MTH-696A: TOPICS IN GEOMETRIC MECHANICS ASSIGNMENT 3

DR. LEO BUTLER

A. Let M be the configuration space of the planar double pendulum. See figure 1.

(a) The configuration space of the planar double pendulum is the set of pairs of angles $(x, y) \mod 2\pi$. What is the standard name for this surface?

Solution. The torus, $\mathbf{T}^2$.

(b) Determine the kinetic energy of the system depicted. The bobs of masses m and M move on the inflexible rods of lengths l and L and the suspension point P is fixed.

Solution. Let $z = l\exp(ix)$ and $w = L\exp(iy)$ so the position of the body of mass m is z and that of mass M is $Z = z + w$. The velocities are $\dot{z} = zi\dot{x}$ and $\dot{Z} = zi\dot{x} + wi\dot{y}$. We get that $|\dot{z}| = l|\dot{x}|$ and $|\dot{Z}|^2 = l^2|\dot{x}|^2 + 2lL\cos(x-y)\dot{x}\dot{y} + L^2|\dot{y}|^2$. The kinetic energy is $\frac{1}{2}m\dot{z}^2 + \frac{1}{2}M|\dot{Z}|^2$.

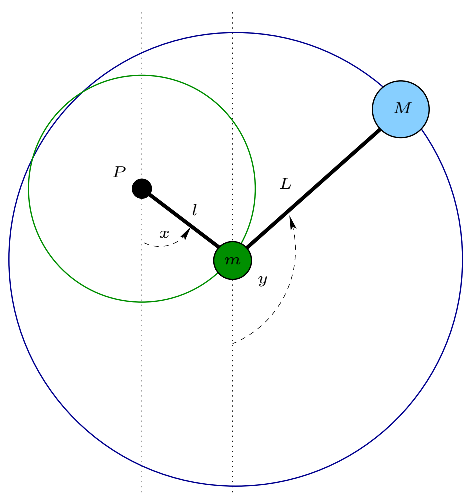

FIGURE 1. Planar double pendulum.

B. Let M be an n-manifold and let g be a Riemannian metric on M.

(a) Suppose that x_i and y_j are smooth coordinate systems on M, and $x = \phi(y)$ for a smooth diffeomorphism ϕ. Show that

$$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \det\Phi \cdot \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n\,,$$

where

$$\Phi = \left[\frac{\partial \phi_i}{\partial y_j}\right]$$

is the Jacobian matrix of the coordinate transformation.

Solution. We have that $\mathrm{d}x_i = \sum_{j=1}^n \frac{\partial \phi_i}{\partial y_j}\,\mathrm{d}y_j$. Then $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \sum_{j_1,\ldots,j_n} \left(\Phi^1_{j_1}\,\mathrm{d}y_{j_1}\right) \wedge \cdots \wedge \left(\Phi^1_{j_n}\,\mathrm{d}y_{j_n}\right) = \det(\Phi)\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$.

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(b) In a local system of coordinates x_i on M, one can define an n-form

$$\Omega = \sqrt{\det \mathbf{g}}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\,,$$

where

$$\mathbf{g} = [g_{ij}]$$

is the matrix of $g = \sum_{i,j=1}^{n} g_{ij}\, \mathrm{d}x_i \cdot \mathrm{d}x_j$ in the local coordinates. Use the first part to show that if M is orientable, then Ω is a tensor field on M. What goes wrong if M is not orientable?

Solution. Let Ω' (resp. $\mathbf{g}' = [g'_{ij}]$) denote Ω (resp. $\mathbf{g} = [g_{ij}]$) in the y coordinates. We want to show that $\phi^*\Omega' = \Omega$, i.e. that the formula for Ω holds in all coordinate systems. Indeed, we compute that $g = \sum_{ij} g_{ij}\, \mathrm{d}x_i \cdot \mathrm{d}x_j = \sum_{i,j,k,l} g_{ij} \Phi^i_k \Phi^j_l\, \mathrm{d}y_k \cdot \mathrm{d}y_l = \sum_{k,l} g'_{kl}\, \mathrm{d}y_k \cdot \mathrm{d}y_l$. This shows that $\mathbf{g}' = \Phi' \mathbf{g} \Phi$. Then $\Omega = \sqrt{\det \mathbf{g}}\, \det(\Phi)\, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n = \sqrt{\det \mathbf{g}'}\, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n = \Omega'$. We have used the positivity of $\det \Phi$ for the last step.

(c) The n-form defined above is non-degenerate. Prove this.

Solution. Since g is a metric, $\mathbf{g}$ is positive definite so its determinant is always positive.

(d) The n-form Ω is called a Riemannian volume form. Compute the Riemannian 2-form for the sphere in $\mathbf{R}^3$ of radius r. [Hint: it is easiest to use spherical coordinates on $\mathbf{R}^3$.]

Solution. Let (θ, ϕ) be spherical coordinates, so that $(x, y, z) = f(\theta, \phi) = r(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$ for $0 < \phi < \pi$ and $0 < \theta < 2\pi$. The vector fields $X = \mathrm{d}f \cdot \frac{\partial}{\partial\theta}$ and $Y = \mathrm{d}f \cdot \frac{\partial}{\partial\phi}$ determine the Riemannian metric by $g = \langle X, X\rangle\, \mathrm{d}\theta^2 + 2\langle X, Y\rangle\, \mathrm{d}\theta \cdot \mathrm{d}\phi + \langle Y, Y\rangle\, \mathrm{d}\phi^2$. We compute that $X = r(-\sin\theta \sin\phi, \cos\theta \sin\phi, 0)$ and $Y = r(\cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi)$. This gives

$$\mathbf{g} = \begin{bmatrix} r^2 \sin^2\phi & 0 \\ 0 & r^2 \end{bmatrix} \qquad \Omega = r^2 \sin\phi\, \mathrm{d}\theta \wedge \mathrm{d}\phi\,. \tag{1}$$