

The Maslov cocycle, smooth structures and real-analytic complete integrability

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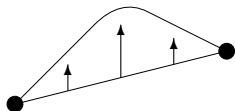
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Hamiltonian mechanics

Principle of Least Action

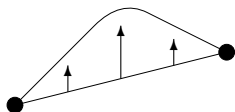
$$\int_a^b L(t, x, \dot{x}) dt \rightarrow \min$$



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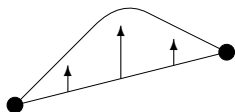
Minimisers solve Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

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Equivalent to **Hamilton's equations**

$$\dot{x} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial x}$$

Hamiltonian mechanics

Poisson bracket $\{, \}$:

$$\{F, H\} = \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial p} - \frac{\partial H}{\partial x} \cdot \frac{\partial F}{\partial p}$$

which implies Hamilton's equations

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial p}, \quad \dot{p} = \{p, H\} = -\frac{\partial H}{\partial x}$$

on

$$T^*M = \{(x, p) : x \text{ is a point on } M, p \text{ is a momentum vector}\}.$$

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Theorem (Liouville-Arnold-Mineur)

Let $H = F_1, \dots, F_n$ be n Poisson commuting, independent functions. If

$$T_c = F_1^{-1}(c_1) \cap \dots \cap F_n^{-1}(c_n)$$

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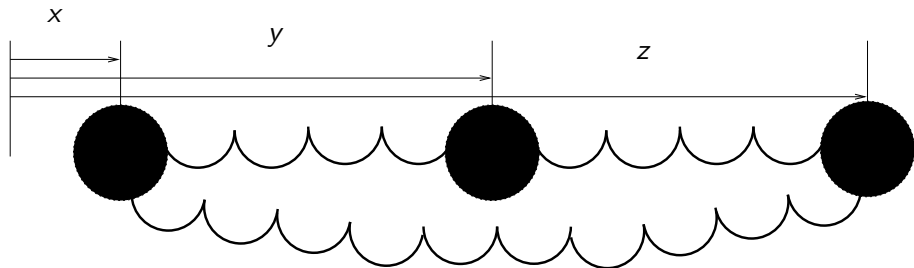
$$T_c = F_1^{-1}(c_1) \cap \dots \cap F_n^{-1}(c_n)$$

is a compact, non-empty, regular level set, then

- T_c is diffeomorphic to the n -torus \mathbf{T}^n ;
- there is an open neighbourhood of T_c diffeomorphic to $\mathbf{T}^n \times \mathbf{R}^n$;
- there are coordinates (θ, I) such that
 - each $F_j = F_j(I)$;
 - each vector field is linear

$$\dot{\theta} = \frac{\partial F_j(I)}{\partial I} \qquad \dot{I} = 0.$$

Toda Lattices



Potential energy:

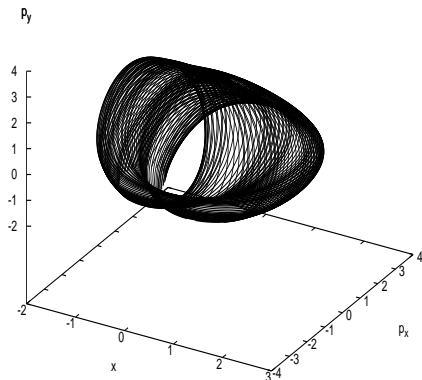
$$V = \exp(x - y) + \exp(y - z) + \exp(z - x).$$

An example

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + e^{x-y} + e^{y-z} + e^{z-x}$$

$$F = p_x p_y p_z - e^{x-y} p_z - e^{z-x} p_y - e^{y-z} p_x$$

$$G = p_x + p_y + p_z$$

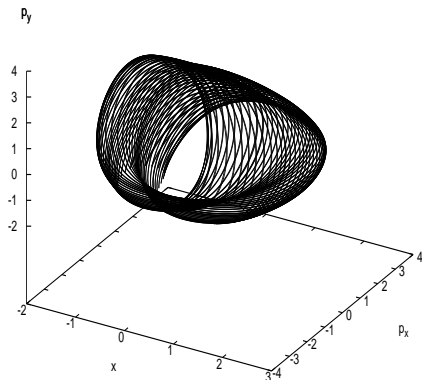


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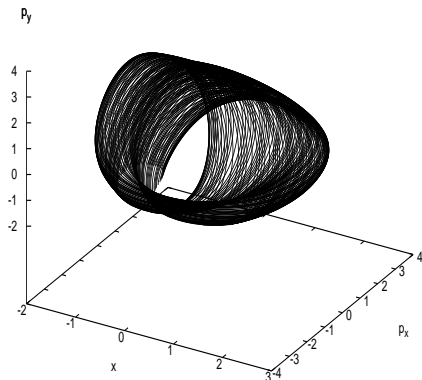


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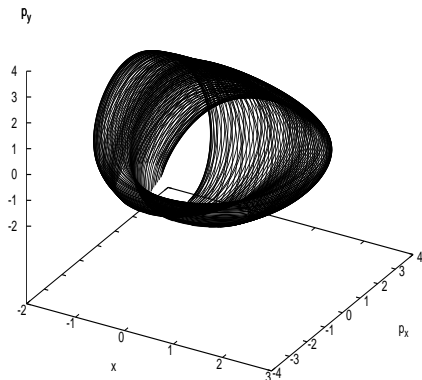


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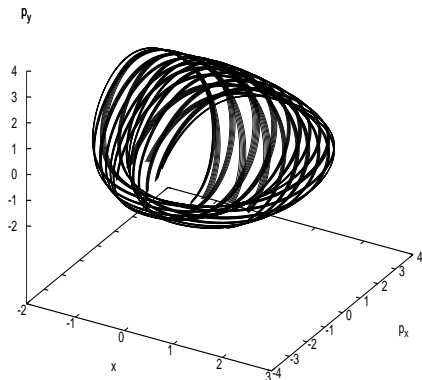


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An example: what accounts for these integrals?

$$L = \begin{bmatrix} p_x & e^{x-y} & e^{z-x} \\ e^{x-y} & p_y & e^{y-z} \\ e^{z-x} & e^{y-z} & p_z \end{bmatrix} \quad M = \frac{1}{2} \times \begin{bmatrix} 0 & e^{x-y} & -e^{z-x} \\ -e^{x-y} & 0 & e^{y-z} \\ e^{z-x} & -e^{y-z} & 0 \end{bmatrix}$$

Hamilton's equations are equivalent to

$$\dot{L} = [L, M]$$

which implies that

$$\begin{aligned} \text{Trace } L \\ = G, \end{aligned}$$

$$\begin{aligned} \text{Trace } L^2 \\ = 2H, \end{aligned}$$

$$\begin{aligned} \text{Trace } L^3 \\ = F. \end{aligned}$$

are integrals.

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \rightarrow \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F , then

- 1 the regular levels of H are graph manifolds;*
- 2 the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

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In addition,

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In addition,

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Graph manifolds

Building Blocks

- 1 $S_k = 2$ -disk with k open disks removed;
- 2 $M_k = S_k \times S^1$ or $S_k \tilde{\times} S^1$;
- 3 $M =$ union of M_{k_1}, \dots, M_{k_n} glued along toral boundary.

Graph manifolds

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Examples

- 1 $S^2 \times S^1 = D^2 \times S^1 \cup_{id} D^2 \times S^1$
- 2 $S^3 = D^2 \times S^1 \cup_{id} S^1 \times D^2$
- 3 T^3, \dots
- 4 no hyperbolic manifolds

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Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic riemannian manifold. If the geodesic flow is real-analytically integrable, then

- 1 $b_1(M) \leq n$;
- 2 $\pi_1(M)$ is almost abelian;
- 3 there is an injection $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

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Theorem (Paternain)

If $H : (M^{2n}, \omega) \rightarrow \mathbf{R}$ is completely integrable with non-degenerate integrals, then its topological entropy vanishes.

Bolsinov & Taimanov's Example

Theorem (Bolsinov & Taimanov)

Let Σ be a \mathbf{T}^2 -bundle over \mathbf{T}^1 . There is a natural riemannian metric on Σ whose geodesic flow is completely integrable. If the gluing map is hyperbolic, then the geodesic flow has positive topological entropy.

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Let Σ be a \mathbf{T}^2 -bundle over \mathbf{T}^1 . There is a natural riemannian metric on Σ whose geodesic flow is completely integrable. If the gluing map is hyperbolic, then the geodesic flow has positive topological entropy.

There are two real analytic integrals, but the third is only smooth.

Geometrisation Conjecture

3 model geometries in 2 dimensions:

 E^2 S^2 H^2

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$$\mathbf{E}^2$$

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$$S^2$$

8 model geometries in 3 dimensions:

$$\mathbf{E}^3$$

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$$S^2 \times \mathbf{E}^1$$

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Conjecture (Thurston)

Each compact 3-manifold admits a canonical decomposition into complete, finite-volume geometric 3-manifolds.

Results in 3 degrees of freedom

Definition (Semisimplicity)

Let

$$\begin{array}{ccccc}
 \mathbf{T}^n \hookrightarrow & L & \xrightarrow{\text{incl.}} & T^*M & \xrightarrow{\pi} & M \\
 & \downarrow f & & & & \\
 & B & & & &
 \end{array}$$

be a lagrangian fibration. If $\Gamma = T^*M - L$ is a closed, nowhere dense tamely-embedded polyhedron, then we say (f, L, B) is **semisimple**.

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Examples.

- 1 a real-analytic singular lagrangian fibration
- 2 a smooth singular lagrangian fibration with a real-analytic singular set
- 3 the Bolsinov-Taimanov example

Results in 3 degrees of freedom

Theorem (B. 2005)

Assume the GC. A compact 3-manifold admits a riemannian metric with a semisimple geodesic flow iff it admits one of five geometric structures:

 $\mathbf{E}^3,$ $S^2 \times \mathbf{E}^1,$ $S^3,$ $Nil,$ $Sol.$

Sketch of proof

\Rightarrow There is a subspace $L_0 \subset L$ s.t. $\pi_1(L_0) \rightarrow \pi_1(M)$ is almost onto and

$$\begin{array}{ccccc} \mathbf{T}^3 & \hookrightarrow & L_0 & \longrightarrow & M \\ & & \downarrow f_0 = f|_{L_0} & & \\ & & B_0 & & \end{array}$$

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- 3 If $\pi_2(M) = 0$, Hempel-Jaco implies M is the connect sum of a geometric manifold with a homotopy sphere.
- 4 Otherwise, (Agol) GC implies that M admits $S^2 \times \mathbf{E}$ geometry.

Sketch of proof



The real-analytic complete integrability of the geodesic flows of the compact geometries modeled on \mathbf{E}^3 , $S^2 \times \mathbf{E}$, S^3 is well-known.

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The semisimplicity of the geodesic flows of *Nil*-geometry is proven by (B.) and that of *Sol* by (Bolsinov-Taimanov). [▶ To B-T](#) [▶ Theorem](#)

Exotic spheres and Tori

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Wall showed that the smooth structures on the topological n -torus is the group

$$\sum_{i=0}^n H^i(\mathbf{T}^n; \Gamma_i) \quad \Gamma_i = \begin{cases} \text{group of exotic } i\text{-spheres,} & i \neq 4, \\ 0 & i = 4. \end{cases}$$

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The group of smooth structures on \mathbf{T}^7 is the group of 28 smooth structures on S^7 .

Exotic tori and integrability

Theorem (B. 2007)

Let Σ be a smooth n -dimensional topological torus. If T^Σ admits a real-analytically completely integrable convex hamiltonian, then Σ is diffeomorphic to the standard n -torus.*

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Remark

Real-analyticity may be weakened to (f, L, B) -semisimplicity plus $\pi_1(B)$ is finite.

Exotic tori and integrability

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Let Σ be a smooth n -dimensional topological torus. If $T^*\Sigma$ admits a real-analytically completely integrable convex hamiltonian, then Σ is diffeomorphic to the standard n -torus.

Proof.

Claim: real-analytic integrability implies there is a lagrangian torus F s.t. ρ is a diffeomorphism:

$$\begin{array}{ccc}
 F \subset & \xrightarrow{\text{incl.}} & T^*\Sigma \\
 & \searrow \rho & \downarrow \pi \\
 & & \Sigma.
 \end{array}$$

We show $\deg \rho \neq 0$, use a result of Viterbo's that $\deg \rho \neq 0$ implies ρ is a covering map, then use a cohomology argument to show ρ is 1 - 1. \square

Exotic tori and integrability

Proof.

- 1 Taimanov's argument implies that there is a lagrangian torus F s.t.

$$\pi_1(F) \xrightarrow{\rho_*} \pi_1(\Sigma)$$

has a finite index image.

Exotic tori and integrability

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- 2 Wall: smooth structures on a topological (=PL) n -torus are classified by a class

$$\sigma \in \bigoplus H^i(\Sigma; \Gamma_i) \quad \Gamma_i = \pi_i(PL/O) \text{ finite,}$$

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- 3 Pullback to a covering

$$\begin{array}{ccccc}
 & & \phi & & \\
 & & \curvearrowright & & \\
 F' & \xrightarrow{\iota_{F'} = \text{incl.}} & T^*\Sigma' & \xrightarrow{P = \text{proj.}} & T^*\Sigma & \xleftarrow{\iota_F = \text{incl.}} & F \\
 & \searrow \rho' = \pi' \cdot \iota_{F'} & \downarrow \pi' = \text{proj.} & & \downarrow \pi = \text{proj.} & \swarrow \rho = \pi \cdot \iota_F & \\
 & & \Sigma' & \xrightarrow{p} & \Sigma & &
 \end{array}$$

Maslov cycle

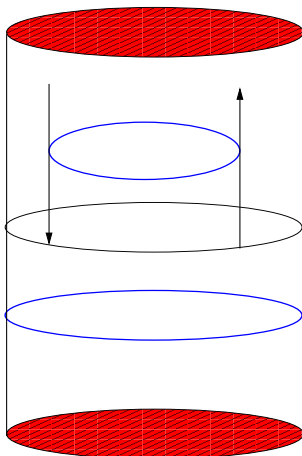


Figure: The Maslov cycle

Exotic tori and integrability

4 Theorem. (Viterbo, 1991)

Let Σ' be diffeomorphic to the standard n -torus. If F' is a lagrangian torus, and $\deg \rho' \neq 0$, then the Maslov class of F' vanishes.

$$\begin{array}{ccc}
 F' \hookrightarrow & \xrightarrow{\iota_{F'} = \text{incl.}} & T^*\Sigma' \\
 & \searrow \rho' = \pi' \cdot \iota_{F'} & \downarrow \pi' = \text{proj.} \\
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(Bialy & Polterovich) If F' is convex, then ρ' is a covering map.

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5 $\implies F \xrightarrow{\rho} \Sigma$ is a covering map.

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Let

$$\begin{aligned}\alpha &= \lambda|_F \\ &= \alpha_0 + \alpha_1\end{aligned}$$

$$\begin{aligned}\lambda &= \text{Liouville one-form of } T^*\Sigma \\ \alpha_0 &\in \wedge^1(F)^\Delta, \alpha_1 = dh,\end{aligned}$$

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6 For each $\gamma \in \Delta$, $x \in F$,

$$\begin{aligned}\alpha_x - \gamma^* \alpha_{\gamma(x)} &= dh_x^\gamma \\ &= (d_x \rho)^*(x - \gamma(x))\end{aligned}$$

$$\begin{aligned}h^\gamma &= h - h \circ \gamma, \\ &\text{from definition of } \alpha.\end{aligned}$$

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7 Since $\exists x = x_\gamma$ s.t.

$$dh_x^\gamma = 0 \quad \implies \gamma(x) = x.$$

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where $\Delta = \text{Deck}(\rho)$.

6 For each $\gamma \in \Delta$, $x \in F$,

$$\begin{aligned} \alpha_x - \gamma^* \alpha_{\gamma(x)} &= dh_x^\gamma & h^\gamma &= h - h \circ \gamma, \\ &= (d_x \rho)^*(x - \gamma(x)) & & \text{from definition of } \alpha. \end{aligned}$$

7 Since $\exists x = x_\gamma$ s.t.

$$dh_x^\gamma = 0 \quad \implies \gamma(x) = x.$$

8 Therefore $\Delta = 1$ and ρ is a diffeomorphism.

Witten-Kreck-Stolz spaces

Let $k, l \neq 0$ be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} : |z| = 1 \right\} \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \quad \text{Witten-Kreck-Stolz space.}$$

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Theorem (Kreck-Stolz)

There are 28 smooth structures on $M_{1,4}$. These structures are represented by $M_{32t+1,4}$ for $t = 0, \dots, 27 \pmod{28}$.

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Theorem (B. 2007)

The riemannian metric on $M_{k,l}$ induced by the round metrics on $S^3 \times S^5$ is real-analytically completely integrable.

Eschenburg-Krugger Spaces

Let $k, l \in \mathbf{Z}^3$ with $\sum k_i = \sum l_i$. Let

$$U_{kl} = \left\{ \left[\begin{array}{ccc} z^{k_0} & & \\ & z^{k_1} & \\ & & z^{k_2} \end{array} \right] \oplus \left[\begin{array}{ccc} z^{l_0} & & \\ & z^{l_1} & \\ & & z^{l_2} \end{array} \right] : |z| = 1 \right\} \subset U_2 \times U_3.$$

This group acts on SU_3 by

$$u \cdot g = u_1 g u_2^{-1}.$$

The action is free if $k - \sigma(l)$ is a primitive vector in \mathbf{Z}^3 for all permutations σ .

$$M_{k,l} = SU_3 / U_{kl}$$

Eschenburg space.

Condition C

Let $k, l \in \mathbf{Z}^3$. Define

$$A = \begin{bmatrix} k_0 - l_0 & k_0 - l_1 & k_0 - l_2 \\ k_1 - l_0 & k_1 - l_1 & k_1 - l_2 \\ k_2 - l_0 & k_2 - l_1 & k_2 - l_2 \end{bmatrix}$$

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Definition

(k, l) satisfy condition C if A has at least one column or row with pairwise coprime, non-zero entries.

Eschenburg-Krugger Spaces

Theorem (Kruggel)

Let $(k, l), (k', l')$ both satisfy condition C. $M_{k,l}$ is orientation-preserving homeomorphic to $M_{k',l'}$ iff

- 1 the ranks of H^4 are equal, the linking form is isometric, and the first Pontryagin classes are equal; and
- 2 the second Kreck-Stolz invariants $s_2 \in \mathbf{Q}/\mathbf{Z}$ are equal.

$M_{k,l}$ is orientation-preserving diffeomorphic to $M_{k',l'}$ iff, in addition,

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Theorem (Paternain-Spatzier 1994, Bazaikin 2000, B. 2007)

Every Eschenburg space admits a real-analytically completely integrable geodesic flow.

(Numerically,) the topological Eschenburg-Kruggel space with $k = (-1, -1, 2)$ and $l = (-2, 0, 2)$ admits 28 distinct smooth structures.

28 smooth structures on $E_{-1,-1;-2,0}$

k_0	k_1	l_0	l_1	$s_1 \bmod 1$	k_0	k_1	l_0	l_1	$s_1 \bmod 1$
-29	10	-28	6	1	-21	-6	-18	-10	0.5
-38	-29	-66	22	0.96428	-5	-5	-6	-4	0.46428
-54	9	-52	4	0.92857	-13	2	-8	-6	0.42857
-17	-17	-18	-16	0.89285	-14	-5	-16	-2	0.39285
-6	-3	-8	0	0.85714	-9	-6	-12	-2	0.35714
-17	-14	-22	-8	0.82142	-38	-11	-48	8	0.32142
-14	-5	-18	2	0.78571	-22	-19	-40	12	0.28571
-1	-1	-2	0	0.75	-22	-1	-14	-12	0.25
-33	-6	-42	20	0.71428	-25	-1	-22	-6	0.21428
-46	-13	-32	-30	0.67857	-54	-9	-68	30	0.17857
-22	5	-20	0	0.64285	-39	-6	-32	-16	0.14285
-13	2	-14	6	0.60714	-29	-14	-32	-10	0.10714
-38	-11	-40	-8	0.57142	-11	1	-12	4	0.07142
-22	-1	-26	12	0.53571	-9	-9	-10	-8	0.03571

Questions

- 1 Are there completely integrable geodesic flows on exotic tori? Must these have positive entropy?
- 2 Is it possible to give a description of the smooth invariant which determines whether or not a topological manifold admits a semisimple geodesic flow?
- 3 Must a real-analytically integrable system in dimensions 6 and more have zero entropy?