Positive-Entropy Integrable Systems and the Toda Lattice

Leo Butler

School of Mathematics,
University of Edinburgh

14 March 2007

Topological Entropy

Ingredients:

Topological Entropy

Ingredients:

- (X, d) is a compact metric space;

Topological Entropy

Ingredients:

- (X, d) is a compact metric space;
- $f : X \to X$ is a map;

Topological Entropy

Ingredients:

- (X, d) is a compact metric space;
- $f : X \to X$ is a map;
- Bowen metric: $d_f^n(x, y) = \max\{\ d(f^k x, f^k y)\ : 0 \leq k \leq n\ \}$

Topological Entropy

Ingredients:

- (X, d) is a compact metric space;
- $f : X \to X$ is a map;
- Bowen metric: $d_f^n(x, y) = \max\{\ d(f^k x, f^k y)\ : 0 \leq k \leq n\ \}$

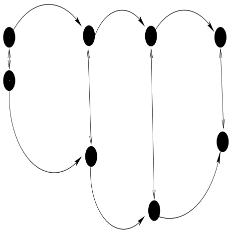


Figure: The Bowen metric.

Topological Entropy

Ingredients:

- (X, d) is a compact metric space;
- $f : X \to X$ is a map;
- Bowen metric: $d_f^n(x, y) = \max\{ \ d(f^k x, f^k y) \ : 0 \leq k \leq n \ \}$

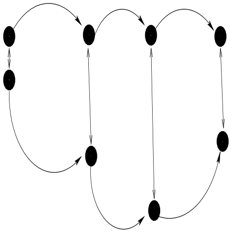


Figure: The Bowen metric.

- $s(\epsilon, n)$ is the minimum $\#$ of d_f^n-balls of radius ϵ to cover X;

Topological Entropy

Ingredients:

- (X, d) is a compact metric space;
- $f : X \to X$ is a map;
- Bowen metric: $d_f^n(x, y) = \max\{\ d(f^k x, f^k y)\ : 0 \leq k \leq n\ \}$

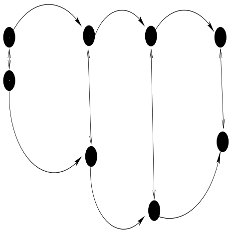


Figure: The Bowen metric.

- $s(\epsilon, n)$ is the minimum $\#$ of d_f^n-balls of radius ϵ to cover X;
- $$h_{top}(f) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log s(\epsilon, n)}{n}.$$

Topological Entropy: The Horseshoe

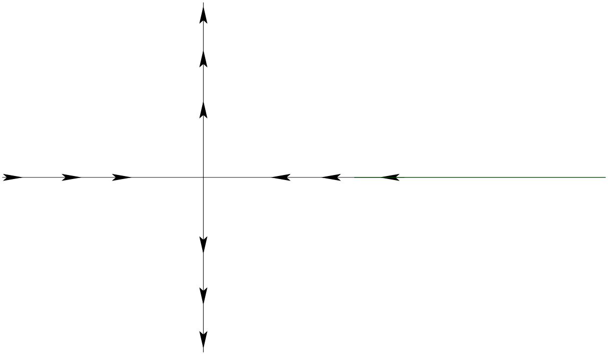
An example:

- $f : S^2 \to S^2$ has a hyperbolic fixed point;

Topological Entropy: The Horseshoe

An example:

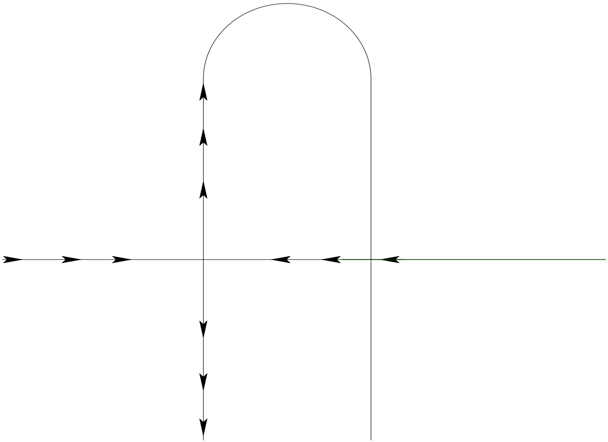
- $f : S^2 \to S^2$ has a hyperbolic fixed point;
- semi-locally:



Topological Entropy: The Horseshoe

An example:

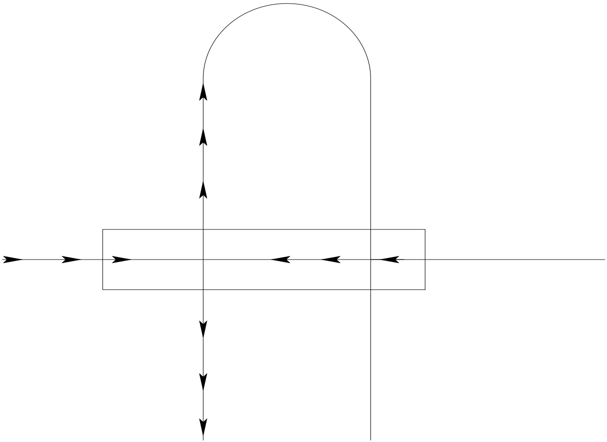
- $f : S^2 \to S^2$ has a hyperbolic fixed point;
- semi-locally:



Topological Entropy: The Horseshoe

An example:

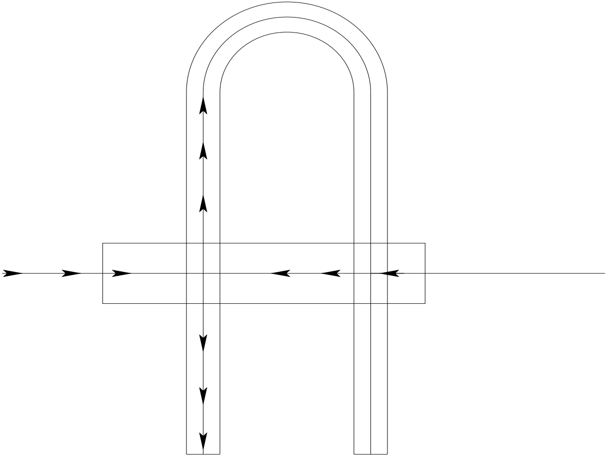
- $f : S^2 \to S^2$ has a hyperbolic fixed point;
- semi-locally:



Topological Entropy: The Horseshoe

An example:

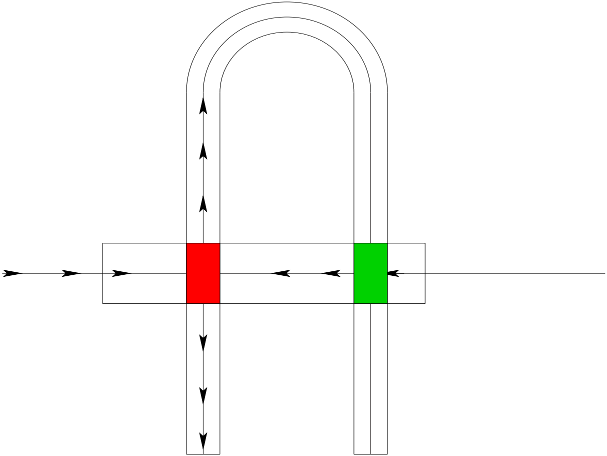
- $f : S^2 \to S^2$ has a hyperbolic fixed point;
- semi-locally:



Topological Entropy: The Horseshoe

An example:

- $f : S^2 \to S^2$ has a hyperbolic fixed point;
- semi-locally:

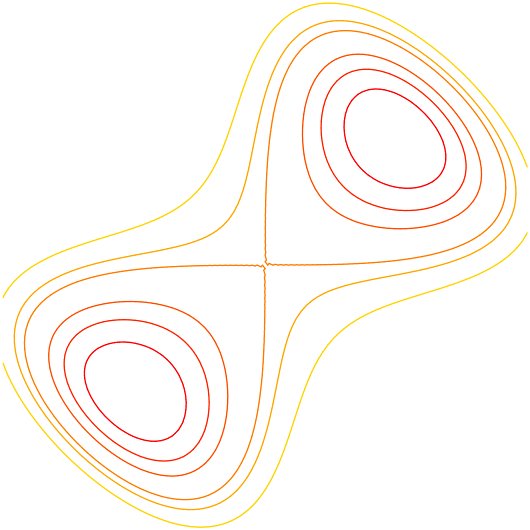


Integrability

- A typical phase portrait:

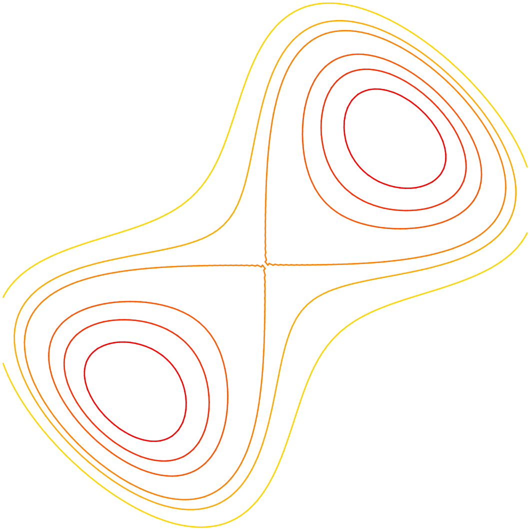
Integrability

- A typical phase portrait:



Integrability

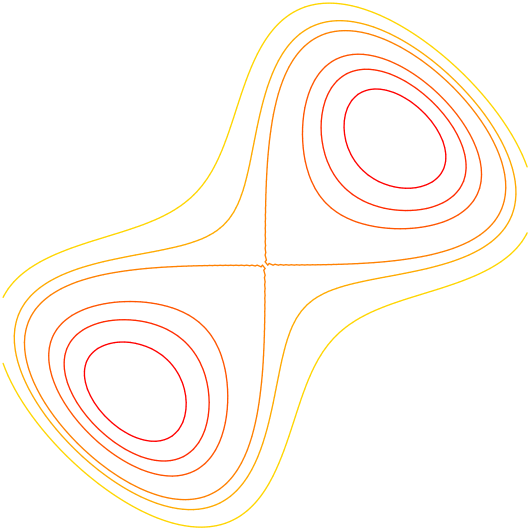
- A typical phase portrait:



1. fibred by tori

Integrability

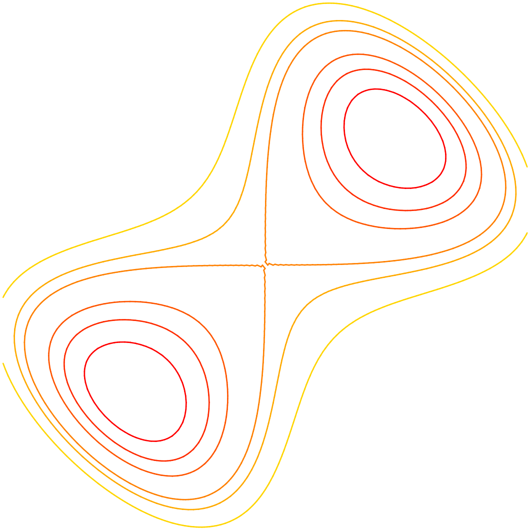
- A typical phase portrait:



1. fibred by tori
2. tori degenerate

Integrability

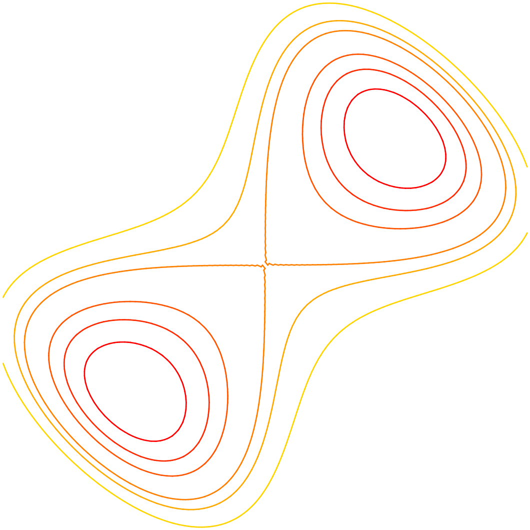
- A typical phase portrait:



1. fibred by tori
2. tori degenerate
3. degenerations controlled

Integrability

- A typical phase portrait:



1. fibred by tori
2. tori degenerate
3. degenerations controlled

   by Morse-like behaviour

Topology, Integrability and Entropy

Theorem (Fomenko, Zieschang, Matveev 1986-9)

If $H : (M^4, \omega) \to \mathbf{R}$ *is integrable with a non-degenerate (or real-analytic) integral* F*, then*

1. *the regular levels of* H *are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

Topology, Integrability and Entropy

Theorem (Fomenko, Zieschang, Matveev 1986-9)

If $H : (M^4, \omega) \to \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F, then

1. *the regular levels of H are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

Non-degenerate means that $F|H^{-1}(c)$ is a Morse-Bott function.

Topology, Integrability and Entropy

Theorem (Fomenko, Zieschang, Matveev 1986-9)

If $H : (M^4, \omega) \to \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F, then

1. *the regular levels of H are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

Non-degenerate means that $F|H^{-1}(c)$ is a Morse-Bott function.

Theorem (Paternain, Moser)

In addition,

3. *the topological entropy vanishes.*

Graph manifolds

Building Blocks

1. S_k = 2-disk with k open disks removed;
2. $M_k = S_k \times S^1$ or $S_k \tilde{\times} S^1$;
3. M = union of $M_{k_1}, \ldots, M_{k_n}$ glued along toral boundary.

Graph manifolds

Building Blocks

1. S_k = 2-disk with k open disks removed;
2. $M_k = S_k \times S^1$ or $S_k \tilde{\times} S^1$;
3. M = union of $M_{k_1}, \ldots, M_{k_n}$ glued along toral boundary.

Examples

1. $S^2 \times S^1$
2. S^3, $L_{p,q}$
3. $\mathbf{T}^3$,...

What about 3 or more degrees of freedom?

What about 3 or more degrees of freedom?

Theorem (Taimanov 1988)

Let (M^n, g) *be a compact, real-analytic manifold. If the geodesic flow is real-analytically integrable, then*

What about 3 or more degrees of freedom?

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;

What about 3 or more degrees of freedom?

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;
2. $\pi_1(M)$ *is almost abelian;*

What about 3 or more degrees of freedom?

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;
2. $\pi_1(M)$ *is almost abelian;*
3. *there is an injection* $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

What about 3 or more degrees of freedom?

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;
2. $\pi_1(M)$ *is almost abelian;*
3. *there is an injection* $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

Theorem (Paternain)

If $H : (M^{2n}, \omega) \to \mathbf{R}$ *is completely integrable with non-degenerate integrals, then its topological entropy vanishes.*

Bolsinov-Taimanov's Example

- Configuration space: $Sol = \mathbf{R} \star \mathbf{R}^2$

$$x.\mathbf{y} = (e^x y_0, e^{-x} y_1).$$

Bolsinov-Taimanov's Example

•Configuration space: $Sol = \mathbf{R} \star \mathbf{R}^2$

$$x.\mathbf{y} = (e^x y_0, e^{-x} y_1).$$

•Left-invariant Metric:

$$\mathbf{g} = dx^2 + e^{-2x} dy_0^2 + e^{2x} dy_1^2.$$

Bolsinov-Taimanov's Example

•Configuration space: $Sol = \mathbf{R} \star \mathbf{R}^2$

$$x.\mathbf{y} = (e^x y_0, e^{-x} y_1).$$

•Left-invariant Metric:

$$\mathbf{g} = dx^2 + e^{-2x} dy_0^2 + e^{2x} dy_1^2.$$

•Hamiltonian and integrals:

$$2H = p_x^2 + e^{2x} p_{y_0}^2 + e^{-2x} p_{y_1}^2$$

Bolsinov-Taimanov's Example

• Configuration space: $Sol = \mathbf{R} \star \mathbf{R}^2$

$$x.\mathbf{y} = (e^x y_0, e^{-x} y_1).$$

• Left-invariant Metric:

$$\mathbf{g} = dx^2 + e^{-2x} dy_0^2 + e^{2x} dy_1^2.$$

• Hamiltonian and integrals:

$$2H = p_x^2 + e^{2x} p_{y_0}^2 + e^{-2x} p_{y_1}^2$$
$$I_1 = p_{y_0}, \qquad I_2 = p_{y_1}.$$

Bolsinov-Taimanov's Example

•Configuration space: $Sol = \mathbf{R} \star \mathbf{R}^2$

$$x.\mathbf{y} = (e^x y_0, e^{-x} y_1).$$

•Left-invariant Metric:

$$\mathbf{g} = dx^2 + e^{-2x} dy_0^2 + e^{2x} dy_1^2.$$

•Hamiltonian and integrals:

$$2H = p_x^2 + e^{2x} p_{y_0}^2 + e^{-2x} p_{y_1}^2$$
$$I_1 = p_{y_0}, \qquad I_2 = p_{y_1}.$$

•Discrete group: $u = \frac{1+\sqrt{5}}{2}$

$$\Delta = \{(u^k, \mathbf{y}) \ : \ y_0 = \frac{n + m\sqrt{5}}{2}, y_1 = \frac{n - m\sqrt{5}}{2}, \ k, m, n \in \mathbf{Z} \}.$$

Bolsinov-Taimanov's Example

Theorem (Bolsinov, Taimanov)

The hamiltonian $H : T^*(Sol/\Delta) \to \mathbf{R}$ *is completely integrable. Its flow has positive topological entropy.*

Bolsinov-Taimanov's Example

Theorem (Bolsinov, Taimanov)

The hamiltonian $H : T^*(Sol/\Delta) \to \mathbf{R}$ *is completely integrable. Its flow has positive topological entropy.*

Proof.

The integrals are

$$2H = p_x^2 + e^{2x} p_{y_0}^2 + e^{-2x} p_{y_1}^2,$$

$$I = p_{y_0} p_{y_1}, \qquad J = \exp(-I^{-2}) \times \sin \frac{2\pi \ln |p_{y_1}|}{\ln u}.$$

□

Bolsinov-Taimanov's Example

Theorem (Bolsinov, Taimanov)

The hamiltonian $H : T^*(Sol/\Delta) \to \mathbf{R}$ is completely integrable. Its flow has positive topological entropy.

Proof.

Anosov diffeomorphism on $\{p_{y_1} = p_{y_2} = 0, p_x = 1\}$:

$$\begin{aligned} x &= x_0 + t, & p_x &= 1, \\ y_0 &= y_{0,0}, & p_{y_0} &= 0, \\ y_1 &= y_{1,0}, & p_{y_1} &= 0, \quad \mod \Delta. \end{aligned}$$

□

Bolsinov-Taimanov's Example

Theorem (Bolsinov, Taimanov)

The hamiltonian $H : T^*(Sol/\Delta) \to \mathbf{R}$ *is completely integrable. Its flow has positive topological entropy.*

Proof.

Anosov diffeomorphism on $\{p_{y_1} = p_{y_2} = 0, p_x = 1\}$:
At $t = \ln u$

$$\begin{array}{rclrcll} x & \equiv & x_0, & p_x & \equiv & 1, & \\ y_0 & \equiv & u^{-1} y_{0,0}, & p_{y_0} & \equiv & 0, & \\ y_1 & \equiv & u y_{1,0}, & p_{y_1} & \equiv & 0, & \mod \Delta. \end{array}$$

or

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y}(0) \bmod \mathbf{Z}^2.$$

□

Toda Lattice

A non-linear coupled oscillator

$$\ddot{x}_i = \exp(x_i - x_{i+1}) - \exp(x_{i-1} - x_i), \qquad i = 1, \ldots, n \bmod n. \quad (*)$$

Toda Lattice

A non-linear coupled oscillator

$$\ddot{x}_i = \exp(x_i - x_{i+1}) - \exp(x_{i-1} - x_i), \qquad i = 1, \ldots, n \bmod n. \quad (*)$$

Flaschka Transform

$$a_i := \exp(x_i/2 - x_{i+1}/2), \qquad b_i := \dot{x}_i,$$

$$L := \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 & \lambda a_n \\ a_1 & b_2 & a_2 & \cdots & 0 & 0 \\ 0 & a_2 & b_3 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & b_{n-1} & a_{n-1} \\ \lambda^{-1} a_n & 0 & & & a_{n-1} & b_n \end{bmatrix}$$

$$M := 1/2 \times (L_+ - L'_+)$$

Toda Lattice

A non-linear coupled oscillator

$$\ddot{x}_i = \exp(x_i - x_{i+1}) - \exp(x_{i-1} - x_i), \qquad i = 1, \ldots, n \bmod n. \quad (*)$$

Flaschka Transform

$$a_i := \exp(x_i/2 - x_{i+1}/2), \qquad b_i := \dot{x}_i,$$

$$L := \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 & \lambda a_n \\ a_1 & b_2 & a_2 & \cdots & 0 & 0 \\ 0 & a_2 & b_3 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & b_{n-1} & a_{n-1} \\ \lambda^{-1} a_n & 0 & & & a_{n-1} & b_n \end{bmatrix} \qquad (*) \implies \dot{L} = [L, M].$$

$$M := 1/2 \times (L_+ - L'_+)$$

Toda Lattices and Positive-Entropy Integrable Systems

Theorem (B.)

Let $F/\mathbf{Q}$ *be a totally-real extension of degree* $n+1$ *with integers* $\mathcal{O}$ *and group of units* $\mathcal{U}$*; let* $\Delta = \mathcal{U}^{+} \star \mathcal{O}$*. There is a bundle* $\mathbf{T}^{n+1} \hookrightarrow \Sigma \to \mathbf{T}^{n}$ *with* $\pi_1(\Sigma) = \Delta$ *such that*

Toda Lattices and Positive-Entropy Integrable Systems

Theorem (B.)

Let $F/\mathbf{Q}$ be a totally-real extension of degree $n+1$ with integers $\mathcal{O}$ and group of units $\mathcal{U}$; let $\Delta = \mathcal{U}^{+} \star \mathcal{O}$. There is a bundle $\mathbf{T}^{n+1} \hookrightarrow \Sigma \to \mathbf{T}^{n}$ with $\pi_1(\Sigma) = \Delta$ such that

1. *for each periodic Toda lattice Ψ of rank n, there is an integrable hamiltonian $H_{\Psi} : T^{*}\Sigma \to \mathbf{R}$;*

Toda Lattices and Positive-Entropy Integrable Systems

Theorem (B.)

Let $F/\mathbf{Q}$ be a totally-real extension of degree $n+1$ with integers $\mathcal{O}$ and group of units $\mathcal{U}$; let $\Delta = \mathcal{U}^{+} \star \mathcal{O}$. There is a bundle $\mathbf{T}^{n+1} \hookrightarrow \Sigma \to \mathbf{T}^{n}$ with $\pi_1(\Sigma) = \Delta$ such that

1. *for each periodic Toda lattice Ψ of rank n, there is an integrable hamiltonian $H_{\Psi} : T^{*}\Sigma \to \mathbf{R}$;*
2. *the flow of H_{Ψ} contains a subsystem isomorphic to $u \in \mathcal{U}$ for all u;*

Toda Lattices and Positive-Entropy Integrable Systems

Theorem (B.)

Let $F/\mathbf{Q}$ be a totally-real extension of degree $n+1$ with integers $\mathcal{O}$ and group of units $\mathcal{U}$; let $\Delta = \mathcal{U}^+ \star \mathcal{O}$. There is a bundle $\mathbf{T}^{n+1} \hookrightarrow \Sigma \to \mathbf{T}^n$ with $\pi_1(\Sigma) = \Delta$ such that

1. *for each periodic Toda lattice Ψ of rank n, there is an integrable hamiltonian $H_\Psi : T^*\Sigma \to \mathbf{R}$;*
2. *the flow of H_Ψ contains a subsystem isomorphic to $u \in \mathcal{U}$ for all u;*
3. *for $n = 1$ and $F = \mathbf{Q}(\sqrt{5})$, the $A_1^{(1)}$ Toda lattice yields the Bolsinov-Taimanov example.*