Topology & Geometry of Integrable Systems

Leo Butler

University of Edinburgh

23 May 2008

Hamiltonian mechanics

Classical Newtonian Mechanics

$$F = \frac{d}{dt} mv.$$

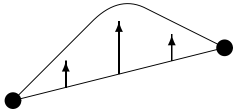
Hamiltonian mechanics

Classical Newtonian Mechanics

$$F = \frac{d}{dt} mv.$$

Principle of Least Action

$$\int_a^b L(t, x, \dot{x})\, dt \to \min$$

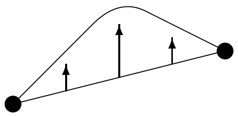


Hamiltonian mechanics

Classical Newtonian Mechanics

$$F = \frac{d}{dt} mv.$$

Principle of Least Action



$$\int_a^b L(t, x, \dot{x})\, dt \to \min$$

Minimizers solve Euler-Lagrange equations

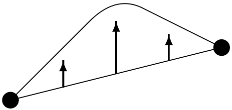
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

Hamiltonian mechanics

Classical Newtonian Mechanics

$$F = \frac{d}{dt} mv.$$

Principle of Least Action



$$\int_a^b L(t, x, \dot{x})\, dt \to \min$$

Minimizers solve Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

Equivalent to Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x}$$

Hamiltonian mechanics

Poisson bracket $\{,\}$:

$$\{F, H\} = \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial p} - \frac{\partial H}{\partial x} \cdot \frac{\partial F}{\partial p}$$

which implies Hamilton's equations

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial p}, \qquad \dot{p} = \{p, H\} = -\frac{\partial H}{\partial x}$$

on

$$T^*M = \{(x, p) \ : \ x \text{ is a point on } M, p \text{ is a momentum vector } \}.$$

Integrability

F and H Poisson commute if $\{F, H\} \equiv 0$.

Integrability

F and H Poisson commute if $\{F, H\} \equiv 0$.

Theorem (Liouville-Arnold-Mineur)

Let $H = F_1, \cdots, F_n$ be n Poisson commuting, independent functions. If

$$T_c = F_1^{-1}(c_1) \cap \cdots \cap F_n^{-1}(c_n)$$

is a compact, non-empty, regular level set, then

Integrability

F and H Poisson commute if $\{F, H\} \equiv 0$.

Theorem (Liouville-Arnold-Mineur)

Let $H = F_1, \cdots, F_n$ be n Poisson commuting, independent functions. If

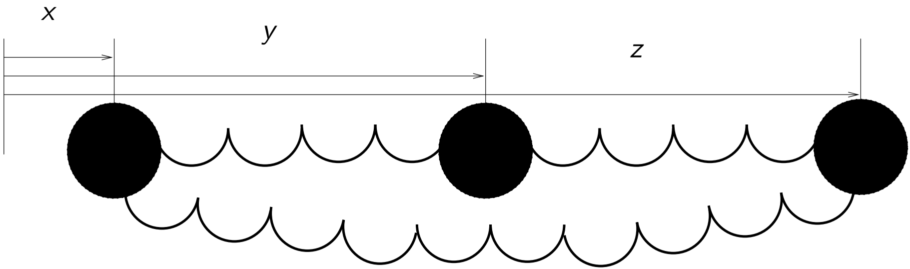
$$T_c = F_1^{-1}(c_1) \cap \cdots \cap F_n^{-1}(c_n)$$

is a compact, non-empty, regular level set, then

- *T_c is diffeomorphic to the n-torus $\mathbf{T}^n$;*
- *there is an open neighbourhood of T_c diffeomorphic to $\mathbf{T}^n \times \mathbf{R}^n$;*
- *there are coordinates (θ, I) such that*
  - *each $F_j = F_j(I)$;*
  - *each vector field is linear*

$$\dot{\theta} = \frac{\partial F_j(I)}{\partial I} \qquad \dot{I} = 0.$$

Toda Lattices



Potential energy:

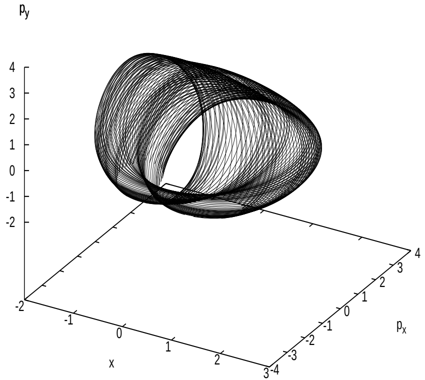
$$V = \exp(x - y) + \exp(y - z) + \exp(z - x).$$

An example

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + e^{x-y} + e^{y-z} + e^{z-x}$$

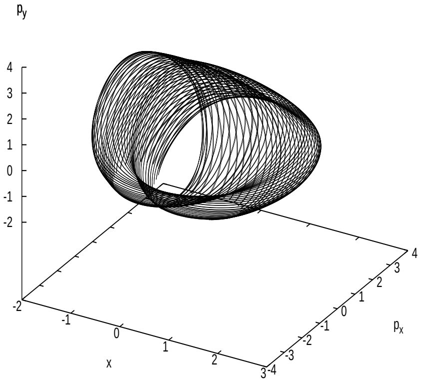
$$F = p_x p_y p_z - e^{x-y} p_z - e^{z-x} p_y - e^{y-z} p_x$$

$$G = p_x + p_y + p_z$$



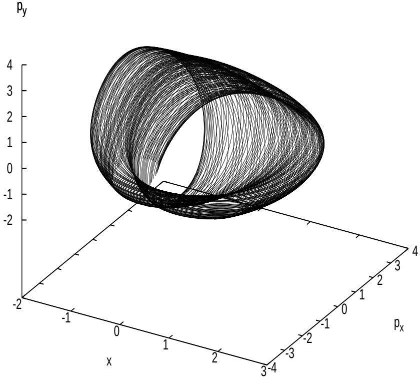
An example

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + e^{x-y} + e^{y-z} + e^{z-x}$$
$$F = p_x p_y p_z - e^{x-y} p_z - e^{z-x} p_y - e^{y-z} p_x$$
$$G = p_x + p_y + p_z$$



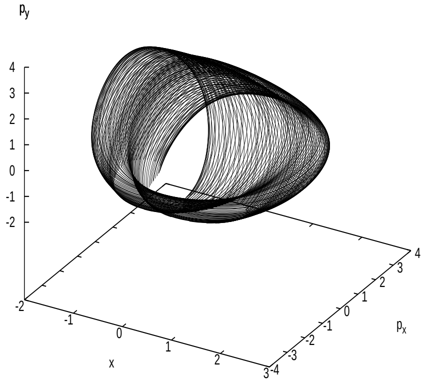
An example

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + e^{x-y} + e^{y-z} + e^{z-x}$$
$$F = p_x p_y p_z - e^{x-y} p_z - e^{z-x} p_y - e^{y-z} p_x$$
$$G = p_x + p_y + p_z$$



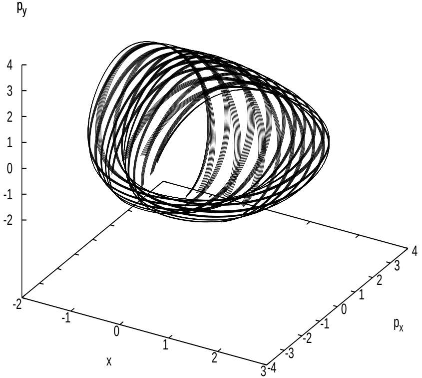
An example

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + e^{x-y} + e^{y-z} + e^{z-x}$$
$$F = p_x p_y p_z - e^{x-y} p_z - e^{z-x} p_y - e^{y-z} p_x$$
$$G = p_x + p_y + p_z$$



An example

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + e^{x-y} + e^{y-z} + e^{z-x}$$
$$F = p_x p_y p_z - e^{x-y} p_z - e^{z-x} p_y - e^{y-z} p_x$$
$$G = p_x + p_y + p_z$$



An example: what accounts for these integrals?

$$L = \begin{bmatrix} p_x & e^{x-y} & e^{z-x} \\ e^{x-y} & p_y & e^{y-z} \\ e^{z-x} & e^{y-z} & p_z \end{bmatrix} \qquad M = \frac{1}{2} \times \begin{bmatrix} 0 & e^{x-y} & -e^{z-x} \\ -e^{x-y} & 0 & e^{y-z} \\ e^{z-x} & -e^{y-z} & 0 \end{bmatrix}$$

Hamilton's equations are equivalent to

$$\dot{L} = [L, M]$$

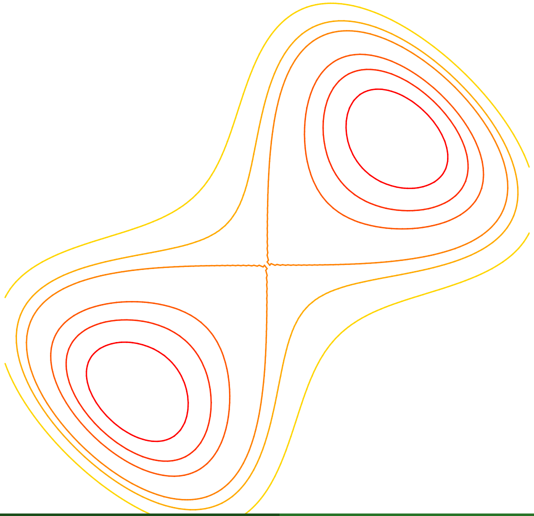
which implies that

$$\mathrm{Trace}\, L = G, \qquad \mathrm{Trace}\, L^2 = 2H, \qquad \mathrm{Trace}\, L^3 = F.$$

are integrals.

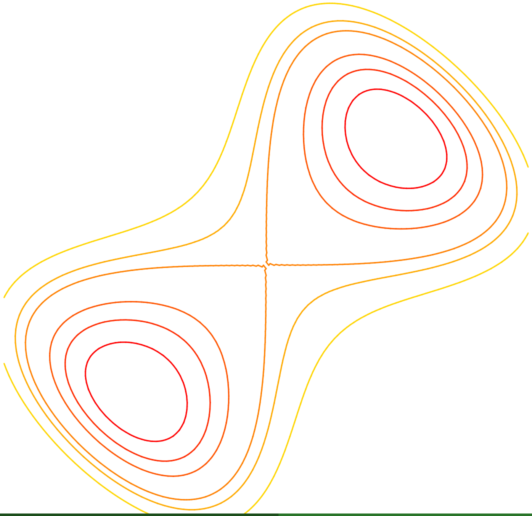
Integrability

- A typical phase portrait:



Integrability

•A typical phase portrait:



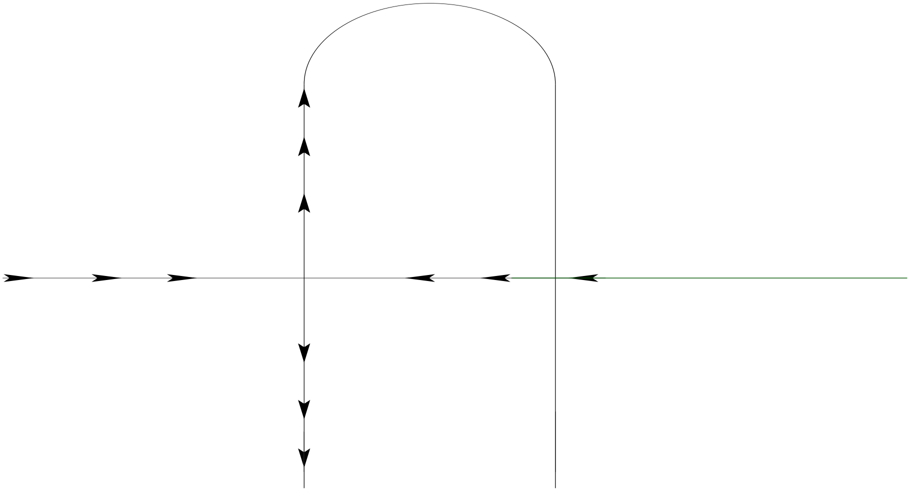
- fibred by tori
- tori degenerate
- degenerations controlled by Morse-like behaviour

To horseshoe To L-A-M

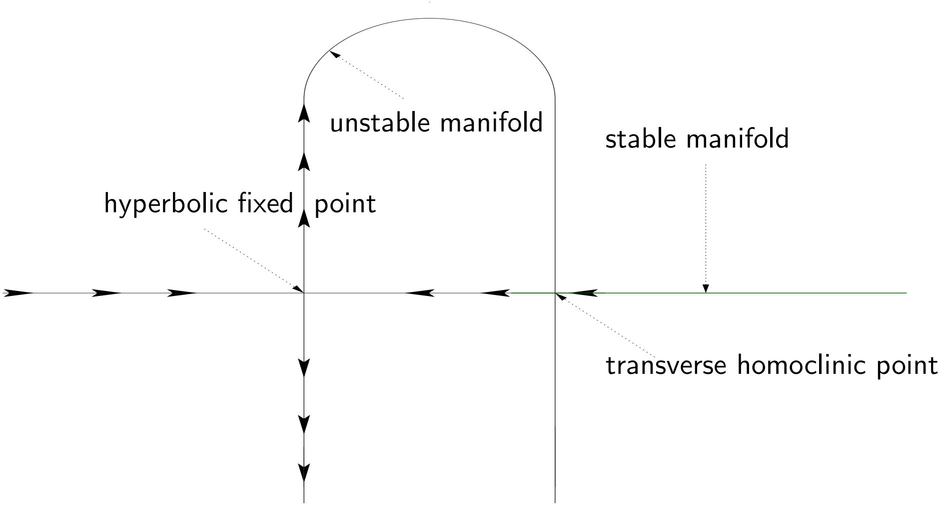
The Horseshoe

An example: $f : S^2 \longrightarrow S^2$ is a smooth map.

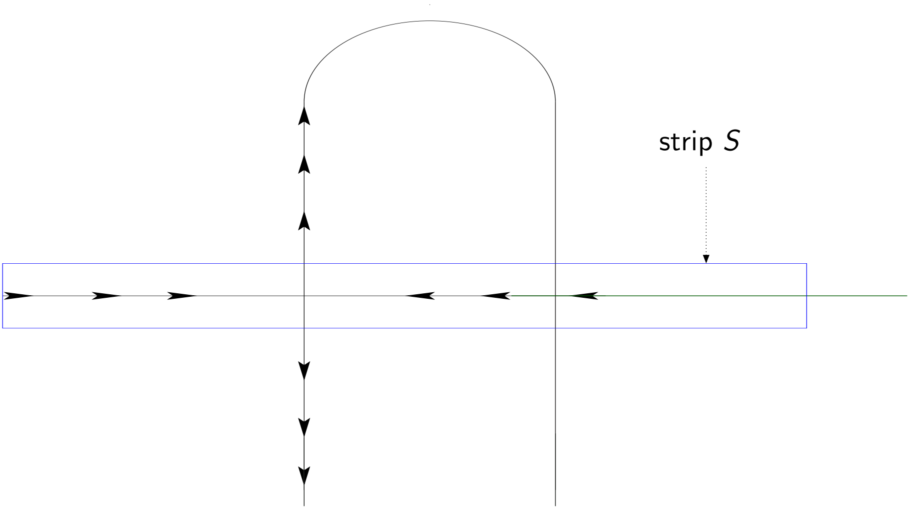
The Horseshoe



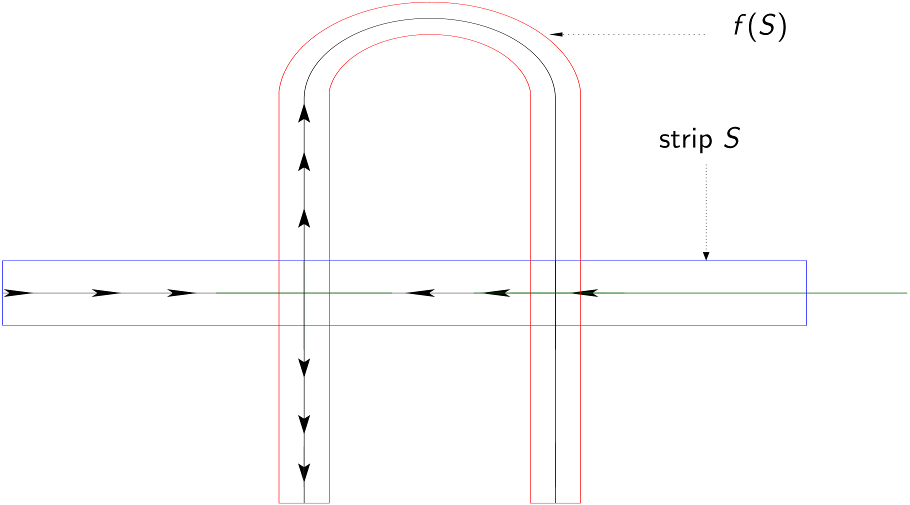
The Horseshoe



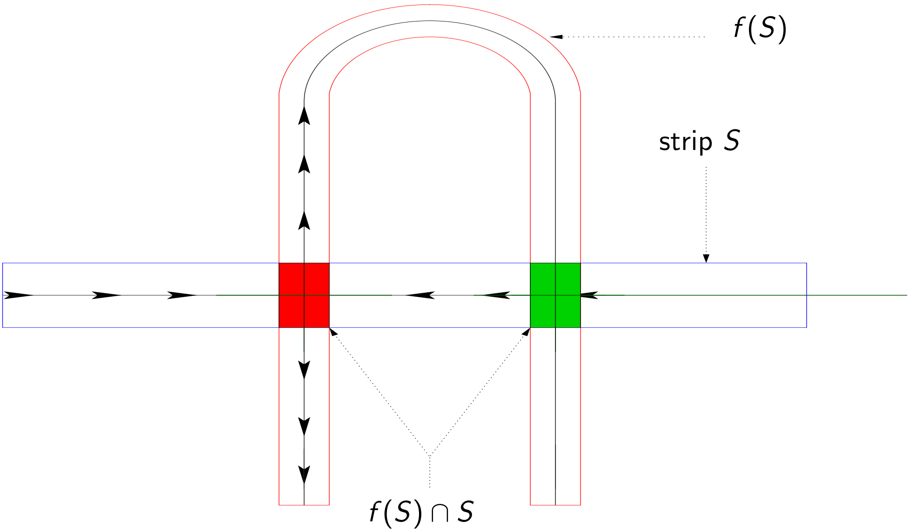
The Horseshoe



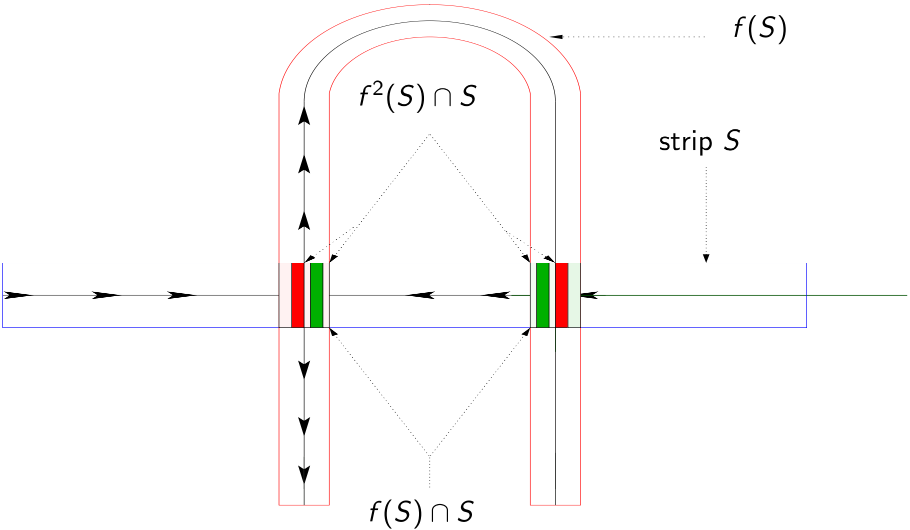
The Horseshoe



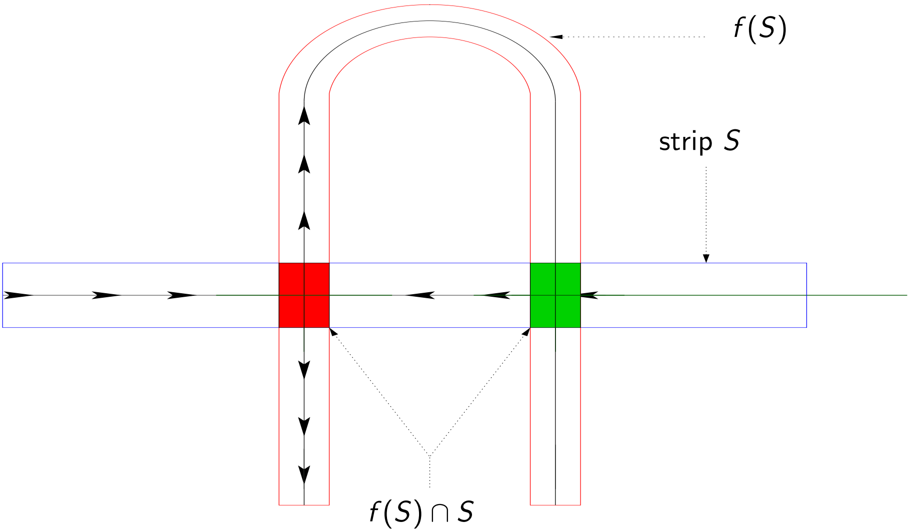
The Horseshoe



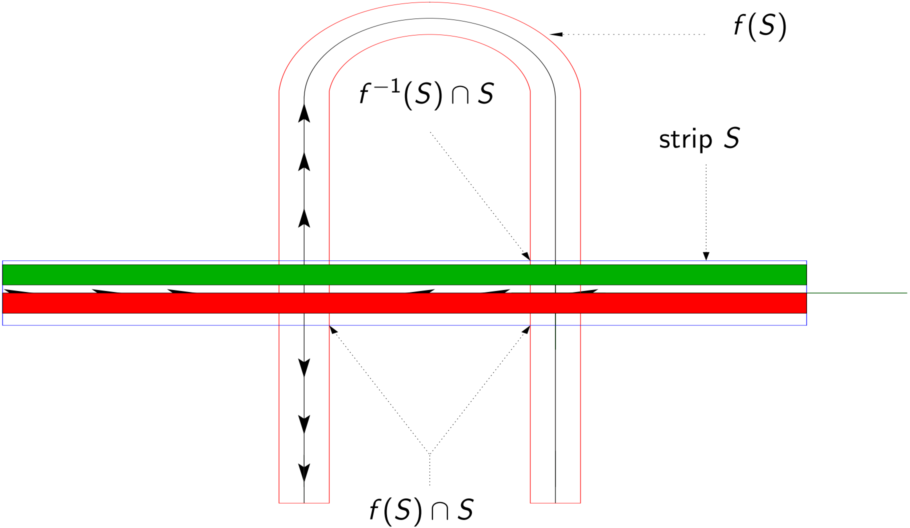
The Horseshoe



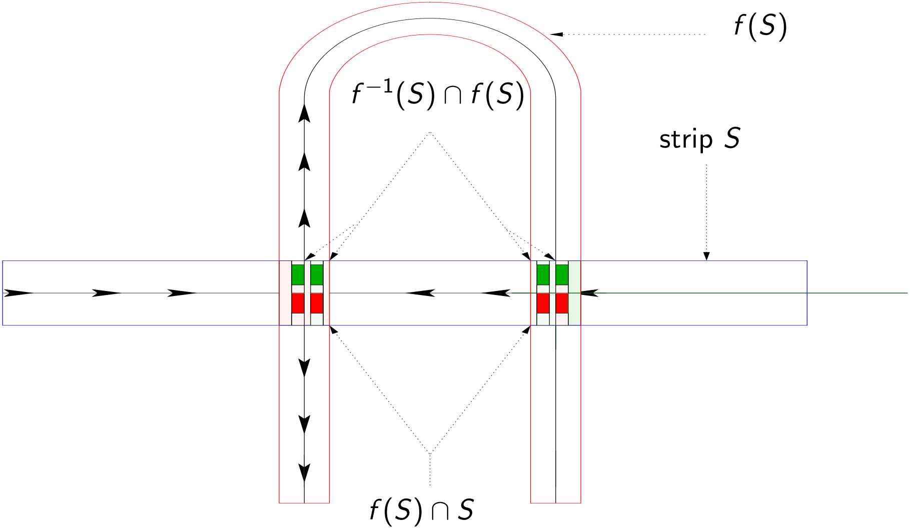
The Horseshoe



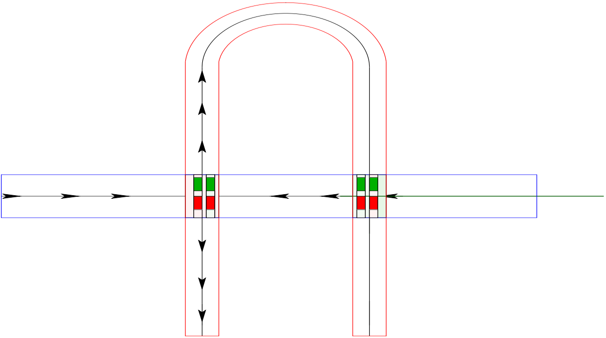
The Horseshoe



The Horseshoe



The Horseshoe



Implications:

1. Invariant Cantor set Λ
2. Invariant analytic function is constant on Λ
3. Invariant analytic function is constant.

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \to \mathbf{R}$ *is integrable with a non-degenerate (or real-analytic) integral* F*, then*

1. *the regular levels of* H *are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \to \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F, then

1. *the regular levels of H are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

Non-degenerate: F is a Morse-Bott function.

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \to \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F, then

1. *the regular levels of H are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

Non-degenerate: F is a Morse-Bott function.

Theorem (Paternain, Moser)

In addition,

3. *the topological entropy vanishes.*

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \to \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F, then

1. *the regular levels of H are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

Non-degenerate: F is a Morse-Bott function.

Theorem (Paternain, Moser)

In addition,

3. *the topological entropy vanishes ($\implies$ no horseshoes).*

Graph manifolds

Building Blocks

1. S_k = 2-disk with k open disks removed;
2. $M_k = S_k \times S^1$ or $S_k \tilde{\times} S^1$;
3. M = union of $M_{k_1}, \ldots, M_{k_n}$ glued along toral boundary.

Graph manifolds

Building Blocks

1. S_k = 2-disk with k open disks removed;
2. $M_k = S_k \times S^1$ or $S_k \tilde{\times} S^1$;
3. M = union of $M_{k_1}, \ldots, M_{k_n}$ glued along toral boundary.

Examples

1. $S^2 \times S^1 = D^2 \times S^1 \cup_{id} D^2 \times S^1$
2. $S^3 = D^2 \times S^1 \cup_{id} S^1 \times D^2$
3. $\mathbf{T}^3$,...
4. no hyperbolic manifolds

What about 3 or more degrees of freedom?

Significantly more difficult problem...

What about 3 or more degrees of freedom?

Significantly more difficult problem...

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic riemannian manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;
2. $\pi_1(M)$ *is almost abelian;*
3. *there is an injection* $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

What about 3 or more degrees of freedom?

Significantly more difficult problem...

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic riemannian manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;
2. $\pi_1(M)$ *is almost abelian;*
3. *there is an injection* $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

Theorem (Paternain)

If $H : (M^{2n}, \omega) \to \mathbf{R}$ *is completely integrable with non-degenerate integrals, then its topological entropy vanishes.*

Bolsinov & Taimanov's Example

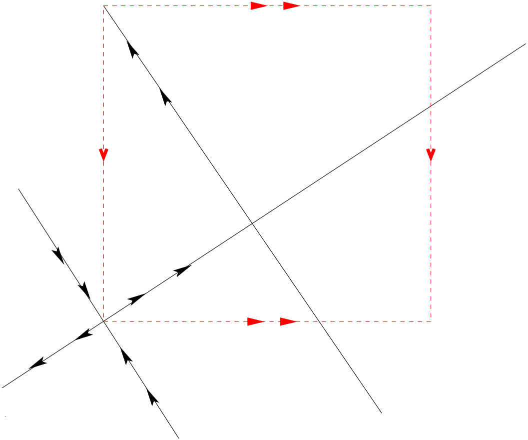
Theorem (Bolsinov & Taimanov)

Let Σ be a $\mathbf{T}^2$-bundle over $\mathbf{T}^1$. There is a natural riemannian metric on Σ whose geodesic flow is completely integrable. If the gluing map is hyperbolic, then the geodesic flow has positive topological entropy.

Positive Entropy

$$\mathbf{y} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y} \bmod \mathbf{Z}^2.$$

To horseshoe
To integrals



Geometrization Conjecture

3 model geometries in 2 dimensions:

$\mathbf{E}^2$ $\mathbf{H}^2$

S^2

Geometrization Conjecture

3 model geometries in 2 dimensions:

$\mathbf{E}^2$	$\mathbf{H}^2$
S^2	

8 model geometries in 3 dimensions:

$\mathbf{E}^3$	$\mathbf{H}^3$
$S^2 \times \mathbf{E}^1$	$\mathbf{H}^2 \times \mathbf{E}^1$
S^3	$\widetilde{SL_2\mathbf{R}}$
Nil	
Sol	

Geometrization Conjecture

3 model geometries in 2 dimensions:

$$\mathbf{E}^2 \qquad \mathbf{H}^2$$
$$S^2$$

8 model geometries in 3 dimensions:

$$\mathbf{E}^3 \qquad \mathbf{H}^3$$
$$S^2 \times \mathbf{E}^1 \qquad \mathbf{H}^2 \times \mathbf{E}^1$$
$$S^3 \qquad \widetilde{SL_2\mathbf{R}}$$
$$Nil$$
$$Sol$$

Conjecture (Thurston)

Each compact 3-manifold admits a canonical decomposition into complete, finite-volume geometric 3-manifolds.

Results in 3 degrees of freedom

Definition (Semisimplicity)

Let

$$\begin{array}{ccccccc} \mathbf{T}^n & \hookrightarrow & L & \xhookrightarrow{\text{incl.}} & T^*M & \xrightarrow{\pi} & M \\ & & \downarrow f & & & & \\ & & B & & & & \end{array}$$

be a lagrangian fibration. If $\Gamma = T^*M - L$ *is a closed, nowhere dense tamely-embedded polyhedron, then we say* (f, L, B) *is* **semisimple**.

Results in 3 degrees of freedom

Definition (Semisimplicity)

Let

$$\begin{array}{ccccccc} \mathbf{T}^n & \hookrightarrow & L & \xhookrightarrow{incl.} & T^*M & \xrightarrow{\pi} & M \\ & & \downarrow f & & & & \\ & & B & & & & \end{array}$$

be a lagrangian fibration. If $\Gamma = T^*M - L$ *is a closed, nowhere dense tamely-embedded polyhedron, then we say* (f, L, B) *is* **semisimple**.

Theorem (B. 2005)

Assume the GC. A compact 3-manifold admits a riemannian metric with a semisimple geodesic flow iff it admits one of five geometric structures:

$$\mathbf{E}^3, \qquad S^2 \times \mathbf{E}^1, \qquad S^3, \qquad Nil, \qquad Sol.$$

Sketch of proof

$\Rightarrow$ There is a subspace $L_0 \subset L$ s.t. $\pi_1(L_0) \to \pi_1(M)$ is almost onto and

$$\begin{array}{ccccc} \mathbf{T}^3 & \hookrightarrow & L_0 & \longrightarrow & M \\ & & \Big\downarrow {\scriptstyle f_0 = f|L_0} & & \\ & & B_0 & & \end{array}$$

is a lagrangian fibration over a surface B_0.

Sketch of proof

$\Rightarrow$

1. There is a subspace $L_1 \subset L_0$ s.t. $\pi_1(L_1) \to \pi_1(M)$ is almost onto and

$$\begin{array}{ccccc} \mathbf{T}^3 & \hookrightarrow & L_1 & \longrightarrow & M \\ & & \Big\downarrow {\scriptstyle f_1 = f|L_1} & & \\ & & B_1 \simeq S^1 & & \end{array}$$

is a lagrangian fibration over a circle $B_1 \subset B_0$.

Sketch of proof

$\Rightarrow$

1. There is a subspace $L_1 \subset L_0$ s.t. $\pi_1(L_1) \to \pi_1(M)$ is almost onto and

$$\begin{array}{ccccc} \mathbf{T}^3 & \hookrightarrow & L_1 & \longrightarrow & M \\ & & \downarrow{\scriptstyle f_1 = f|L_1} & & \\ & & B_1 \simeq S^1 & & \end{array}$$

   is a lagrangian fibration over a circle $B_1 \subset B_0$.

2. $\pi_1(L_1) = \mathbf{Z}^3 \star \mathbf{Z}$ is solvable so $\pi_1(M)$ is almost solvable.

Sketch of proof

$\Rightarrow$

1. There is a subspace $L_1 \subset L_0$ s.t. $\pi_1(L_1) \to \pi_1(M)$ is almost onto and

$$\begin{array}{ccccc} \mathbf{T}^3 & \hookrightarrow & L_1 & \longrightarrow & M \\ & & \downarrow{\scriptstyle f_1 = f|L_1} & & \\ & & B_1 \simeq S^1 & & \end{array}$$

   is a lagrangian fibration over a circle $B_1 \subset B_0$.

2. $\pi_1(L_1) = \mathbf{Z}^3 \star \mathbf{Z}$ is solvable so $\pi_1(M)$ is almost solvable.
3. If $\pi_2(M) = 0$, Hempel-Jaco implies M is the connect sum of a geometric manifold with a homotopy sphere.

Sketch of proof

$\Rightarrow$

1. There is a subspace $L_1 \subset L_0$ s.t. $\pi_1(L_1) \to \pi_1(M)$ is almost onto and

$$\begin{array}{ccccc} \mathbf{T}^3 & \hookrightarrow & L_1 & \longrightarrow & M \\ & & \downarrow{\scriptstyle f_1 = f|L_1} & & \\ & & B_1 \simeq S^1 & & \end{array}$$

   is a lagrangian fibration over a circle $B_1 \subset B_0$.

2. $\pi_1(L_1) = \mathbf{Z}^3 \star \mathbf{Z}$ is solvable so $\pi_1(M)$ is almost solvable.
3. If $\pi_2(M) = 0$, Hempel-Jaco implies M is the connect sum of a geometric manifold with a homotopy sphere.
4. Otherwise, (Agol) GC implies that M admits $S^2 \times \mathbf{E}$ geometry.

Sketch of proof

$\Leftarrow$

The real-analytic complete integrability of the geodesic flows of the compact geometries modeled on $\mathbf{E}^3$, $S^2 \times \mathbf{E}$, S^3 is well-known.

Sketch of proof

$\Leftarrow$

The real-analytic complete integrability of the geodesic flows of the compact geometries modeled on $\mathbf{E}^3$, $S^2 \times \mathbf{E}$, S^3 is well-known.

The semisimplicity of the geodesic flows of *Nil*-geometry is proven by (B.) and that of *Sol* by (Bolsinov-Taimanov). To B-T Theorem

Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

Milnor-Kervaire/Eels-Kuiper proved there are exactly 28 smooth structures on S^7.

Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

Milnor-Kervaire/Eels-Kuiper proved there are exactly 28 smooth structures on S^7.

Wall showed that the smooth structures on the topological n-torus is the group

$$\sum_{i=0}^{n} H^i(\mathbf{T}^n; \Gamma_i) \qquad \Gamma_i = \begin{cases} \text{group of exotic } i\text{-spheres}, & i \neq 4, \\ 0 & i = 4. \end{cases}$$

Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

Milnor-Kervaire/Eels-Kuiper proved there are exactly 28 smooth structures on S^7.

Wall showed that the smooth structures on the topological n-torus is the group

$$\sum_{i=0}^{n} H^i(\mathbf{T}^n; \Gamma_i) \qquad \Gamma_i = \begin{cases} \text{group of exotic } i\text{-spheres}, & i \neq 4, \\ 0 & i = 4. \end{cases}$$

The group of smooth structures on $\mathbf{T}^7$ is the group of 28 smooth structures on S^7.

Exotic tori and integrability

Theorem (B. 2007)

Let Σ be a smooth n-dimensional topological torus. If Σ admits a real-analytically completely integrable geodesic flow, then Σ is diffeomorphic to the standard n-torus.

Exotic tori and integrability

Theorem (B. 2007)

Let Σ be a smooth n-dimensional topological torus. If Σ admits a real-analytically completely integrable geodesic flow, then Σ is diffeomorphic to the standard n-torus.

Proof.

Claim: real-analytic integrability implies there is lagrangian torus F s.t. ρ is a diffeomorphism:

$$\begin{array}{ccc} F & \overset{\text{incl.}}{\hookrightarrow} & T^*\Sigma \\ & \underset{\rho}{\searrow} & \downarrow \pi \\ & & \Sigma. \end{array}$$

We show $\deg \rho \neq 0$, use a result of Viterbo's that $\deg \rho \neq 0$ implies ρ is a covering map, then use a cohomology argument to show ρ is $1-1$. □

Witten-Kreck-Stolz spaces

Let $k, l \neq 0$ be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} \ : \ |z| = 1 \right\} \qquad \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \qquad \text{Witten-Kreck-Stolz space.}$$

Witten-Kreck-Stolz spaces

Let $k, l \neq 0$ be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} \; : \; |z| = 1 \right\} \qquad \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \qquad \text{Witten-Kreck-Stolz space.}$$

Theorem (Kreck-Stolz)

There are 28 smooth structures on $M_{1,4}$. These structures are represented by $M_{32t+1,4}$ for $t = 0, \ldots, 27 \bmod 28$.

Witten-Kreck-Stolz spaces

Let $k, l \neq 0$ be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} \ : \ |z| = 1 \right\} \qquad \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \qquad \text{Witten-Kreck-Stolz space.}$$

Theorem (Kreck-Stolz)

There are 28 smooth structures on $M_{1,4}$. These structures are represented by $M_{32t+1,4}$ for $t = 0, \ldots, 27 \bmod 28$.

Theorem (B. 2007)

The riemannian metric on $M_{k,l}$ induced by the round metrics on $S^3 \times S^5$ is real-analytically completely integrable.

Questions

1. Are there completely integrable geodesic flows on exotic tori? Must these have positive entropy?
2. Is it possible to give a description of the smooth invariant which determines whether or not a topological manifold admits a semisimple geodesic flow?
3. Must a real-analytically integrable system in dimensions 6 and more have zero entropy?