A construction of integrable systems with positive entropy

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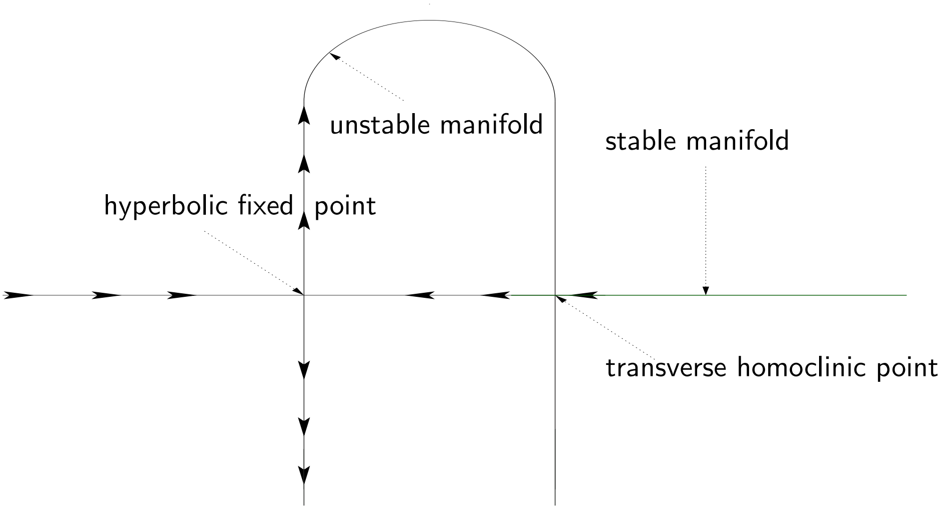
The Horseshoe

An example: $f : S^2 \to S^2$ is a smooth map.



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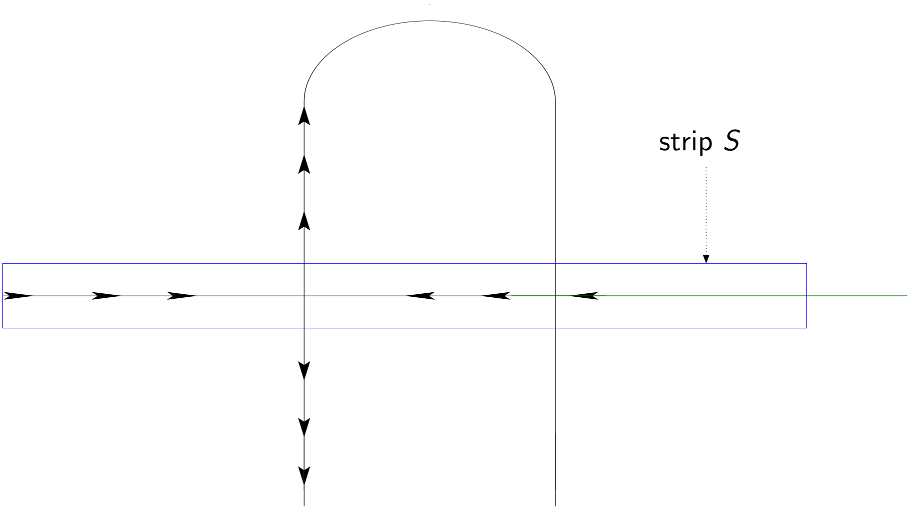

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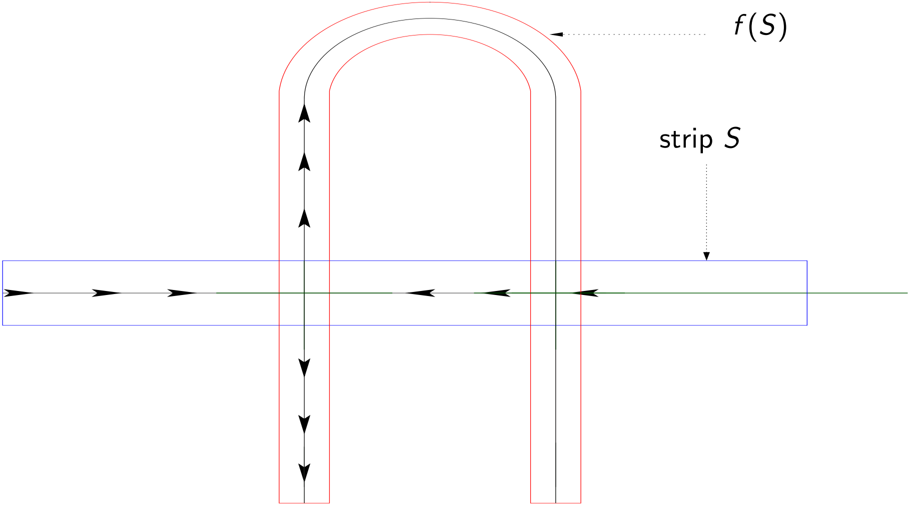

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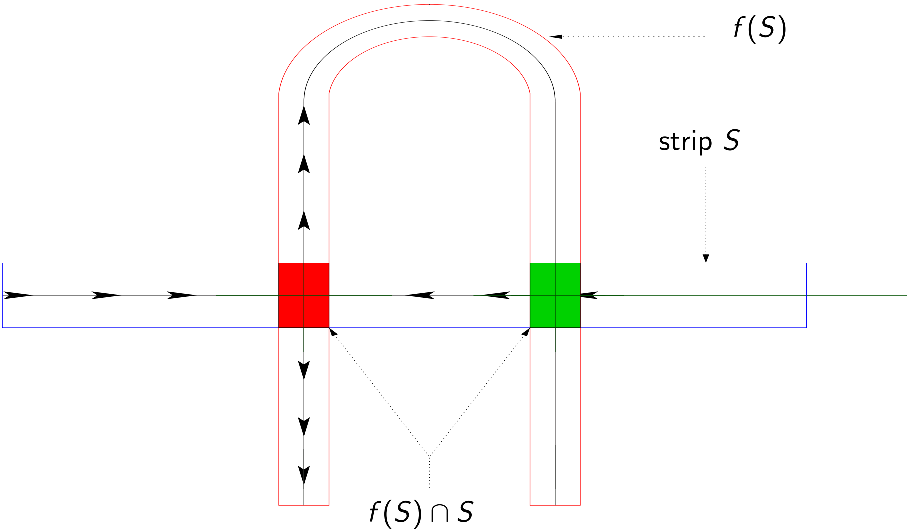

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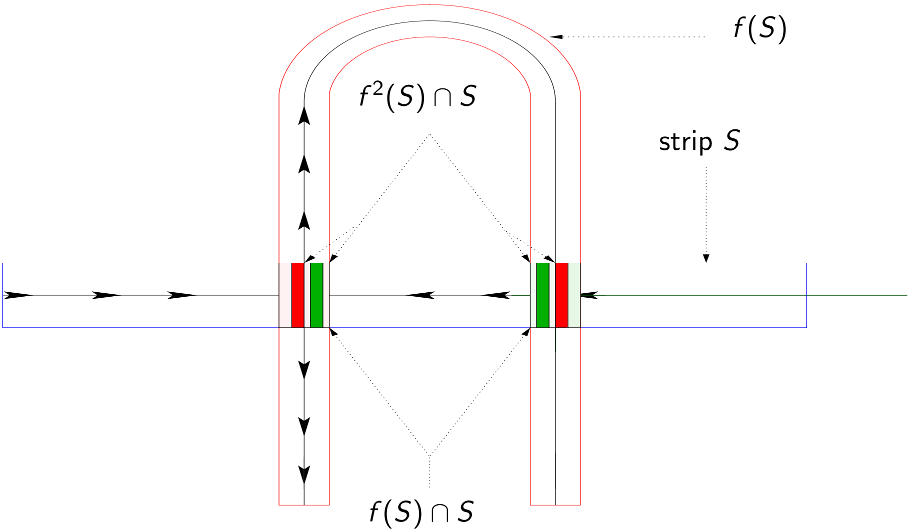

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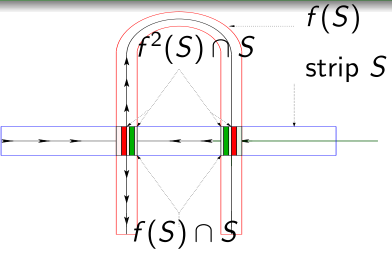

Implications:

1. Invariant Cantor set Λ
2. Invariant analytic function is constant on Λ
3. Invariant analytic function is constant.

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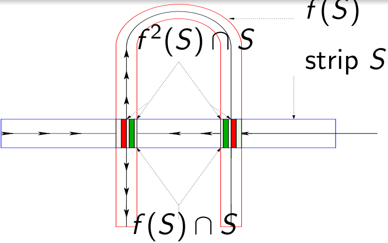

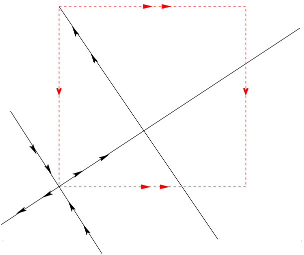

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Example: $\mathbf{T}^2$ automorphism

$$\mathbf{y} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y} \bmod \mathbf{Z}^2.$$

Integrability

First integral: an F constant along trajectories of $X_H = \{\cdot, H\}$
F and H Poisson commute if $\{F, H\} \equiv 0$.

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Theorem (Liouville-Arnold-Mineur)

Let $H = F_1, \cdots, F_n$ be n Poisson commuting functions. If

$$T_c = F_1^{-1}(c_1) \cap \cdots \cap F_n^{-1}(c_n)$$

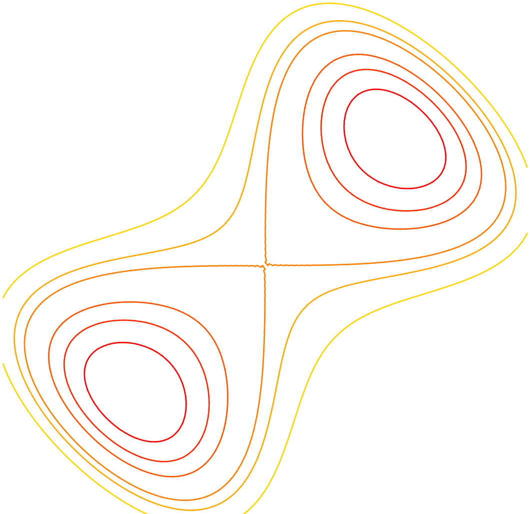
is a compact, non-empty, regular level set, then

- *T_c is diffeomorphic to the n-torus $\mathbf{T}^n$;*
- *there is an open neighbourhood of T_c diffeomorphic to $\mathbf{T}^n \times \mathbf{R}^n$;*
- *there are coordinates (θ, I) such that*
    - *each $F_j = F_j(I)$;*
    - *each vector field is linear*

$$\dot{\theta} = \frac{\partial F_j(I)}{\partial I} \qquad \dot{I} = 0.$$

Integrability

- A typical phase portrait:



- fibred by tori
- tori degenerate
- degenerations controlled by Morse-like behaviour

▸ To horseshoe ▸ To L-A-M

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \to \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F, then

1. *the regular levels of H are graph manifolds;*
2. *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

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Theorem (Paternain, Moser)

In addition,

3. *the topological entropy vanishes.*

What about 3 or more degrees of freedom?

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Theorem (Paternain)

If $H : (M^{2n}, \omega) \to \mathbf{R}$ *is completely integrable with non-degenerate integrals, then its topological entropy vanishes.*

What about 3 or more degrees of freedom?

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;
2. $\pi_1(M)$ *is almost abelian;*
3. *there is an injection* $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

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This implies:

1. Surfaces: $S^2, \mathbf{T}^2$;
2. 3-manifolds:

$b_1 = 0$: $\tilde{M}$ is a homotopy S^3;
$b_1 = 1$: M is finitely covered by a homotopy $S^1 \times S^2$;
$b_1 = 2$: can't happen (Reidemeister);
$b_1 = 3$: M is finitely covered by a homotopy $\mathbf{T}^3$.

Questions

1. Is Taimanov's theorem true in the smooth category?
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Theorem (Bolsinov-Taimanov 2000)

There are smoothly integrable systems with positive topological entropy.

It is unknown if a real-analytically integrable system must have zero entropy.

Bolsinov-Taimanov's Example

- Configuration space: $Sol = \mathbf{R} \star \mathbf{R}^2$

$$x.\mathbf{y} = (e^x y_0, e^{-x} y_1).$$

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- Discrete group: $u = \frac{1+\sqrt{5}}{2}$

$$\Delta = \{(u^k, \mathbf{y}) \ : \ y_0 = \frac{n + m\sqrt{5}}{2}, y_1 = \frac{n - m\sqrt{5}}{2}, \ k, m, n \in \mathbf{Z} \}.$$

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The hamiltonian $H : T^*(Sol/\Delta) \to \mathbf{R}$ *is completely integrable. Its flow has positive topological entropy.*

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Proof.

The integrals are

$$2H = p_x^2 + e^{2x} p_{y_0}^2 + e^{-2x} p_{y_1}^2,$$

$$I = p_{y_0} p_{y_1}, \qquad J = \exp(-I^{-2}) \times \sin \frac{2\pi \ln |p_{y_1}|}{\ln u}.$$

□

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Proof.

Anosov diffeomorphism on $\{p_{y_1} = p_{y_2} = 0, p_x = 1\}$:

$$\begin{array}{rclcrcl} x & = & x_0 + t, & & p_x & = & 1, \\ y_0 & = & y_{0,0}, & & p_{y_0} & = & 0, \\ y_1 & = & y_{1,0}, & & p_{y_1} & = & 0, \quad \mod \Delta. \end{array}$$

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At $t = \ln u$

$$\begin{array}{rclcrcll} x & \equiv & x_0, & & p_x & \equiv & 1, & \\ y_0 & \equiv & u^{-1} y_{0,0}, & & p_{y_0} & \equiv & 0, & \\ y_1 & \equiv & u y_{1,0}, & & p_{y_1} & \equiv & 0, & \mod \Delta. \end{array}$$

or

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y}(0) \bmod \mathbf{Z}^2.$$

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3. Which dynamical systems can be embedded as a subsystem of an integrable system?

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In the Bolsinov-Taimanov *Sol*-manifold

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Answers:

algebraic number theory

- abelian subgroups of $\mathrm{Aut}(\mathbf{T}^n) \to$ algebraic extensions of $\mathbf{Q}$
- "best" subgroup $\to$ group of units.

A dictionary

Definition

Let $\mathbf{Q} \subset L$ be an algebraic number field:

$\mathcal{O}_L \rightarrow$ integers of L;

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abelian $A < \mathrm{Aut}(\mathbf{T}^n)$	$\longrightarrow$	field F generated by eigenvalues
$\mathbf{Z}^n$	$\longrightarrow$	module over $\mathcal{O}_F$
$\mathbf{T}^n$	$\longrightarrow$	sum of copies of $\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{R}/\mathcal{O}_F$
A	$\longrightarrow$	subgroup of $\mathcal{U}_F$.

Toda Lattice

A non-linear coupled oscillator

$$\ddot{x}_i = \exp(x_i - x_{i+1}) - \exp(x_{i-1} - x_i), \qquad i = 1, \ldots, n \bmod n. \qquad (*)$$

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Flaschka Transform

$$a_i := \epsilon_i \exp(x_i/2 - x_{i+1}/2), \qquad b_i := \dot{x}_i,$$

$$L := \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 & \lambda a_n \\ a_1 & b_2 & a_2 & \cdots & 0 & 0 \\ 0 & a_2 & b_3 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & b_{n-1} & a_{n-1} \\ \lambda^{-1} a_n & 0 & & & a_{n-1} & b_n \end{bmatrix}$$

$$M := 1/2 \times (L_+ - L_+')$$

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Toda Lattices and Positive-Entropy Integrable Systems

Theorem (B. 2004, B. 2008)

Let $\mathbf{Q} \subset F \subset E$ *be algebraic number fields. Let* $\Delta = \mathcal{U}_F^+ \star \mathcal{O}_E$. *There is a bundle* $\mathbf{T}^b \hookrightarrow \Sigma \to \mathbf{T}^a$ *with* $\pi_1(\Sigma) = \Delta$ *such that*

1. *for each periodic Toda lattice* Ψ *of rank* n*, there is an integrable hamiltonian* $H_\Psi : T^*\Sigma \to \mathbf{R}$*;*
2. *the flow of* H_Ψ *contains a subsystem isomorphic to* $u \in \mathcal{U}_F$ *for all* u*;*
3. *for* $n = 1$ *and* $F = \mathbf{Q}(\sqrt{5})$*, the* $A_1^{(1)}$ *Toda lattice yields the Bolsinov-Taimanov example.*

Idea of a Proof: Use the Flaschka transform

$\Sigma = \Delta \backslash \mathbf{S}$ where $\mathbf{S}$ is a solvable Lie group and Δ acts by left-translation;

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The momentum map λ of $\mathbf{S}$'s left action and
the momentum map ρ of $\mathbf{V}$'s right action on $T^*\mathbf{S}$ satisfies

$$\mathbf{R}^b = \mathfrak{v}^* \xleftarrow{\ \rho\ } T^*\mathbf{S} \xrightarrow{\ \lambda\ } \mathfrak{s}^* \xleftarrow{\ \mathrm{F.T.}\ } T^*\mathbf{R}^a, \qquad T^*\mathbf{S} \xrightarrow{\text{Lax matrix}} T^*\mathbf{R}^a$$

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λ descends to $T^*\Sigma$
ρ does not.

Reduction of ρ

We use equivariance to reduce ρ.

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The momentum map of $\mathbf{V}$'s right action on $T^*\mathbf{S}$ satisfies

$$\begin{array}{ccccc} T^*\mathbf{S}_0 & \xrightarrow{\ \rho\ } & \mathfrak{v}_0^* & \xrightarrow{\ \cong\ } & \prod(\mathbf{R}-0)^a \\ \downarrow{\scriptstyle \text{mod } \Delta} & & \downarrow{\scriptstyle \text{mod } \mathcal{U}_F^+} & & \downarrow{\scriptstyle \text{mod } \mathbf{Z}^a} \\ T^*\Sigma_0 & \xrightarrow{\ \rho\ } & \mathfrak{v}_0^*/\mathcal{U}_F^+ & \xrightarrow{\ \cong\ } & \prod(\mathbf{R}-0)^a/\mathbf{Z}^a. \end{array}$$

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The subspace $\mathfrak{v}^{\perp} \times \Sigma = \rho^{-1}(0)$ is invariant.

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The entropy of the $A_a^{(1)}$ Toda flow on $\mathfrak{v}^{\perp} \times \Sigma \cap S^*\Sigma$ is

$$h_{top} = \sqrt{\frac{a+1}{2}}.$$

Topological Conjugacy

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The Flaschka transform is non-natural, it depends on a permutation σ of $a+1$ roots:

$$T^*\mathbf{S} \xrightarrow{\lambda} \mathfrak{s}^* \xleftarrow{\text{F.T.}} T^*\mathbf{R}^a$$

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Theorem

Let $\mathrm{h} : \mathcal{U}_F^+ \to \mathbf{R}$ *be the entropy functional:*

$$\mathrm{h}(u) = h_{top}(u) \qquad u \in \mathrm{Aut}(\mathbf{V}/\mathcal{O}_E).$$

If the flows φ_1 *and* φ_2 *are topologically conjugate on their unit sphere bundles, then there is* $f \in \mathrm{Aut}(\mathcal{U}_F)$ *such that*

$$\mathrm{h} \circ f = \mathrm{h}.$$

If F *is totally hyperbolic, then* f *is induced by some* $\beta \in \mathrm{Aut}(F/\mathbf{Q})$.

Continuing Questions

1. These systems have positive topological entropy, but zero Liouville entropy. Are there positive Liouville entropy integrable systems?
2. What about other *Sol*-manifolds? E.g. suspensions of non-maximal abelian groups of automorphisms?
3. How essential is total hyperbolicity?
4. Connections with magneto-hydrodynamics.