MTH-696A: TOPICS IN GEOMETRIC MECHANICS ASSIGNMENT 2

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A. Let M be the configuration space of the spherical pendulum.

(a) Let $c : [0, 1] \to M$ be a smooth curve. Show directly that the work done by the stiff rod along this curve is zero.

Solution. The force F exerted by the rod is perpendicular to the sphere, so for any curve c tangent to the sphere we have $\langle F, \dot{c}\rangle \equiv 0$ which gives $W = \int_0^1 \langle F(c(t)), \dot{c}(t)\rangle \, \mathrm{d}t = 0$.

(b) Determine the kinetic and potential energies of a bob of mass m, assuming the stiff rod has zero mass.

Solution. If the instantaneous velocity of the bob is v when it is at the point x, then the kinetic energy $T = \frac{1}{2}|v|^2$ where $| \bullet |$ is the Euclidean norm in $\mathbf{R}^3$ restricted to $T_x\mathbf{S}^2$. To compute the potential energy, let $F = -ge_3$ be the downward force due to gravity. We choose $S = -e_3$ to be the point of zero potential energy and let $c : [0, 1] \to M$ be a smooth curve connecting S to x. The work done by the gravitational force is $W = \int_0^1 \langle F, \dot{c}(t)\rangle \, \mathrm{d}t = \int_0^1 -g \frac{dx_3}{dt} \, \mathrm{d}t = -g(x_3(1) - x_3(0))$. Thus $W = -g\Delta x_3$ and so the potential energy is $V = g(x_3 + 1)$ since $x_3(0) = -1$.

B. Let V be an n-dimensional vector space, and let $(\mathsf{T}^*(V), \otimes)$, $(\mathsf{S}^*(V), \cdot)$ and $(\Lambda^*(V), \wedge)$ be the tensor, symmetric and exterior algebras of V.

(a) Let $v_1, \ldots, v_n$ be a basis of V. Show that, for each natural number k, the set

$$\{v_{i_1} \otimes \cdots \otimes v_{i_k} \text{ s.t. } 1 \leq i_1, \ldots, i_k \leq n\}$$

is a basis of $\mathsf{T}^k(V)$.

Solution. The claim is certainly true for $k = 1$ when $\mathsf{T}^1(V) = V$. So, assume the claim is true for $i, \ldots, k - 1$. Let $\phi_1, \ldots, \phi_n$ be a dual basis of V^*. For each k-tuple $I = (i_1, \ldots, i_k)$ with $1 \leq i_1, \ldots, i_k \leq n$, let $v_I = v_{i_1} \otimes \cdots \otimes v_{i_k}$ and likewise for ϕ_I. The multilinearity of $\otimes$ implies that the collection of v_I spans $\mathsf{T}^k(V)$. To prove it is a basis, suppose that $\eta = \sum_I \eta_I v_I = 0$. We need to show that each coefficient $\eta_I = 0$. Define, for a monomial $w_J \in \mathsf{T}^k(V)$,

$$\langle \phi_I, w_J\rangle = \prod_{\alpha=1}^{k} \langle \phi_{i_\alpha}, w_{j_\alpha}\rangle.$$

Due to the properties of the $\otimes$, this is a well-defined map that extends from the generators to a linear map. Moreover, we see that $\langle \phi_I, v_J\rangle = \delta_{IJ}$. Therefore $0 = \langle \phi_I, \eta\rangle = \eta_I$ for all I. This proves the linear independence.

(b) Prove the analogous facts for $\Lambda^k(V)$ and $\mathsf{S}^k(V)$.

Solution. The proof is essentially identical, since we have only used multi-linearity of $\otimes$.

(c) Let us say that a k-tensor x is irreducible if there are $a_1, \ldots, a_k \in V$ such that $x = a_1 \otimes \cdots \otimes a_k$. Show that for $k = 2$, there are reducible (= not irreducible) tensors. [Remark: this is true for all $k \geq 2$.]

Solution. Let W, V be finite-dimensional vector spaces of dimension ≥ 2. Let $w_i, i = 1, \ldots, w$ (resp. $v_j, j = 1, \ldots, v$) be a basis of W (resp. V). We know that $w_i \otimes v_j$ is a basis of $W \otimes V$.

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Define a linear map $\phi : W \otimes V \to \mathrm{Mat}_{w\times v}(\mathbf{R})$ by

$$\phi(\eta) = \phi(\sum_{ij} \eta_{ij} w_i \otimes v_j) = \sum_{ij} \eta_{ij} E_{ij}$$

where E_{ij} is the $w \times v$ matrix with zeroes everywhere but in the (i,j) entry, which is 1. This map ϕ is a linear isomorphism. It is clear that if $\eta = a \otimes b$, then the rank of the matrix $\phi(\eta)$ is 1. Since $v, w \geq 2$, there is a matrix x of rank 2 or more in $\mathrm{Mat}_{w\times v}(\mathbf{R})$. Then $y = \phi^{-1}(x)$ cannot equal $a \otimes b$. If we apply this to $\mathsf{T}^{k+1}(V) = \mathsf{T}^k(V) \otimes V$, then we have proven the claim in full generality.

(d) Show that the previous fact is true for both symmetric and skew-symmetric tensors, too.

(i) **Solution**. For symmetric tensors, let $v, w \in V$ be linearly independent and define $\alpha = v{\cdot}v + w{\cdot}w \in \mathsf{S}^2(V)$. Wolg, we can suppose that v, w are orthogonal unit vectors. Define $\phi : \mathsf{S}^2(V) \to \mathrm{Mat}_{n\times n}(\mathbf{R})$ by $\phi(x \cdot y) = xy' + yx'$. This map is well-defined on generators of $\mathsf{S}^2(V)$ and extends to a linear map. Suppose that $A = \phi(\alpha)$ equals $B = \phi(x \cdot y)$ for some $x, y \in V$. By construction, $Av = 2v, Aw = 2w$ and $As = 0$ for all $s \perp v, w$. On the other hand, the image of B lies in spanx, y. Therefore, we must have that $x, y \in$ spanv, w. Then, $2x = Ax = Bx = x(y'x) + y(x'x)$. This implies that y is a scalar multiple of x. Therefore, the rank of B is 1; but the rank of A is 2. Absurd.

(ii) **Solution**. For skew-symmetric tensors, let $u, v, w, x \in V$ be linearly independent and define $\alpha = u \wedge v + w \wedge x \in \Lambda^2(V)$. Define a map $\phi : \Lambda^2(V) \to \mathrm{Mat}_{n\times n}(\mathbf{R})$ by $\phi(a \wedge b) = ab' - ba'$. One verifies that ϕ is well-defined on the generators of $\Lambda^2(V)$ and extends to a linear map. If $A = \phi(\alpha)$ equals $B = \phi(a \wedge b)$ for some $a, b \in V$, then rank of A $(= 4)$ equals that of B $(= 2)$. Absurd.

C. Let us continue with the notation of the previous question. Say that a linear transformation $L : \mathsf{T}^*(V) \to \mathsf{T}^*(V)$ is a derivation if

$$L(x \otimes y) = L(x) \otimes y + x \otimes L(y)$$

for all $x, y \in \mathsf{T}^*(V)$.

(a) Let $A : V \to V$ be a linear transformation and let $\exp(tA) = I + tA + \frac{1}{2}t^2A^2 + \cdots$ be the exponential. Show that $\frac{d\exp(tA)}{dt}|_{t=0}$ induces a derivation of $\mathsf{T}^*(V)$.

Solution. We have that $\exp(tA)x = x + tAx + O(t^2)$. Using the multilinearity of $\otimes$, we have that $(\exp(tA)x) \otimes (\exp(tA)y) = x \otimes y + t(Ax) \otimes y + tx \otimes (Ay) + O(t^2)$. Take the derivative with respect to t at $t = 0$ gives the answer.

(b) Show that there is a bijection between linear transformations $V \to V$ and derivations of $\mathsf{T}^*(V)$. [Hint: show that a derivation is uniquely determined by its action on V.]

Solution. Suppose that L, M are derivations that agree on $V = \mathsf{T}^1(V)$. Suppose that $L = M$ on $\mathsf{T}^{k-1}(V)$ for some $k \geq 2$. Then $L(v_1 \otimes \cdots \otimes v_{k-1} \otimes v_k) = L(v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k + v_1 \otimes \cdots \otimes v_{k-1} \otimes L(v_k) = M(v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k + v_1 \otimes \cdots \otimes v_{k-1} \otimes M(v_k) = M(v_1 \otimes \cdots \otimes v_{k-1} \otimes v_k)$ for all $v_1, \ldots, v_k \in V$. Therefore L and M coincide on a basis of $\mathsf{T}^k(V)$, and since they are linear, they coincide. By induction, they coincide on $\mathsf{T}^*(V)$.

D. Let I be the $n \times n$ identity matrix and

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \qquad J : \mathbf{R}^n \oplus \mathbf{R}^n \to \mathbf{R}^n \oplus \mathbf{R}^n,$$

$$\mathrm{Sp}(\mathbf{R}^{2n}) = \{X \in \mathrm{Mat}_{2n\times 2n}(\mathbf{R}) \text{ s.t. } X'JX = J\}.$$

Prove that $\mathrm{Sp}(\mathbf{R}^{2n})$ is a submanifold of $\mathrm{Mat}_{2n\times 2n}(\mathbf{R})$. [Bonus: show it is a group, too.]

Solution. Define the map $f : \mathrm{Mat}_{2n\times 2n}(\mathbf{R}) \to \mathfrak{so}(2n;\mathbf{R})$ (where $\mathfrak{so}(2n;\mathbf{R})$ is the vector space of real skew symmetric $2n \times 2n$ matrices), by

$$f(x) = x'Jx - J, \qquad x \in \mathrm{Mat}_{2n\times 2n}(\mathbf{R}). \tag{1}$$

Since J is skew symmetric, $y = f(x)$ is too, for all x. Since f is quadratic in the entries of x, it is smooth. In addition,

$$\mathrm{d}\, f_x v = v'Jx - x'Jv, \qquad x, v \in \mathrm{Mat}_{2n\times 2n}(\mathbf{R}) \tag{2}$$

where we have identified $T_x\mathrm{Mat}_{2n\times 2n}(\mathbf{R})$ with $\mathrm{Mat}_{2n\times 2n}(\mathbf{R})$. We want to show that when $x \in f^{-1}(0)$ (i.e. $x'Jx = J$), we have that $\mathrm{d}\, f_x : \mathrm{Mat}_{2n\times 2n}(\mathbf{R}) \to \mathfrak{so}(2n;\mathbf{R})$ is a surjective linear map. The submersion theorem then says that $f^{-1}(0)$ is a smooth submanifold. First, if $x \in f^{-1}(0)$, then $\det(x)^2 = \det(x'Jx) = \det(J) = 1$, so x is invertible. Second, we have that $\mathrm{d}\, f_x v = S \circ R_{Jx} \circ T(v)$ where $T(v) = v'$, $R_{Jx}(y) = yJx$ and $S(z) = z - z'$. The maps T and S are clearly surjective; since Jx is invertible, so is R_{Jx}. This proves that $\mathrm{d}\, f_x$ is surjective.

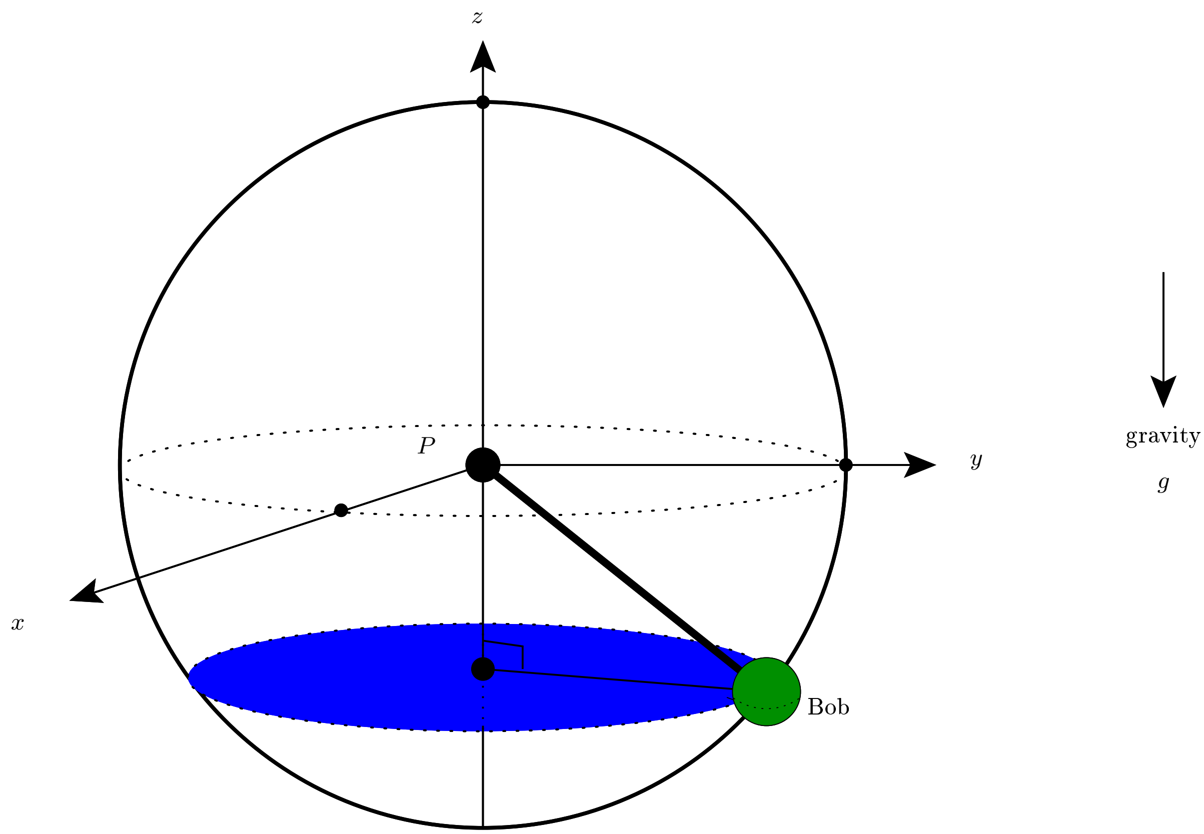

FIGURE 1. The spherical pendulum. The bob (in green) moves freely about the pivot P.