \in M$ and so it defines a skew-symmetric 2-form on $\mathfrak{g}$. It is closed, and therefore exact, so there is a $\mu \in \mathfrak{g}^*$ such that

$$c(\xi, \eta) = \langle \mu, [\xi, \eta] \rangle.$$

If we define

$$H_\xi = h_\xi - \langle \mu, \xi \rangle,$$

then $\xi \mapsto H_\xi$ is a Lie algebra homomorphism. □

Cotangent Bundles

Let (T^*M, Ω). Let $f : M \to M$ be a diffeomorphism of M. Define the canonical lift of f to T^*M by

$$F(x, p) = (f(x), (d_x f^{-1})^* p).$$

Theorem

The canonical lift is a symplectomorphism.

Proof.

First, assume that M is an open subset of $\mathbf{R}^n$. Let $x = (x_i)$ be a coordinate system in a neighbourhood of $x_0 \in M$ and $y_i = f_i(x)$ a coordinate system in a neighbourhood of $y_0 = f(x_0)$. Let p_i (resp. q_i) be linear coordinates on $T_x^* M$ (resp. $T_y^* M$) so that the Liouville 1-form is $\theta_x = \sum_i p_i \,\mathrm{d}x_i$, $\theta_y = \sum_i q_i \,\mathrm{d}y_i$. Observe that $q = (d_x f^{-1})^* p$, i.e. $p = (d_x f)^* q$ and that $\mathrm{d}y_i = \sum_\alpha \frac{\partial f_i}{\partial x_\alpha} \,\mathrm{d}x_\alpha$.

$$F^*\theta = \sum_i F^*(q_i \,\mathrm{d}y_i) = \sum_{i,\alpha} q_i \frac{\partial f_i}{\partial x_\alpha} \,\mathrm{d}x_\alpha = \sum_\alpha [(df)^* q]_\alpha \,\mathrm{d}x_\alpha$$
$$= \sum_\alpha p_\alpha \,\mathrm{d}x_\alpha = \theta.$$

Since $F^*(\mathrm{d}\theta) = \mathrm{d}(F^*\theta) = \mathrm{d}\theta$, we are done. □

Lifts and Hamiltonians

Let X be a vector field on M.

Theorem

The hamiltonian vector field Y of

$$H_X(x,p) = \langle p, X(x) \rangle$$

satisfies

$$\mathrm{d}\pi(Y) = X, \qquad \text{where } \pi : T^*M \to M : (x,p) \mapsto x.$$

We call Y the canonical lift of X.

Proof.

We have that $\mathrm{d}\pi(\dot{x}, \dot{p}) = \dot{x}$. Now, use Hamilton's equations

$$\dot{x} = \frac{\partial H_X}{\partial p} = X(x).$$

□

Lie group actions

Theorem

Let G be a Lie group that acts smoothly on M. Then, the canonical lift of this action is hamiltonian, with momentum map

$$\psi : T^*M \to \mathfrak{g}^* \qquad \langle \psi(x,p), \xi \rangle = \langle p, \xi_M(x) \rangle$$

where $\xi_M(x) = \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot x$ *for all* $x \in M$, $\xi \in \mathfrak{g}$.

Proof.

It suffices to verify that if X, Y are smooth vector fields on M, then

$$\{H_X, H_Y\} = H_{[X,Y]}.$$

□