MTH-696A: TOPICS IN GEOMETRIC MECHANICS ASSIGNMENT 1

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A. Let $\times : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ be the vector (=cross) product. Define the following operator ω

$$\omega_x(u, v) := \langle x, u \times v \rangle, \tag{1}$$

for $x \in \mathbf{R}^3$ and $u, v \in T_x\mathbf{R}^3 \equiv \mathbf{R}^3$. Let e_1, e_2, e_3 be the standard basis of $\mathbf{R}^3$ and let x_i be coordinates induced by this basis (so $x = x_1e_1 + x_2e_2 + x_3e_3$).

(a) Compute $\omega = \sum_{i<j} \omega_{ij}(x)\, \mathrm{d}x_i \wedge \mathrm{d}x_j$.

Solution. We compute that

$$\omega_x(e_1, e_2) = \langle x, e_3 \rangle = x_3 \qquad \omega_x(e_2, e_3) = \langle x, e_1 \rangle = x_1 \qquad \omega_x(e_3, e_1) = \langle x, e_2 \rangle = x_2$$

$$\omega_x = x_3\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 + x_1\, \mathrm{d}x_2 \wedge \mathrm{d}x_3 + x_2\, \mathrm{d}x_3 \wedge \mathrm{d}x_1 \qquad \text{since } \mathrm{d}x_i \text{ is dual to } e_i.$$

Thus,

$$\mathrm{d}\omega_x = \mathrm{d}x_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 = 3\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 .$$

Alternatively: Let $u, v, w \in T_x\mathbf{R}^3$. Because $\mathrm{d}\omega$ is a tensor, it suffices to extend each tangent vector as a "constant" vector field on $\mathbf{R}^3$. Then,

$$\begin{aligned}
\mathrm{d}\omega_x(u, v, w) &= \mathsf{L}_u\omega_x(v, w) + \mathsf{L}_v\omega_x(w, u) + \mathsf{L}_w\omega_x(u, v) \\
&\quad - \omega_x(u, [v, w]) - \omega_x(v, [w, u]) - \omega_x(w, [u, v]) \\
&= \omega_u(v, w) + \omega_v(w, u) + \omega_w(u, v) && \text{since } \omega \text{ is linear in } x \\
&= 3 \det \begin{bmatrix} u & v & w \end{bmatrix} = 3\Omega_x(u, v, w),
\end{aligned}$$

where Ω is the "standard" volume form on $\mathbf{R}^3$.

(b) Compute the exterior derivative of ω, $\mathrm{d}\omega$, and show that this equals $\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$, the standard volume element on $\mathbf{R}^3$.

Solution. See above.

B. Let $\mathbf{S}^2 = \left\{x \in \mathbf{R}^3 \text{ s.t. } |x| = 1\right\}$ be the unit sphere. Let $\eta = \omega|_{\mathbf{S}^2}$ be the restriction of the 2-form ω defined in (1). Show that η is a closed, non-degenerate 2-form.

Solution. Since $\mathbf{S}^2$ is 2-dimensional, any 3-form on it is 0. This shows that $\mathrm{d}\eta = \mathrm{d}\omega\,|\mathbf{S}^2$ is zero. To prove non-degeneracy, let $x \in \mathbf{S}^2$ and $v \in T_x\mathbf{S}^2$. Then $w = x \times v \in T_x\mathbf{S}^2$ and is orthogonal to both x, v. Therefore, if $v \neq 0$, then $A = [x\ v\ w]$ has columns that form a basis of $\mathbf{R}^3$ so $\eta_x(v, w) = \det A \neq 0$.

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