MTH-696A: TOPICS IN GEOMETRIC MECHANICS ASSIGNMENT 9

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We will say that $(M, \{,\})$ is a Poisson manifold if, for all smooth function f and g, $\{f, g\} = \mathcal{P}(df, dg)$ for some $(2, 0)$ skew-symmetric tensor field $\mathcal{P}$, and $\{,\}$ satisfies the Jacobi identity. We call $\mathcal{P}$ the associated Poisson tensor.

A. Let $(M, \{,\})$ be a Poisson manifold. Show that $\mathcal{P}$ is preserved by all Hamiltonian vector fields: $\mathsf{L}_X P = 0$ for all hamiltonian vector fields X. (Hint: show that the Jacobi identity implies this.)

B. Let $\mathfrak{g}$ be a Lie algebra with a non-trivial centre. Prove that the centre of Poisson algebra $(C^\infty(\mathfrak{g}^*), \cdot, \{,\})$ with the canonical Poisson structure is non-trivial.

C. Let $(M, \{,\})$ be a Poisson manifold with associated Poisson tensor $\mathcal{P}$.

(a) Show that, for each $x \in M$, the subspace $H_x = \{X_f(x) \mid f \in C^\infty(M)\}$ equals the subspace $\operatorname{Im} \mathcal{P}_x$ where $\mathcal{P}_x : T_x^* M \to T_x M$.

(b) Prove that H_x is even dimensional and that $\mathcal{P}_x$ induces a non-degenerate skew-symmetric 2-form on H_x.

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