A Weak Liouville-Arnol′d Theorem

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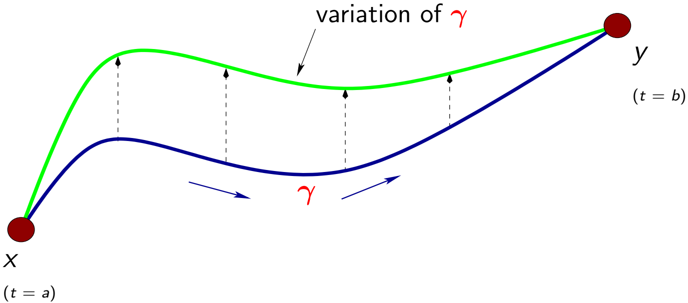

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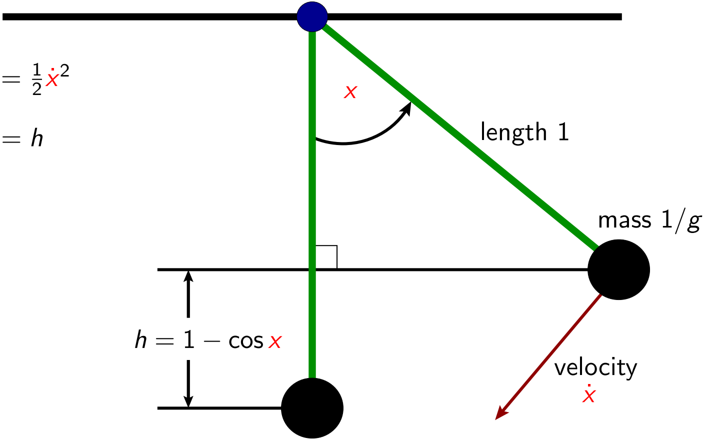

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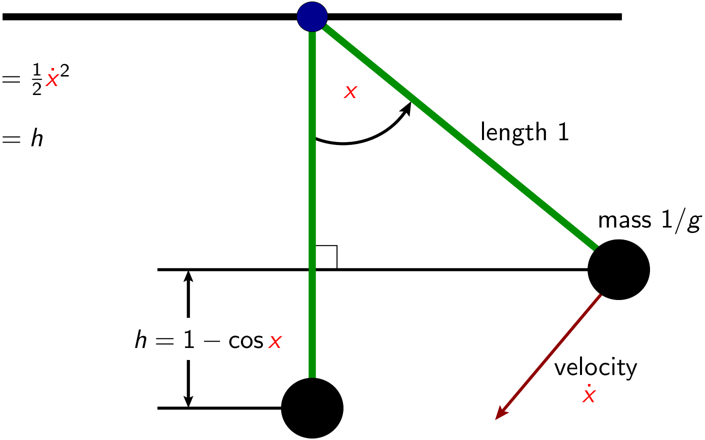

From which one has the *Lagrangian*:

$$L(x, \dot{x}) = T - K.$$

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& the *action* of a curve γ:

$$A[\gamma] = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt$$

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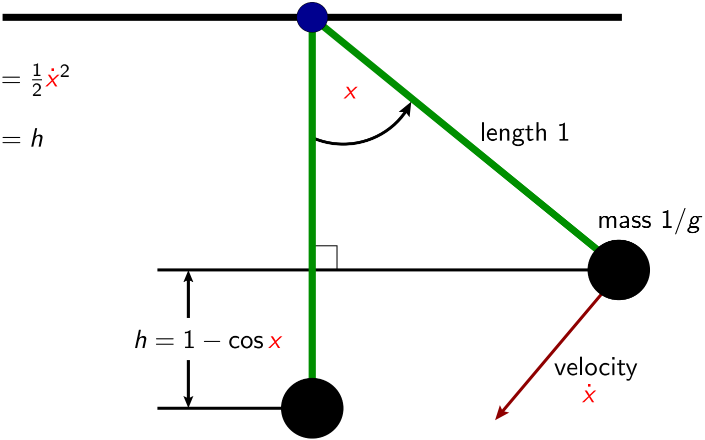

$$A = \int_a^b \left\{ \frac{1}{2}\dot{x}(t)^2 - (1 - \cos(x(t)) \right\} dt$$

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For the pendulum:

$$\ddot{x} = -\sin(x)$$

The Principal of Least Action, II

Solutions of the *Euler-Lagrange Equations* *locally* minimise the action.

The simple pendulum

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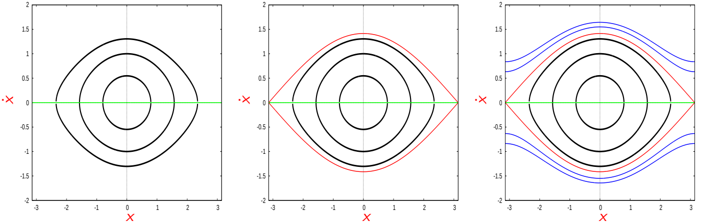

(a) Local but not global minimisers. (b) A global minimiser. (c) A family of global minimisers.

An Historical Note, I

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In 1913, the Berlin Academy reversed its previous decision and found Leibniz had priority.

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Dirichlet $\rightarrow$ analogous principal for PDE, used by Riemann.

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Weierstrass showed these *principals* are generally *wrong*.

Tonelli Lagrangians

A *Tonelli Lagrangian* is a C^r $L : TM \times S^1 \to \mathbf{R}$:

1. L is strictly fibre-wise convex:

$$\frac{\partial}{\partial v}\frac{\partial L}{\partial v} > 0.$$

2. L is superlinear:

$$\lim_{|v|\to\infty} L(x, v, \theta)/|v| = \infty \qquad \forall x \in M, \theta \in S^1.$$

3. The Euler-Lagrange flow of L is complete.

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Remark. Tonelli's theorem generalises Hopf-Rinow.

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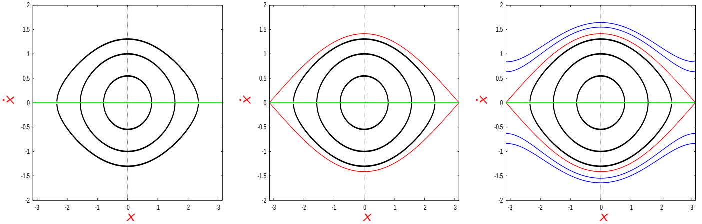

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integral $\rightarrow$ real homology class;

curve $\rightarrow$ measure.

Mather: For each real homology class h, there is an action-minimising invariant probability measure with rotation vector h.

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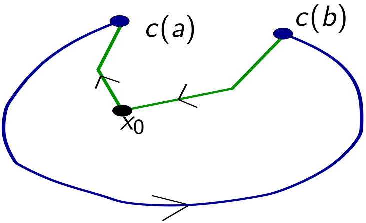

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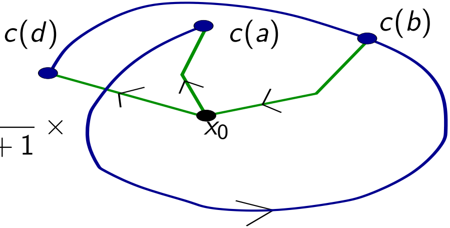

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The *rotation vector of* μ, $\rho(\mu)$,

$$\langle \rho(\mu), \lambda \rangle := \int_{TM \times S^1} \lambda \, d\mu \qquad \lambda \in H^1(M).$$

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Mather (1991):

- the set of action minimisers with fixed rotation vector is convex; (call this set $\mathfrak{M}_h(L)$)
- if $\mu \in \mathfrak{M}_h(L)$ is an extreme point, and $P \in \mathrm{supp}(\mu)$, then the Schwartzman asymptotic homology of the orbit through P contains h.

Mather's β function

$$\beta(h) := \inf \{A[\mu] \,:\, \rho(\mu) = h\}$$
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- β is Lipschitz, convex and superlinear;
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- the *convex dual* of β, $\alpha : H^1(M) \to \mathbf{R}$, satisfies

$$\alpha(c) := \max_h \langle c, h\rangle - \beta(h)$$
$$= -\min \{A_c[\mu] \,:\, \mu \in \mathfrak{M}(L)\} \quad A_c[\mu] = A[\mu] - \langle c, \rho(\mu)\rangle$$

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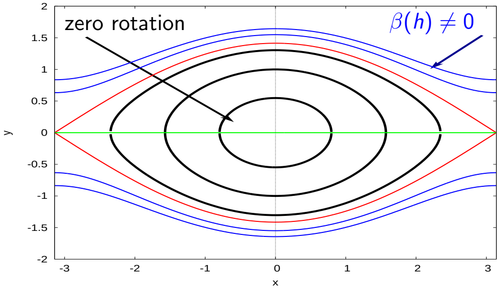
- $\mathfrak{M}_c(L) = \{\mu \,:\, A_c[\mu] = \alpha(c)\}$ satisfies

$$\mathrm{supp}(\mathfrak{M}_c(L)) = \text{Lipschitz graph over its projection to } M \times S^1$$
$$= \text{union of minimising orbits.}$$

The pendulum and α/β



Fathi: $\alpha(c)$ satisfies $\alpha(c) = \inf_{\theta \in c} \sup_x \{E(x, \theta(x))\}$.

Differentiability of α & β

By construction

$$\begin{aligned} \alpha(c) + \beta(h) &\geq \langle c, h \rangle && \forall c, h \text{ and} \\ h \in \partial\alpha(c), c \in \partial\beta(h) && \iff & \text{equality.} \end{aligned}$$

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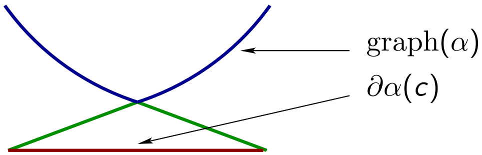
α is *differentiable* at c iff the rotation set of $\mathrm{supp}(\mathfrak{M}_c(L)) = \{h\}$;



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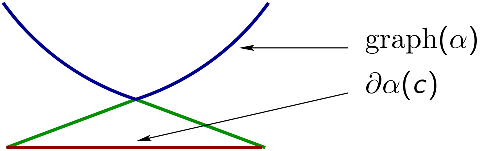

conjecturally, differentiability of α is related to integrability.

Complete Integrability

From the tangent to cotangent bundles:

Euler-Lagrange equations of Tonelli Lagrangian L

$$\equiv$$

Hamilton's equations of H (energy)

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Benefit: Rich geometric structure

$$\Omega = \sum_i^n dp_i \wedge dq^i \qquad \mathfrak{P} = \{\, , \,\} = -\sum_i^n \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}.$$

$=$ *symplectic form* $\qquad =$ *Poisson tensor*

Function $H \implies$ vector field
$X_H = \{H, \,\} = \sum_i^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$.

Liouville-Arnol′d Theorem

Let $F = (H = F_1, \ldots, F_n) : T^*M^n \to \mathbf{R}^n$ be a smooth map whose components Poisson commute. If $f \in \mathbf{R}^n$ is a regular value, then there is a neighbourhood $U \in f$ of a connected component $T \subset F^{-1}(f)$ such that

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1. (*action-angle coordinates*)

   there is a fibre-preserving symplectomorphism

$$\begin{array}{ccccccc} & & T & \xrightarrow{\equiv} & \mathbf{T}^n & & \\ & & \downarrow & & \downarrow & & \\ F^{-1}(U) & \xleftarrow{\supset} & W & \xrightarrow{\equiv} & \mathbf{T}^n \times V & \xrightarrow{\subset} & T^*\mathbf{T}^n \\ & \searrow^{F} & \downarrow F|W & & \text{proj.}\downarrow & & \\ & & U & \xrightarrow{\equiv} & V & & \end{array}$$

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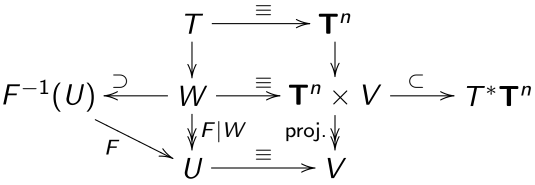
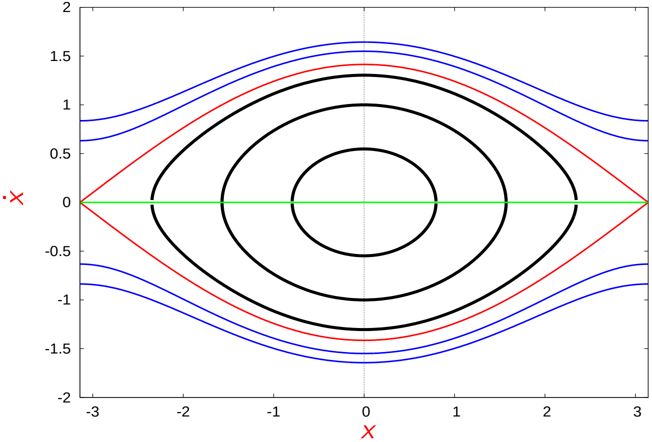


2. each vector-field *linearises*

$$X_{F_k} = \begin{cases} \dot{\theta}^i = \frac{\partial F_k}{\partial I_i} \\ \dot{I}_i = 0 \end{cases}$$

Liouville-Arnol′d Theorem



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Weak integrability (Sorrentino)

A Hamiltonian is *weakly integrable* if it admits dim M functionally independent integrals.

Compact Riemannian Homogeneous Spaces

B. & G. Paternain

Let G be a compact semi-simple Lie group of rank at least 2. In any neighbourhood of the bi-invariant metrics, there are left-invariant metrics with positive topological entropy that are not completely integrable.

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These metrics are *weakly integrable*.

A Weak Liouville-Arnol′d Theorem

Sorrentino, B.-Sorrentino

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6. if $\dim M \leq 3$, then M is a torus;
7. if $\dim H^1(M) \geq \dim M$, then H is completely integrable in a neighbourhood of Λ_c and $M = \mathbf{T}^n$.

Concluding Question/Conjecture

There are *no known examples* of completely integrable Tonelli Hamiltonians on T^*M^n for manifolds M admitting a metric of negative curvature.

Question. Is this a theorem?

Concluding Question/Conjecture

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Question. If $\dim H^1(M) > \dim M$, does this imply that there are *no* weakly integrable Tonelli Hamiltonians on T^*M?

Credit

Joint work with Alfonso Sorrentino.

Thank you

for your attention.