# MTH-696A: TOPICS IN GEOMETRIC MECHANICS ASSIGNMENT 4

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**A.** Let $\mathrm{G} = \mathrm{O}(n; \mathbf{R})$ and let $\mathfrak{g} = \mathfrak{o}(n; \mathbf{R})$ be the Lie algebra of G.

(a) Show that $\mathfrak{g}$ can be identified with the skew-symmetric real $n \times n$ matrices with Lie bracket $[x, y] = xy - yx$.

(b) Show that the Cartan-Killing trace form $\kappa(x, y) := -\mathrm{Tr}\,(xy)$ is positive definite on $\mathfrak{g}$.

**B.** Let us continue with the notation of the previous question. Let G act on $V = \mathrm{Mat}_{n\times n}(\mathbf{R})$ by

$$g \cdot x := gx \qquad\qquad g \in \mathrm{G}, x \in V.$$

(a) Show that this defines a smooth action of G on $V$.

(b) Let

$$\mathbf{x}_k = \mathbf{diag}(\underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{n-k \text{ times}})$$

and let $V_k = \mathrm{G} \cdot \mathbf{x}_k$. Show that the columns of each $x \in V_k$ form an orthonormal basis of a $k$-dimensional subspace of $\mathbf{R}^n$. Conversely, for each orthonormal basis $\hat{x} = [x_1, \ldots, x_k]$ of a $k$-dimensional plane in $\mathbf{R}^n$, there is a unique $x \in V_k$ whose columns provide the same orthonormal basis. Conclude that $V_k$ can be identified with the set of all orthonormal bases of all $k$-dimensional subspaces of $\mathbf{R}^n$. [This manifold is known as the Stiefel manifold of $k$-frames in $\mathbf{R}^n$.]

(c) Show that $V_1$ is diffeomorphic to $\mathbf{S}^{n-1}$ the unit sphere in $\mathbf{R}^n$.

(d) Show that $V_2$ is diffeomorphic to the unit tangent bundle of $\mathbf{S}^{n-1}$.

**C.** Let $e_1, \ldots, e_n$ be the standard basis of $V = \mathbf{R}^n$ and let $f_1, \ldots, f_n$ be the dual basis of $V^*$. Let $A \in \mathrm{Hom}(V; V)$ be a linear transformation.

(a) For an ordered subset $I \subset \{1, \ldots, n\}$, of cardinality $k$, let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. For $k = n - 1$, note that there is a bijection between the subsets $I$ and elements $i$ ($\{i\} = I^c$). Let $\phi_i = e_I$, using this bijection. Relative to the basis $\phi_1, \ldots, \phi_n$, show that he induced linear transformation $\Lambda^{n-1}(A) : \Lambda^{n-1}(V) \to \Lambda^{n-1}(V)$ has the matrix

$$C = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$$

where $C_{ij}$ is the determinant of the $(i, j)$ minor of the matrix of $A$. Let $D_{ij} = (-1)^{i+j} C_{ji}$.

(b) Show that $AD = (\det A)I$.

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