Powers of Π

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What kind of number is π?

Lambert (ca. 1768): π is irrational.π is irrational.

Proof: continued fraction expansion of $\tan\alpha$ shows α and $\tan\alpha$ cannot both be rational. But $\tan(\frac{\pi}{4}) = 1$.

Lindemann (ca. 1882): π is transcendental.π is transcendental.

(Recall: α is algebraic if it is a root of a polynomial with rational coefficients; otherwise, α is transcendental.)

Proof: $\exp(\alpha)$ and α cannot be both algebraic. If α is algebraic, then so is $i\alpha$. But $\exp(i\pi) = -1$.

Corollary: every natural power of π is transcendental.

Morbus Cyclometricus

Problem: is there a square, constructible by ruler and compass, whose area is π?

One of the oldest and enduring problems in maths: Rhind papyrus (ca. 3500BC) makes mention of it.

London Maths Society barred 'circle squarers'.

Values of π

And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about. *1 Kings 7:23 (King James Version)*

Old testament (ca. 950BC): $\pi \cong 3$

Egypt/Mesopotamia (ca. 950BC): $\pi \cong 25/8$

Mathematician	Date	Value of π	Comment
Archimedes	(ca. 287–212BC):	$\frac{223}{71} < \pi < \frac{22}{7}$	
Ptolemy	(ca. 150 AD)	3.1416	
Zu Chongzhi	(430-501 AD)	355/113	6 places
al-Khwarizmi	(ca. 800)	3.1416	al jabr
al-Kashi	(ca. 1430)	14 places	
Vite	(1540-1603)	9 places	
Roomen	(1561-1615)	17 places	
Van Ceulen	(ca. 1600)	35 places	

Gregory - Leibnitz' Formula for $\frac{\pi}{4}$

Recall

$$\frac{d}{dt}\arctan(t) = \frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

so the fundamental thm of calculus plus geometric series

$$\begin{aligned}\arctan(x) &= \int_0^x \frac{dt}{1+t^2} \qquad\qquad \forall x \in [-1,1] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \times x^{2k+1}\end{aligned}$$

which implies

$$\boxed{\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.}$$

Identities

Again, recall

$$f(x) = x \qquad x \in [-\pi, \pi]$$

this function has Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \exp(-ikx)\, dx = \frac{(-1)^k}{ik}, \qquad c_0 = 0.$$

Parseval's identity asserts

$$\frac{1}{2\pi} \int x^2 dx = \sum_{k \in \mathbf{Z}} |c_k|^2$$

We compute each side to obtain

$$\boxed{\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}}$$

The series

Theorem

For each integer $n \geq 1$, there is a rational number p_n such that

$$\boxed{p_n \pi^{2n} = \sum_{k=1}^{\infty} \frac{1}{k^{2n}}}$$

proof, pt. 1

We have seen this for $n = 1$: $p_1 = 1/6$. For $n = 2$, observe

$$f(x) = x^2 \qquad\qquad x \in [-\pi, \pi]$$

this function has Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \exp(-ikx)\, dx = \frac{2(-1)^k}{k^2}, \qquad c_0 = \frac{\pi^2}{3}.$$

Parseval's identity asserts

$$\frac{1}{2\pi} \int x^4 dx = \sum_{k \in \mathbf{Z}} |c_k|^2$$

We compute each side to obtain $\dfrac{\pi^4}{5} = \dfrac{\pi^4}{9} + 8 \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$ or

$$\boxed{\frac{\pi^4}{90} = \sum_{k=1}^{\infty} \frac{1}{k^4}}$$

proof, pt. 2

For $n \geq 3$, we proceed likewise.

Problems.

1. This proof leaves us with no insight!
2. Moreover, what does p_n equal? Formulae?
3. What about $\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}}$?

A second try: $x\cot(x)$

Lemma

For $|x| < 1$*, we have*

$$\cot(x) = \sum_{k\in\mathbf{Z}} \frac{1}{x - k\pi} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2\pi^2}$$

Let $g(x)$ be the LHS-RHS. Since $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$ is π-periodic, $g(x)$ is too. Since

$$\cot(x) = \frac{1}{x} + O(1)$$

at $x = 0$, so g is holomorphic at $x = 0$. Hence, by periodicity, g is holomorphic.

Now show that g is bounded. Hence Liouville's theorem applies. But $g(0) = 0$.

A second try: $x\cot(x)$

Lemma

For $|x| < 1$, we have

$$x\cot(x) = 1 - 2\sum_{n=1}^{\infty}\left(\frac{1}{\pi^{2n}} \times \sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right)\ x^{2n}$$

We see that

$$\frac{2x^2}{x^2 - k^2\pi^2} = -\frac{2x^2}{k^2\pi^2} \times \frac{1}{1 - \frac{x^2}{k^2\pi^2}} = -2\sum_{n=1}\frac{x^{2n}}{k^{2n}} \times \frac{1}{\pi^{2n}}.$$

Now rearrange.

A second try: $x\cot(x)$

Theorem

For $|x| < 1$, we have

$$x\cot(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n}$$

where the rational Bernoulli numbers B_n are defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

We see that for $z = 2ix$

$$x\cot(x) = ix\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = \frac{z}{2}\frac{e^z + 1}{e^z - 1} = \frac{z}{2} + \frac{z}{e^z - 1}$$

$$= \frac{z}{2} + \sum_{n=0}^{\infty} (i)^n \frac{B_n 2^n}{(n)!} x^n \quad \text{split into real and imaginary.}$$

Bernoulli numbers

Some facts

1. $B_0 = 1$, $B_1 = -\frac{1}{2}$and $B_{2n+1} = 0$ for all n;
2. $$\sum_{k=0}^{n-1} \frac{B_k}{k!(n-k)!} = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$
3. The first 20 Bernoulli numbers

n	0	1	2	4	6	8	10
B_n	1	-1/2	1/6	$-\frac{1}{30}$	1/42	$-\frac{1}{30}$	5/66

n	12	14	16	18	20
B_n	$-\frac{691}{2730}$	7/6	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

A formula for $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$

Theorem

For each $n \geq 1$,

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \times \frac{2^{2n-1} B_{2n}}{(2n)!} \times \pi^{2n}.$$

$$\begin{aligned} x\cot(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n} \\ &= 1 - 2\sum_{n=1}^{\infty} \left(\frac{1}{\pi^{2n}} \times \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \right) x^{2n} \end{aligned}$$

▸ Bernoulli ▸ sum

Knots

In 3-d, we have knots.

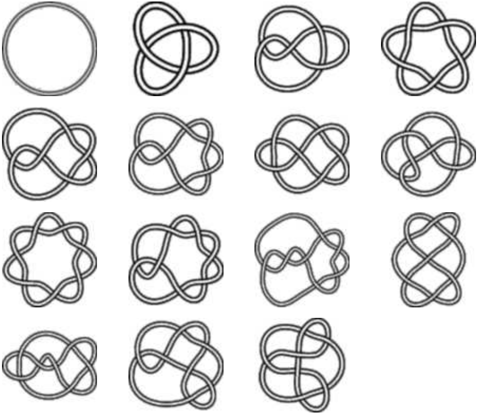

Figure: Knots with fewer than 8 crossings. Courtesy of the Knot Atlas.

Exotic spheres

(Milnor 1956): In dimensions 7 or more, we also have bizarre knots: topological n-spheres that are smooth manifolds not diffeomorphic to the standard n-sphere.

Let bP_{n+1} be the set of topological n-spheres that bound a parallelizable manifold, identified up to 'h-cobordism'.

Theorem (Milnor-Kervaire 1963)

bP_{n+1} is a finite cyclic group under connected sum; the trivial element is S^n; it is trivial for n odd. The order is

$$|bP_{n+1}| = \begin{cases} 1 \text{ or } 2 & n \equiv 1 \bmod 4 \\ 2^{2m-2}(2^{2m-1}-1) \times \text{numerator}\left(\frac{4B_m}{m}\right) & n+1 = 4m. \end{cases}$$

Exotic Spheres

n	7	8	9	10	11	12	13	14	15
bP_{n+1}	28	1	2	1	992	1	1	1	8128
Θ_n	28	2	8	6	992	1	3	2	16256
n	16	17	18	19	20				
bP_{n+1}	1	2	1	261632	1				
Θ_n	2	16	16	523264	24				

Figure: The order of the group of exotic n-spheres Θ_n, and its subgroup bP_{n+1}.

Highlighted table

Maple code to compute $|bP_{4n}|$

```
> # B computes the n-th Bernoulli number
> # EX computes the order of bP_{4m}
>
> B:=proc(n::integer)
> local c;
> c:=coeftayl( z*coth(z), z=0, 2*n );
> c:=(-1)^(n-1) * c * factorial(2*n) / 2^(2*n);
> return c;
> end proc:
>
> EX:=proc(m::integer)
> return 2^(2*m-2) * (2^(2*m-1) - 1) * numer(4*B(m)/m);
> end proc:
>
> seq( [4*m-1, EX(m) ], m=2..5 );
                        [7, 28], [11, 992], [15, 8128], [19, 261632]
```

The order of bP_{4n}

n	7	8	9	10	11	12	13	14	15
bP_{n+1}	28	1	2	1	992	1	1	1	8128
Θ_n	28	2	8	6	992	1	3	2	16256
n	16	17	18	19	20				
bP_{n+1}	1	2	1	261632	1				
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```
> seq( [4*m-1, EX(m) ], m=2..5 );
                [7, 28], [11, 992], [15, 8128], [19, 261632]
```

Brieskorn spheres

Let $a_0, \dots, a_k$ be positive integers and let

$$V_a = \{\mathbf{z} \in \mathbf{C}^{2k+2} \ : \ z_0^{a_0} + \cdots + z_k^{a_k} = 0\} \cap S^{2k+1}.$$

For $a_0 = p, a_1 = q$,

$$V_a = \{\mathbf{z} \in \mathbf{C}^2 \ : \ z_0 = \omega r \exp(it), z_1 = r \exp(itp/q), t \in [0, 2q\pi]\}$$

where $r = \sqrt{1 - r^2}^{q/p}$ and $\omega^q = -1$.
When $p = 3, q = 2$, we get

Knots

Figure: The 3_1 torus knot

Let

$$V_r = \{\mathbf{z} \in \mathbf{C}^5 \ : \ z_0^2 + z_1^2 + z_2^2 + z_3^3 + z_4^{6r-1} = 0\} \cap S^9.$$

Brieskorn spheres

When $p = 3, q = 2$, we get

Knots

Figure: The 3_1 torus knot

Let

$$V_r = \{\mathbf{z} \in \mathbf{C}^5 \ : \ z_0^2 + z_1^2 + z_2^2 + z_3^3 + z_4^{6r-1} = 0\} \cap S^9.$$

Theorem (Brieskorn 1966)

For $r = 1, \ldots,$ each of the 28 exotic 7-spheres is a Brieskorn sphere V_r.