MTH-696A: TOPICS IN GEOMETRIC MECHANICS ASSIGNMENT 2

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A. Let M be the configuration space of the spherical pendulum.

(a) Let $c : [0, 1] \to M$ be a smooth curve. Show directly that the work done by the stiff rod along this curve is zero.

(b) Determine the kinetic and potential energies of a bob of mass m, assuming the stiff rod has zero mass.

B. Let V be an n-dimensional vector space, and let $(\mathsf{T}^*(V), \otimes)$, $(\mathsf{S}^*(V), \cdot)$ and $(\Lambda^*(V), \wedge)$ be the tensor, symmetric and exterior algebras of V.

(a) Let $v_1, \ldots, v_n$ be a basis of V. Show that, for each natural number k, the set

$$\{v_{i_1} \otimes \cdots \otimes v_{i_k} \text{ s.t. } 1 \le i_1, \ldots, i_k \le n\}$$

is a basis of $\mathsf{T}^k(V)$.

(b) Prove the analogous facts for $\Lambda^k(V)$ and $\mathsf{S}^k(V)$.

(c) Let us say that a k-tensor x is irreducible if there are $a_1, \ldots, a_k \in V$ such that $x = a_1 \otimes \cdots \otimes a_k$. Show that for $k = 2$, there are reducible (= not irreducible) tensors. [Remark: this is true for all $k \ge 2$.]

(d) Show that the previous fact is true for both symmetric and skew-symmetric tensors, too.

(i)

(ii)

C. Let us continue with the notation of the previous question. Say that a linear transformation $L : \mathsf{T}^*(V) \to \mathsf{T}^*(V)$ is a derivation if

$$L(x \otimes y) = L(x) \otimes y + x \otimes L(y)$$

for all $x, y \in \mathsf{T}^*(V)$.

(a) Let $A : V \to V$ be a linear transformation and let $\exp(tA) = I + tA + \frac{1}{2}t^2A^2 + \cdots$ be the exponential. Show that $\frac{d\exp(tA)}{dt}|_{t=0}$ induces a derivation of $\mathsf{T}^*(V)$.

(b) Show that there is a bijection between linear transformations $V \to V$ and derivations of $\mathsf{T}^*(V)$. [Hint: show that a derivation is uniquely determined by its action on V.]

D. Let I be the $n \times n$ identity matrix and

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \qquad J : \mathbf{R}^n \oplus \mathbf{R}^n \to \mathbf{R}^n \oplus \mathbf{R}^n,$$

$$\mathrm{Sp}(\mathbf{R}^{2n}) = \{X \in \mathrm{Mat}_{2n\times 2n}(\mathbf{R}) \text{ s.t. } X'JX = J\}.$$

Prove that $\mathrm{Sp}(\mathbf{R}^{2n})$ is a submanifold of $\mathrm{Mat}_{2n\times 2n}(\mathbf{R})$. [Bonus: show it is a group, too.]

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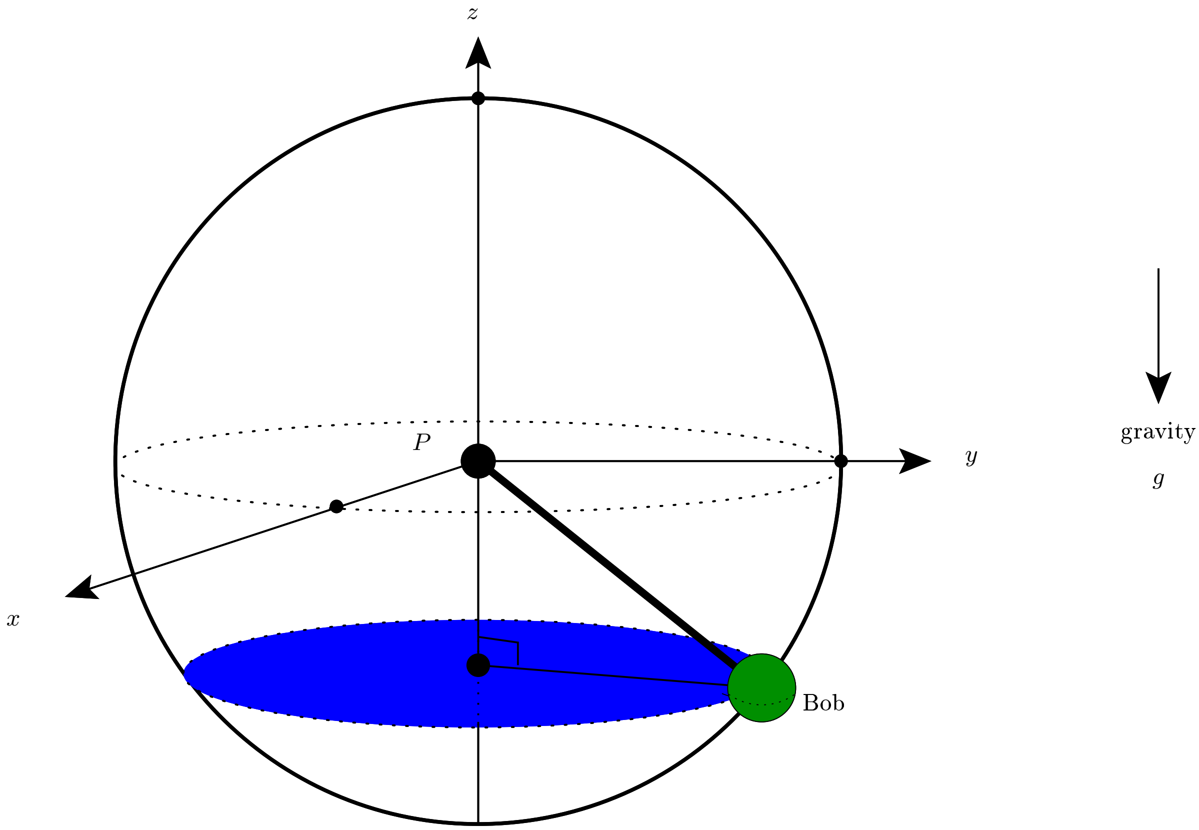

FIGURE 1. The spherical pendulum. The bob (in green) moves freely about the pivot P.