Topics in Geometric Mechanics: Week 1

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Introduction

What is geometric mechanics? At first blush, it is an *approach* to studying mechanics:

- classical Newtonian mechanics;
- Lagrangian mechanics;
- Hamiltonian mechanics;
- Nonholonomic mechanics.

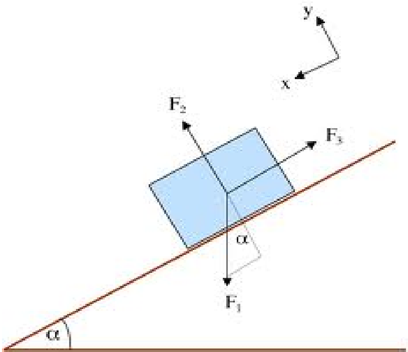
Let us look briefly at each in turn.

Newtonian Mechanics

Classical mechanics

$$F = \frac{d(mv)}{dt} = \frac{dp}{dt}$$

- m is the *mass* of the body;
- v is the body's velocity;
- F is the force acting;
- $p = mv$ is the momentum.



Newtonian Mechanics

Classical mechanics - clarifications

$$\underbrace{F}_{\text{what is a force?}} = \underbrace{\frac{d\overbrace{(mv)}^{\text{what is mass? velocity?}}}{dt}}_{\text{what is the rate of change of velocity?}} = \frac{\overbrace{dp}^{\text{what is momentum?}}}{dt}$$

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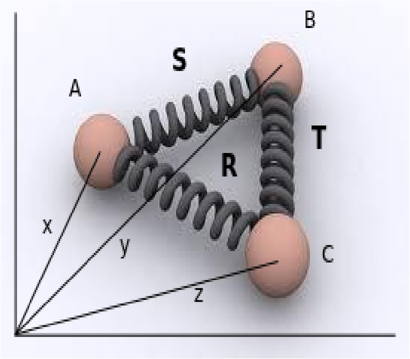
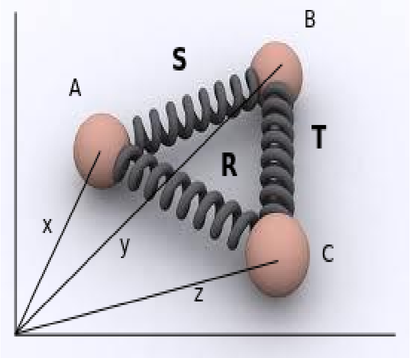
An Example - springs & masses



$$A\ddot{x} = -S(x-y) - R(x-z)$$
$$B\ddot{y} = -S(y-x) - T(y-z)$$
$$C\ddot{z} = -R(z-x) - T(z-y)$$

An Example - springs & masses

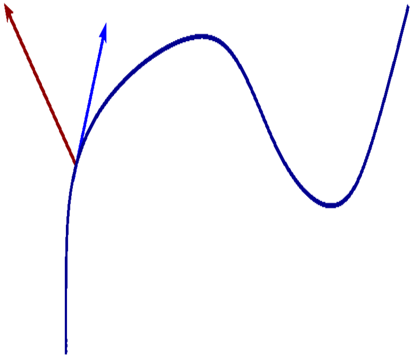


$$\frac{d}{dt}\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = -\begin{bmatrix} +S+R & -S & -R \\ -S & +S+T & -T \\ -R & -T & +R+T \end{bmatrix}\begin{bmatrix} x \\ y \\ x \end{bmatrix}$$

$$\frac{d}{dt}\mathbf{M}\dot{\mathbf{q}} = -\mathbf{Q}\mathbf{q}$$

What is a force?

A *Force* is something we *integrate* over a path to compute *Work* done.



$$W = \int_\sigma \langle F, d\sigma \rangle$$

What is a force?

A Force is a differential 1-form.

- a force F is *conservative* if the line integral $\oint_\sigma \langle F, \mathrm{d}\sigma \rangle$ vanishes for all closed loops.
- a 1-form F is *exact* if it integrates to zero along all closed loops, in which case $F = \mathrm{d}f$ for some smooth function f.

An example - spring force

Let's compute the exterior derivative of the force 1-form F for the spring-mass system:

$$\begin{aligned} F &= \sum_i (\mathbf{Q}\mathbf{q})_i \,\mathrm{d}\mathbf{q}_i = \sum_{i,j} \mathbf{Q}_{ij}\mathbf{q}_j \,\mathrm{d}\mathbf{q}_i \\ \mathrm{d}F &= \sum_{i,j} \mathbf{Q}_{ij} \,\mathrm{d}\mathbf{q}_j \wedge \mathrm{d}\mathbf{q}_i = 0 \end{aligned}$$

since the first is symmetric in i, j and the second is skew symmetric.

An example - spring force

This shows F is *closed*. It is exact since every closed 1-form on $\mathbf{R}^n$ is exact. We compute, for the path $\sigma(t) = t\mathbf{q}$,

$$f(\mathbf{q}) = \int_{\mathbf{0}}^{\mathbf{q}} \langle F(\sigma(t)), \dot{\sigma}(t)\rangle \,\mathrm{d}t = \int_0^1 \langle F(t\mathbf{q}), \mathbf{q}\rangle \,\mathrm{d}t$$

$$= -\int_0^1 \langle \mathbf{Q}\mathbf{q}, \mathbf{q}\rangle t \,\mathrm{d}t = -\int_0^1 t \,\mathrm{d}t \times \langle \mathbf{Q}\mathbf{q}, \mathbf{q}\rangle = -\frac{1}{2}\langle \mathbf{Q}\mathbf{q}, \mathbf{q}\rangle.$$

By definition $U = -f$ is the *potential energy* of the system.

What is a velocity?

1. The velocity of a point with position $p(t)$ at time t is $\dot{p}(t)$.

2. We measure velocity by a measuring rod, f (a scalar valued function).

$$\dot{f}(t) = \frac{d}{dt} f \circ p(t)$$

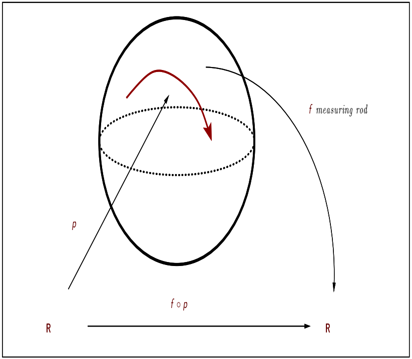


Figure: How to measure velocity

What is a velocity?

We need to do calculus. To do calculus, we need differentiable manifolds.

What is a velocity?

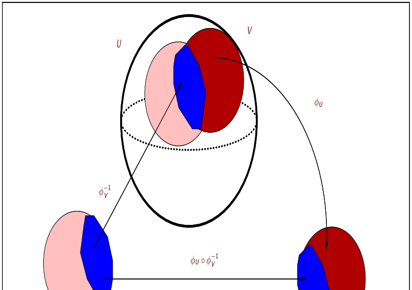
We need to do calculus. To do calculus, we need differentiable manifolds.

1. A differentiable manifold M is a (Hausdorff) topological space, an open covering $C = \{U\}$, and homeomorphisms $\phi_U : U \to \mathbf{R}^n$.

2. The homeomorphisms ϕ_U satisfy:

$$\phi_U \circ \phi_V^{-1} : \phi_V(U \cap V) \to \phi_U(U \cap V)$$

is a diffeomorphism for all $U, V \in C$.



Examples of manifolds

1. $\mathbf{R}^n$
2. $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ (the unit sphere)
3. The group of $n \times n$ matrices of non-zero determinant, $\mathrm{GL}(\mathbf{R}^n)$.
4. The group of solid rotations.
5. The set of all lines in $\mathbf{R}^3$.

A worked example

Let $\mathbf{S}^2 \subset \mathbf{R}^3$ be the unit sphere.

- regular level set;
- stereographic projection;
- graph.

A worked example

Let A be the set of all lines $\mathbf{R}^3$.

- each line ℓ can be written as $\ell = \{p + tv \,\mathrm{s.t.}\, t \in \mathbf{R}\}$ where:
  1. v has unit length;
  2. p and v are orthogonal;
  3. v is unique up to a change of sign (= orientation of the line).
- $A \simeq \left\{(v, p) \in \mathbf{R}^3 \times \mathbf{R}^3 \,\mathrm{s.t.}\, |v| = 1, \langle v, p\rangle = 0\right\} / \sim .$

Vector fields and differential forms

Let M be a manifold with a coordinate system $x : U \subset M \to \mathbf{R}^n$.

- We get functions x_i defined on U;
- We get vector fields

$$\frac{\partial}{\partial x_i}$$

- We get differential forms

$$\mathrm{d}x_i$$

- Each vector field V on M has

$$V|U = V_1(x)\frac{\partial}{\partial x_1} + \cdots + V_n(x)\frac{\partial}{\partial x_n}$$

- Each differential 1-form F has

$$F|U = F_1(x)\,\mathrm{d}x_1 + \cdots + F_n(x)\,\mathrm{d}x_n$$

Vector fields and differential forms

We compute that

$$\langle \mathrm{d}x_i , \frac{\partial}{\partial x_j} \rangle = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

$$\langle \mathrm{d}x_i , V \rangle = \mathsf{L}_V x_i = \sum_j V_j \times \langle \mathrm{d}x_i , \frac{\partial}{\partial x_j} \rangle = V_i$$

$$\langle F, V \rangle = \sum_i F_i(x) V_i(x).$$