Topics in Geometric Mechanics: Week 11

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Recall

We left off with:

- Poisson algebras;
- Poisson structures;

Poisson manifolds

Definition

Let M be a smooth manifold, $\{,\}$ a Poisson bracket on $C^\infty(M)$. We call $(M, \{,\})$ a Poisson manifold. We call $\mathcal{P}$,

$$\mathcal{P}(df, dg) := \{f, g\}, \qquad \forall f, g \in C^\infty(M)$$

a Poisson structure.

Example

Let $M = \mathbf{R}^3$ with its dot and cross products. Define, at $x \in \mathbf{R}^3$,

$$\mathcal{P}(df, dg)_x = x \cdot (\nabla f \times \nabla g) \qquad \forall f, g \in C^\infty(\mathbf{R}^3).$$

Theorem

The Poisson structure is a $(2,0)$ skew-symmetric tensor field on M. Conversely, given a skew-symmetric $(2,0)$ tensor field $\mathcal{P}$, the equation

$$\mathcal{P}(df, dg) := \{f, g\}, \qquad \forall f, g \in C^{\infty}(M)$$

defines a Poisson bracket iff

$$J^{ijk} = \sum_{\alpha} \mathcal{P}^{i\alpha} \frac{\partial \mathcal{P}^{jk}}{\partial x_{\alpha}} + \mathcal{P}^{j\alpha} \frac{\partial \mathcal{P}^{ki}}{\partial x_{\alpha}} + \mathcal{P}^{k\alpha} \frac{\partial \mathcal{P}^{ij}}{\partial x_{\alpha}}$$

where (x_i) is a coordinate system and $i, j, k = 1, \ldots, \dim M$.

Properties

Theorem

If $\mathcal{P}$ is a Poisson structure and X is a hamiltonian vector field with respect to $\mathcal{P}$, then $\mathsf{L}_X\mathcal{P} = 0$.

Proof.

Let $f \in C^\infty(M)$ and $X = \mathcal{P} \cdot df$. Then $\{f, g\} = \mathcal{P}(df, dg) = \mathsf{L}_X g$ for all $g \in C^\infty(M)$. Then

$$\begin{aligned}(\mathsf{L}_X\mathcal{P})(dg, dh) &= \mathsf{L}_X(\mathcal{P}(dg, dh,)) - \mathcal{P}(\mathsf{L}_X dg, dh) - \mathcal{P}(dg, \mathsf{L}_X dh) \\ &= \{f, \{g, h\}\} - \{\{f, g\}, h\} - \{g, \{f, h\}\} \\ &= \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0\end{aligned}$$

$\forall g, h \in C^\infty(M)$.

Since the cotangent space at any point is spanned by differentials of functions, this shows that $\mathsf{L}_X\mathcal{P} = 0$. $\square$

Symplectic manifolds

Theorem

A Poisson structure $\mathcal{P}$ is nowhere degenerate iff the $(0, 2)$ tensor field $\Omega = \mathcal{P}^{-1}$ is a symplectic form.

Proof.

First, note that if $X = \mathcal{P} \cdot df$, then $\iota_X \Omega = \Omega \cdot X = df$ since $\Omega \cdot \mathcal{P} = 1$. We need Ω to be non-degenerate and closed. The former is clear. For the latter, let $p \in M$ and $x, y, z \in T_p M$. There are smooth functions f, g, h such that $x = \mathcal{P}_p \cdot df_p, y = \mathcal{P}_p \cdot dg_p, z = \mathcal{P}_p \cdot dh_p$. Let $X = \mathcal{P} \cdot df$, etc. so that $x = X(p)$, etc.

$$
\begin{aligned}
\left[(d\iota_X + \iota_X d)\Omega\right](Y, Z) &= \left[\mathsf{L}_X \Omega\right](Y, Z) \\
\left[(d\iota_Y + \iota_Y d)\Omega\right](Z, X) &= \left[\mathsf{L}_Y \Omega\right](Z, X) \\
\left[(d\iota_Z + \iota_Z d)\Omega\right](X, Y) &= \left[\mathsf{L}_Z \Omega\right](X, Y) \\
\implies 3d\Omega(X, Y, Z) &= 0.
\end{aligned}
$$

□

Poisson maps

Definition

A map $\phi : (M, \{,\}_M) \to (N, \{,\}_N)$ is a Poisson map iff it preserves Poisson brackets

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi \qquad \forall f, g \in C^\infty(N).$$

Examples

- Let $\varphi_t : M \to M$ be the time-t map of the hamiltonian flow of $f \in C^\infty(M)$. Then $\phi = \varphi_t$ is a Poisson map.
- Let $\iota : \mathfrak{h} \hookrightarrow \mathfrak{g}$ be a subalgebra, let $\pi = \iota^* : \mathfrak{g}^* \to \mathfrak{h}^*$. Then π is a Poisson map.
- Let $f : M \to N$ be a diffeomorphism. Then $\phi = \mathrm{d}f^* : T^*M \to T^*N$ is a Poisson map.

Momentum map

Definition

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ a Lie algebra of smooth hamiltonians on $(M, \{,\})$. An homomorphism

$$\Psi : \mathfrak{g} \to \mathfrak{h} \qquad \xi \mapsto h_\xi$$

induces

$$\psi : M \to \mathfrak{g}^* \qquad \langle \psi(x), \xi \rangle = h_\xi(x).$$

We call ψ a momentum map.

Examples

1. Let $h \in C^\infty(M)$ and $\mathfrak{h} = \mathbf{R} \cdot h$, $\mathfrak{g} = \mathbf{R}$, and $\Psi(1) = h$. Then $\psi(x) = h(x)$. This is the tautological momentum map.

2. For $\mathfrak{g} = \mathfrak{sp}(\mathbf{R}^{2n})$, $\mathfrak{h} = S^2(\mathbf{R}^{2n})$ from above,

$$\psi(z) = \frac{1}{2} z z' J \qquad \psi : \mathbf{R}^{2n} \to \mathfrak{sp}(\mathbf{R}^{2n})^* \equiv \mathfrak{sp}(\mathbf{R}^{2n}).$$

3. For $\mathfrak{g} = \mathfrak{so}(\mathbf{R}^n)$ acting on $\mathbf{R}^{2n} = T^*\mathbf{R}^n$,

$$\psi(z) = \frac{1}{2}(p x' - x p') \qquad \psi : \mathbf{R}^{2n} \to \mathfrak{so}(\mathbf{R}^n)^* \equiv \mathfrak{so}(\mathbf{R}^n).$$

Theorem

Let $\psi : M \to \mathfrak{g}^*$ *be a momentum map. Then* ψ *is a Poisson map.*

Momentum maps and symplectic manifolds

Let (M, Ω) be a symplectic manifold, G a Lie group that acts by Poisson diffeomorphisms. For $\xi \in \mathfrak{g}$, define the vector field on M

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot x \qquad\qquad \forall x \in M.$$

Theorem

If $H^1(M)$ and $H^2(\mathfrak{g})$ vanishes, then there is a momentum map $\psi : M \to \mathfrak{g}^*$ such that the hamiltonian of ξ_M is $\psi^*\xi$ for all $\xi \in \mathfrak{g}$.

Proof.

If $H^1(M)$ vanishes, then all closed 1-forms are exact. The locally hamiltonian vector field ξ_M therefore has a hamiltonian $h = h_\xi$. We get a linear map $\Psi : \mathfrak{g} \to C^\infty(M)$, $\xi \mapsto h_\xi$. We know that $h_{[\xi,\eta]}$ is a hamiltonian of $[\xi, \eta]_M$ and that $\{h_\xi, h_\eta\}$ is a hamiltonian, too. Therefore,

$$c(\xi, \eta) = h_{[\xi,\eta]}(x) - \{h_\xi, h_\eta\}(x)$$

is independent of $x \in M$ and so it defines a skew-symmetric 2-form on $\mathfrak{g}$. It is closed, and therefore exact, so there is a $\mu \in \mathfrak{g}^*$ such that

$$c(\xi, \eta) = \langle \mu, [\xi, \eta] \rangle.$$

If we define

$$H_\xi = h_\xi - \langle \mu, \xi \rangle,$$

then $\xi \mapsto H_\xi$ is a Lie algebra homomorphism. $\square$