

# Integrability and smooth structures

Leo Butler

University of Edinburgh

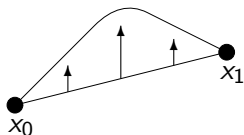
3 March 2010

- 1 Introduction
  - Hamiltonian mechanics
  - Complete Integrability
- 2 Topological Obstructions to Integrability
  - Motivating Questions
  - 2 degrees of freedom
  - 3 or more degrees of freedom
- 3 First counterexamples
- 4 Results in 3 degrees of freedom
  - Geometrisation Conjecture
  - Geometrisation and Integrability
- 5 Exotic Spheres and Topological Manifolds
  - Exotic Spheres and Tori
  - Exotic Manifolds and Integrability
- 6 Conclusions
- 7 Continuing Questions

# Hamiltonian mechanics

## Principle of Least Action

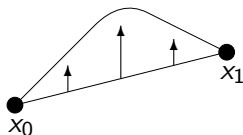
$$\int_a^b L(t, x, \dot{x}) dt \rightarrow \min$$



# Hamiltonian mechanics

Principle of Least Action

$$\int_a^b L(t, x, \dot{x}) dt \rightarrow \min$$



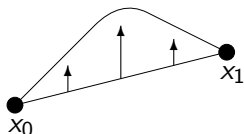
Stationary curves solve Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

# Hamiltonian mechanics

Principle of Least Action

$$\int_a^b L(t, x, \dot{x}) dt \rightarrow \min$$



Stationary curves solve Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

Equivalent to Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial x}$$

# Historical Aside: Priority

Maupertuis (1698-1759) is generally credited with the discovery of the principal of least action (1741-46): “Nature is thrifty in all its actions”

# Historical Aside: Priority

Maupertuis (1698-1759) is generally credited with the discovery of the principal of least action (1741-46): “Nature is thrifty in all its actions”  
König (1751) claimed priority for Leibniz based on a *copy* of a letter dated 1707.

# Historical Aside: Priority

Maupertuis (1698-1759) is generally credited with the discovery of the principle of least action (1741-46): “Nature is thrifty in all its actions”

König (1751) claimed priority for Leibniz based on a *copy* of a letter dated 1707.

Euler: fraud

King of Prussia: fraud

Voltaire: nonsense

# Historical Aside: Priority

Maupertuis (1698-1759) is generally credited with the discovery of the principle of least action (1741-46): “Nature is thrifty in all its actions”  
König (1751) claimed priority for Leibniz based on a *copy* of a letter dated 1707.

Euler: fraud

King of Prussia: fraud

Voltaire: nonsense

Berlin Academy (1900s) re-investigated: Leibniz

**Poisson bracket**  $\{, \}$ :

$$\{F, H\} = \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial p} - \frac{\partial H}{\partial x} \cdot \frac{\partial F}{\partial p}$$

which implies Hamilton's equations

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial p}, \quad \dot{p} = \{p, H\} = -\frac{\partial H}{\partial x}$$

on

$$T^*M = \{(x, p) : x \text{ is a point on } M, p \text{ is a momentum vector}\}.$$

# Integrability

$F$  and  $H$  Poisson commute if  $\{F, H\} \equiv 0$ .

# Integrability

$F$  and  $H$  Poisson commute if  $\{F, H\} \equiv 0$ .

## Theorem (Liouville-Arnold)

Let  $H = F_1, \dots, F_n$  be  $n$  Poisson commuting, independent functions on  $T^*M^n$ . If

$$T_c = F_1^{-1}(c_1) \cap \dots \cap F_n^{-1}(c_n)$$

is a compact, connected, non-empty, regular level set, then

# Integrability

$F$  and  $H$  Poisson commute if  $\{F, H\} \equiv 0$ .

## Theorem (Liouville-Arnold)

Let  $H = F_1, \dots, F_n$  be  $n$  Poisson commuting, independent functions on  $T^*M^n$ . If

$$T_c = F_1^{-1}(c_1) \cap \dots \cap F_n^{-1}(c_n)$$

is a compact, connected, non-empty, regular level set, then

- $T_c$  is diffeomorphic to the  $n$ -torus  $\mathbf{T}^n$  ;
- there is an open neighbourhood of  $T_c$  diffeomorphic to  $\mathbf{T}^n \times \mathbf{R}^n$ ;
- there are coordinates  $(\theta, I)$  such that
  - each  $F_j = F_j(I)$ ;
  - each vector field is linear

$$\dot{\theta} = \frac{\partial F_j(I)}{\partial I} \quad \dot{I} = 0.$$

# What accounts for these integrals?

# What accounts for these integrals?

'Symmetries'

# What accounts for these integrals?

'Symmetries'

$\implies$  Must the configuration space be 'uncomplicated'?

# What accounts for these integrals?

'Symmetries'

$\implies$  Must the configuration space be 'uncomplicated'?

Almost all manifolds have a trivial symmetry group

# What accounts for these integrals?

'Symmetries'

$\implies$  Must the configuration space be 'uncomplicated'?

Almost all manifolds have a trivial symmetry group

$\implies$  ?

## A Configuration Space:

$M$  is a compact, boundaryless,  $n$ -dimensional smooth manifold.

## A Configuration Space:

$M$  is a compact, boundaryless,  $n$ -dimensional smooth manifold.

### Definition (Convexity)

*A smooth hamiltonian  $H : T^*M \rightarrow \mathbf{R}$  is convex if  $H|_{T_x^*M}$  is a convex function for each  $x \in M$ .*

## A Configuration Space:

$M$  is a compact, boundaryless,  $n$ -dimensional smooth manifold.

### Definition (Convexity)

*A smooth hamiltonian  $H : T^*M \rightarrow \mathbf{R}$  is convex if  $H|_{T_x^*M}$  is a convex function for each  $x \in M$ .*

### Question

*Which manifolds can be the configuration space of an integrable convex hamiltonian?*

# Results

Some results are known in 2 and 3 degrees of freedom.

# Surfaces

## Theorem (Uniformisation Theorem)

*Each smooth surface admits a constant curvature metric which is locally isometric to one of*

$\mathbf{E}^2$ ,

$\mathbf{H}^2$

or  $S^2$ .

# Surfaces

## Theorem (Uniformisation Theorem)

*Each smooth surface admits a constant curvature metric which is locally isometric to one of*

$\mathbf{E}^2$ ,  $\mathbf{H}^2$  or  $S^2$ .

## Theorem (Kozlov)

*Let  $H : T^*M^2 \rightarrow \mathbf{R}$  be a convex, real-analytic function. If  $H$  enjoys an independent real-analytic integral, then  $M$  admits one of the following geometries*

$\mathbf{E}^2$ , or  $S^2$ .

# General results

## Theorem (Fomenko with Zieschang, Matveev)

*If  $H : (M^4, \omega) \rightarrow \mathbf{R}$  is integrable with a non-degenerate (or real-analytic) integral  $F$ , then*

- 1** *the regular levels of  $H$  are graph manifolds;*
- 2** *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

# General results

## Theorem (Fomenko with Zieschang, Matveev)

If  $H : (M^4, \omega) \rightarrow \mathbf{R}$  is integrable with a *non-degenerate* (or real-analytic) integral  $F$ , then

- 1 the regular levels of  $H$  are graph manifolds;
- 2 the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.

*Non-degenerate*:  $F$  is a Morse-Bott function.

# General results

## Theorem (Fomenko with Zieschang, Matveev)

If  $H : (M^4, \omega) \rightarrow \mathbf{R}$  is integrable with a *non-degenerate* (or real-analytic) integral  $F$ , then

- 1 the regular levels of  $H$  are graph manifolds;
- 2 the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.

**Non-degenerate:**  $F$  is a Morse-Bott function.

## Theorem (Paternain, Moser)

In addition,

- 3 the topological entropy vanishes.

# What about 3 or more degrees of freedom?

## Theorem (Taimanov 1988)

*Let  $H : T^*M \rightarrow \mathbf{R}$  be a convex, real-analytic hamiltonian. If  $H$  is real-analytically integrable, then*

# What about 3 or more degrees of freedom?

## Theorem (Taimanov 1988)

Let  $H : T^*M \rightarrow \mathbf{R}$  be a convex, real-analytic hamiltonian. If  $H$  is real-analytically integrable, then

- 1  $b_1(M) \leq n$ ;
- 2  $\pi_1(M)$  is almost abelian;
- 3 there is an injection  $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$ .

# What about 3 or more degrees of freedom?

## Theorem (Taimanov 1988)

Let  $H : T^*M \rightarrow \mathbf{R}$  be a convex, real-analytic hamiltonian. If  $H$  is real-analytically integrable, then

- 1  $b_1(M) \leq n$ ;
- 2  $\pi_1(M)$  is almost abelian;
- 3 there is an injection  $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$ .

V. Matveev has proven a similar theorem for projectively equivalent riemannian metrics.

# What about 3 or more degrees of freedom?

## Theorem (Taimanov 1988)

Let  $H : T^*M \rightarrow \mathbf{R}$  be a convex, real-analytic hamiltonian. If  $H$  is real-analytically integrable, then

- 1  $b_1(M) \leq n$ ;
- 2  $\pi_1(M)$  is almost abelian;
- 3 there is an injection  $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$ .

V. Matveev has proven a similar theorem for projectively equivalent riemannian metrics.

$n = 2$ : Kozlov's result.

$n = 3$ :  $M$  must be finitely-covered by one of:

a homotopy  $S^3$ ;

a homotopy  $S^2 \times S^1$ ;

a homotopy  $\mathbf{T}^3$ .

## Theorem (Paternain)

*If  $H : (M^{2n}, \omega) \rightarrow \mathbf{R}$  is completely integrable with non-degenerate integrals, then its topological entropy vanishes.*

*If  $M^{2n} = T^*N^n$  and  $H$  is convex, then  $N$  is rationally elliptic.*

## Theorem (Paternain)

*If  $H : (M^{2n}, \omega) \rightarrow \mathbf{R}$  is completely integrable with non-degenerate integrals, then its topological entropy vanishes.*

*If  $M^{2n} = T^*N^n$  and  $H$  is convex, then  $N$  is rationally elliptic.*

N.T. Zung has also shown how non-degenerate integrable systems can be classified up to topological and symplectic type.

# Bolsinov & Taimanov's Example

## Theorem

*Let  $\Sigma$  be the mapping torus of an Anosov automorphism of  $\mathbf{T}^2$ . There is a natural riemannian metric on  $\Sigma$  whose geodesic flow is completely integrable. The geodesic flow has positive topological entropy.*

# Bolsinov & Taimanov's Example

## Theorem

*Let  $\Sigma$  be the mapping torus of an Anosov automorphism of  $\mathbf{T}^2$ . There is a natural riemannian metric on  $\Sigma$  whose geodesic flow is completely integrable. The geodesic flow has positive topological entropy.*

There are two real analytic integrals, but the third is only smooth.

▶ compare

# Bolsinov & Taimanov's Example

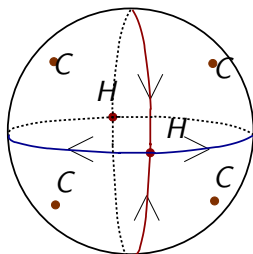
## Theorem

Let  $\Sigma$  be the mapping torus of an Anosov automorphism of  $\mathbf{T}^2$ . There is a natural riemannian metric on  $\Sigma$  whose geodesic flow is completely integrable. The geodesic flow has positive topological entropy.

There are two real analytic integrals, but the third is only smooth.

▶ compare

Generalises to mapping tori of maximal abelian torsion-free groups of toral automorphisms. The Toda lattices explain the complete integrability.



# Geometrisation Conjecture

3 model geometries in 2 dimensions:

 $E^2$  $H^2$  $S^2$

# Geometrisation Conjecture

3 model geometries in 2 dimensions:

$$\mathbf{E}^2$$

$$\mathbf{H}^2$$

$$S^2$$

8 model geometries in 3 dimensions:

$$\mathbf{E}^3$$

$$\mathbf{H}^3$$

$$S^2 \times \mathbf{E}^1$$

$$\mathbf{H}^2 \times \mathbf{E}^1$$

$$S^3$$

$$\widetilde{SL_2\mathbf{R}}$$

*Nil*

*Sol*

# Geometrisation Conjecture

3 model geometries in 2 dimensions:

$$\mathbf{E}^2$$

$$\mathbf{H}^2$$

$$S^2$$

8 model geometries in 3 dimensions:

$$\mathbf{E}^3$$

$$\mathbf{H}^3$$

$$S^2 \times \mathbf{E}^1$$

$$\mathbf{H}^2 \times \mathbf{E}^1$$

$$S^3$$

$$\widetilde{SL_2\mathbf{R}}$$

$$Nil$$

$$Sol$$

## Conjecture (Thurston)

*Each compact 3-manifold admits a canonical decomposition into complete, finite-volume geometric 3-manifolds.*

## Definition (Semisimplicity)

Let

$$\begin{array}{ccccc}
 \mathbf{T}^n \hookrightarrow & L & \xrightarrow{\text{incl.}} & T^*M & \xrightarrow{\pi} M \\
 & \downarrow f & & & \\
 & B & & & 
 \end{array}$$

be a lagrangian fibration. If  $\Gamma = T^*M - L$  is a closed, nowhere dense tamely-embedded polyhedron, then we say  $(f, L, B)$  is **semisimple**.

## Definition (Semisimplicity)

Let

$$\begin{array}{ccccc}
 \mathbf{T}^{n\mathbb{C}} & \longrightarrow & L & \xrightarrow{\text{incl.}} & T^*M & \xrightarrow{\pi} & M \\
 & & \downarrow f & & & & \\
 & & B & & & & 
 \end{array}$$

be a lagrangian fibration. If  $\Gamma = T^*M - L$  is a closed, nowhere dense tamely-embedded polyhedron, then we say  $(f, L, B)$  is **semisimple**.

### Examples.

- 1 a real-analytic singular lagrangian fibration
- 2 a smooth singular lagrangian fibration with a real-analytic singular set
- 3 the Bolsinov-Taimanov example

# Results in 3 degrees of freedom

## Theorem (B. 2005)

*Assume the GC. A compact 3-manifold admits a convex hamiltonian with a semisimple flow iff it admits one of five geometric structures:*

 $\mathbf{E}^3,$  $S^2 \times \mathbf{E}^1,$  $S^3,$  $Nil,$  $Sol.$ 

▶ compare to surfaces

# Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

# Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

Milnor-Kervaire/Eels-Kuiper proved there are exactly 28 smooth structures on  $S^7$ .

Milnor-Kervaire also showed that the smooth structures on  $S^n$  form a *finite* group for all  $n$ .

# Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

Milnor-Kervaire/Eels-Kuiper proved there are exactly 28 smooth structures on  $S^7$ .

Milnor-Kervaire also showed that the smooth structures on  $S^n$  form a *finite* group for all  $n$ .

Wall showed that the smooth structures on the topological  $n$ -torus is the set

$$\sum_{i=0}^n H^i(\mathbf{T}^n; \Gamma_i) \quad \Gamma_i = \begin{cases} \text{group of exotic } i\text{-spheres,} & i \neq 4, \\ 0 & i = 4. \end{cases}$$

# Exotic spheres and Tori

Milnor (1956) discovered 7 non-diffeomorphic smooth structures on the topological 7-sphere.

Milnor-Kervaire/Eels-Kuiper proved there are exactly 28 smooth structures on  $S^7$ .

Milnor-Kervaire also showed that the smooth structures on  $S^n$  form a *finite* group for all  $n$ .

Wall showed that the smooth structures on the topological  $n$ -torus is the set

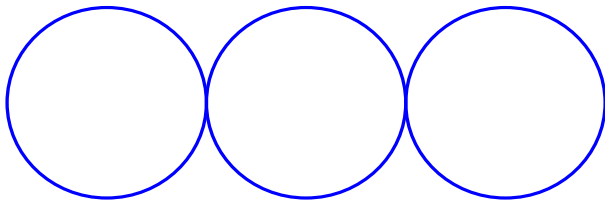
$$\sum_{i=0}^n H^i(\mathbf{T}^n; \Gamma_i) \quad \Gamma_i = \begin{cases} \text{group of exotic } i\text{-spheres,} & i \neq 4, \\ 0 & i = 4. \end{cases}$$

## Example.

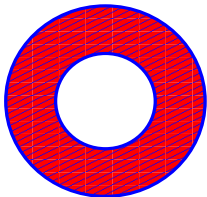
The set of smooth structures on  $\mathbf{T}^7$  is the set of 28 smooth structures on  $S^7$ . Each smooth structure is of the form:

$$M^7 = \mathbf{T}^7 \# \Sigma^7 \quad \Sigma^7 = \text{a homotopy 7-sphere.}$$

$k$ -skeleton



attaching map



# Exotic tori and integrability

## Theorem (B. 2007)

*Let  $\Sigma$  be a smooth  $n$ -dimensional topological torus. If  $T^*\Sigma$  admits a real-analytically completely integrable convex hamiltonian, then  $\Sigma$  is diffeomorphic to the standard  $n$ -torus.*

# Exotic tori and integrability

## Theorem (B. 2007)

*Let  $\Sigma$  be a smooth  $n$ -dimensional topological torus. If  $T^*\Sigma$  admits a real-analytically completely integrable convex hamiltonian, then  $\Sigma$  is diffeomorphic to the standard  $n$ -torus.*

## Remark

*Real-analyticity may be weakened to  $(f, L, B)$ -semisimplicity plus  $\pi_1(B)$  is finite.*

## Remark

*It is unknown if irreducible smooth structures must have positive entropy.*

# Exotic tori and integrability

## Theorem (B. 2007)

Let  $\Sigma$  be a smooth  $n$ -dimensional topological torus. If  $T^*\Sigma$  admits a real-analytically completely integrable convex hamiltonian, then  $\Sigma$  is diffeomorphic to the standard  $n$ -torus.

## Proof.

Sketch: real-analytic integrability implies there is a lagrangian torus  $F$  s.t.  $\rho$  is a diffeomorphism:

$$\begin{array}{ccc}
 F \subset & \xrightarrow{\text{incl.}} & T^*\Sigma \\
 & \searrow \rho & \downarrow \pi \\
 & & \Sigma.
 \end{array}$$

We show  $\deg \rho \neq 0$ , use a result of Viterbo's that  $\deg \rho \neq 0$  implies  $\rho$  is a covering map, then use a cohomology argument to show  $\rho$  is  $1 - 1$ .  $\square$

# Exotic tori and integrability

## Proof.

- 1 Taimanov:  $\exists$  a lagrangian torus  $F$  s.t.

$$\mathbf{Z}^n = \pi_1(F) \xrightarrow{\rho_*} \pi_1(\Sigma) = \mathbf{Z}^n$$

has a finite index image. Therefore  $\deg \rho \neq 0$ .

# Exotic tori and integrability

## Proof.

- 1 Taimanov:  $\exists$  a lagrangian torus  $F$  s.t.

$$\mathbf{Z}^n = \pi_1(F) \xrightarrow{\rho_*} \pi_1(\Sigma) = \mathbf{Z}^n$$

has a finite index image. Therefore  $\deg \rho \neq 0$ .

- 2 Wall: smooth structures on a topological (=PL)  $n$ -torus are classified by a class

$$\sigma \in \sum H^i(\Sigma; \Gamma_i) \quad \Gamma_i = \text{finite,}$$

which is natural with respect to covering maps.

# Exotic tori and integrability

## Proof.

- 1 Taimanov:  $\exists$  a lagrangian torus  $F$  s.t.

$$\mathbf{Z}^n = \pi_1(F) \xrightarrow{\rho_*} \pi_1(\Sigma) = \mathbf{Z}^n$$

has a finite index image. Therefore  $\deg \rho \neq 0$ .

- 2 Wall: smooth structures on a topological (=PL)  $n$ -torus are classified by a class

$$\sigma \in \sum H^i(\Sigma; \Gamma_i) \quad \Gamma_i = \text{finite,}$$

which is natural with respect to covering maps.

- 3 Pullback to a finite covering

$$\begin{array}{ccccc}
 & & \phi & & \\
 & & \curvearrowright & & \\
 F' \hookrightarrow & & & & F \\
 \downarrow \iota_{F'} = \text{incl.} & \xrightarrow{P = \text{proj.}} & T^* \Sigma' & \xrightarrow{\quad} & T^* \Sigma \xleftarrow{\iota_F = \text{incl.}} \\
 \downarrow \rho' = \pi' \cdot \iota_{F'} & & \downarrow \pi' = \text{proj.} & & \downarrow \pi = \text{proj.} \\
 & & \Sigma' & \xrightarrow{p} & \Sigma
 \end{array}$$

# Maslov cycle

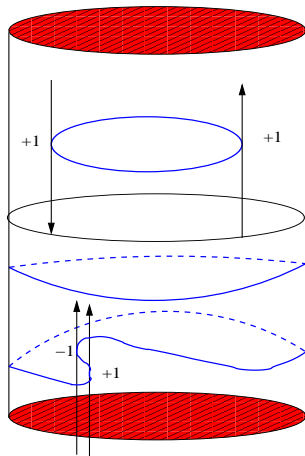


Figure: The Maslov cycle

# Exotic tori and integrability

## 4 Theorem. (Viterbo, 1991)

Let  $\Sigma'$  be diffeomorphic to the standard  $n$ -torus. If  $F'$  is a lagrangian torus, and  $\deg \rho' \neq 0$ , then the Maslov class of  $F'$  vanishes.

$$\begin{array}{ccc}
 F' \hookrightarrow & \xrightarrow{\iota_{F'} = \text{incl.}} & T^*\Sigma' \\
 & \searrow \rho' = \pi' \cdot \iota_{F'} & \downarrow \pi' = \text{proj.} \\
 & & \Sigma'
 \end{array}$$

(Bialy & Polterovich) If  $F'$  is convex, then  $\rho'$  is a covering map.

# Exotic tori and integrability

## 4 Theorem. (Viterbo, 1991)

Let  $\Sigma'$  be diffeomorphic to the standard  $n$ -torus. If  $F'$  is a lagrangian torus, and  $\deg \rho' \neq 0$ , then the Maslov class of  $F'$  vanishes.

$$\begin{array}{ccc}
 F' \hookrightarrow & \xrightarrow{\iota_{F'} = \text{incl.}} & T^*\Sigma' \\
 & \searrow \rho' = \pi' \cdot \iota_{F'} & \downarrow \pi' = \text{proj.} \\
 & & \Sigma'
 \end{array}$$

(Bialy & Polterovich) If  $F'$  is convex, then  $\rho'$  is a covering map.

5  $\implies F \xrightarrow{\rho} \Sigma$  is a smooth covering map. ▶ diagram

# The two possibilities

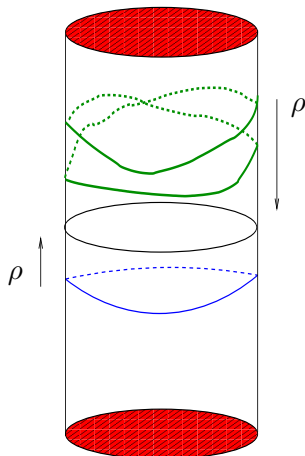


Figure: Two lagrangian tori with vanishing Maslov class

# Exotic tori and integrability

Let

$$\begin{aligned}\alpha &= \lambda|_F \\ &= \alpha_0 + \alpha_1\end{aligned}$$

$$\begin{aligned}\lambda &= \text{Liouville one-form of } T^*\Sigma \\ \alpha_0 &\in \wedge^1(F)^\Delta, \alpha_1 = dh,\end{aligned}$$

where  $\Delta = \text{Deck}(\rho)$ .

# Exotic tori and integrability

Let

$$\begin{aligned}\alpha &= \lambda|_F \\ &= \alpha_0 + \alpha_1\end{aligned}$$

$$\begin{aligned}\lambda &= \text{Liouville one-form of } T^*\Sigma \\ \alpha_0 &\in \wedge^1(F)^\Delta, \alpha_1 = dh,\end{aligned}$$

where  $\Delta = \text{Deck}(\rho)$ .

**6** For each  $\gamma \in \Delta$ ,  $x \in F$ ,

$$\begin{aligned}\alpha_x - \gamma^* \alpha_{\gamma(x)} &= dh_x^\gamma \\ &= (d_x \rho)^*(x - \gamma(x))\end{aligned}$$

$$\begin{aligned}h^\gamma &= h - h \circ \gamma, \\ &\text{from definition of } \alpha.\end{aligned}$$

# Exotic tori and integrability

Let

$$\begin{aligned}\alpha &= \lambda|_F \\ &= \alpha_0 + \alpha_1\end{aligned}$$

$$\begin{aligned}\lambda &= \text{Liouville one-form of } T^*\Sigma \\ \alpha_0 &\in \wedge^1(F)^\Delta, \alpha_1 = dh,\end{aligned}$$

where  $\Delta = \text{Deck}(\rho)$ .

**6** For each  $\gamma \in \Delta$ ,  $x \in F$ ,

$$\begin{aligned}\alpha_x - \gamma^* \alpha_{\gamma(x)} &= dh_x^\gamma & h^\gamma &= h - h \circ \gamma, \\ &= (d_x \rho)^*(x - \gamma(x)) & &\text{from definition of } \alpha.\end{aligned}$$

**7** Since  $\exists x = x_\gamma$  s.t.

$$dh_x^\gamma = 0 \quad \implies \gamma(x) = x.$$

# Exotic tori and integrability

Let

$$\begin{aligned}\alpha &= \lambda|_F \\ &= \alpha_0 + \alpha_1\end{aligned}$$

$$\begin{aligned}\lambda &= \text{Liouville one-form of } T^*\Sigma \\ \alpha_0 &\in \wedge^1(F)^\Delta, \alpha_1 = dh,\end{aligned}$$

where  $\Delta = \text{Deck}(\rho)$ .

**6** For each  $\gamma \in \Delta$ ,  $x \in F$ ,

$$\begin{aligned}\alpha_x - \gamma^* \alpha_{\gamma(x)} &= dh_x^\gamma & h^\gamma &= h - h \circ \gamma, \\ &= (d_x \rho)^*(x - \gamma(x)) & &\text{from definition of } \alpha.\end{aligned}$$

**7** Since  $\exists x = x_\gamma$  s.t.

$$dh_x^\gamma = 0 \quad \implies \gamma(x) = x.$$

**8** Therefore  $\Delta = 1$  and  $\rho$  is a diffeomorphism.

# Witten-Kreck-Stolz spaces

Let  $k, l \neq 0$  be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} : |z| = 1 \right\} \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \quad \text{Witten-Kreck-Stolz space.}$$

# Witten-Kreck-Stolz spaces

Let  $k, l \neq 0$  be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} : |z| = 1 \right\} \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \quad \text{Witten-Kreck-Stolz space.}$$

## Theorem (Kreck-Stolz)

*There are 28 smooth structures on  $M_{1,4}$ . These structures are represented by  $M_{32t+1,4}$  for  $t = 0, \dots, 27 \pmod{28}$ .*

# Witten-Kreck-Stolz spaces

Let  $k, l \neq 0$  be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} : |z| = 1 \right\} \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \quad \text{Witten-Kreck-Stolz space.}$$

## Theorem (Kreck-Stolz)

*There are 28 smooth structures on  $M_{1,4}$ . These structures are represented by  $M_{32t+1,4}$  for  $t = 0, \dots, 27 \pmod{28}$ .*

## Theorem (B. 2007)

*The riemannian metric on  $M_{k,l}$  induced by the round metrics on  $S^3 \times S^5$  is real-analytically completely integrable.*

# Eschenburg-Krugger Spaces

Let  $k, l \in \mathbf{Z}^3$  with  $\sum k_i = \sum l_i$ . Let

$$U_{kl} = \left\{ \begin{bmatrix} z^{k_0} & & \\ & z^{k_1} & \\ & & z^{k_2} \end{bmatrix} \oplus \begin{bmatrix} z^{l_0} & & \\ & z^{l_1} & \\ & & z^{l_2} \end{bmatrix} : |z| = 1 \right\} \subset U_3 \times U_3.$$

This group acts on  $SU_3$  by

$$u \cdot g = u_1 g u_2^{-1}.$$

The action is free if  $k - \sigma(l)$  is a primitive vector in  $\mathbf{Z}^3$  for all permutations  $\sigma$ .

$$M_{k,l} = SU_3 / U_{kl}$$

Eschenburg space.

# Condition C

Let  $k, l \in \mathbf{Z}^3$ . Define

$$A = \begin{bmatrix} k_0 - l_0 & k_0 - l_1 & k_0 - l_2 \\ k_1 - l_0 & k_1 - l_1 & k_1 - l_2 \\ k_2 - l_0 & k_2 - l_1 & k_2 - l_2 \end{bmatrix}$$

# Condition C

Let  $k, l \in \mathbf{Z}^3$ . Define

$$A = \begin{bmatrix} k_0 - l_0 & k_0 - l_1 & k_0 - l_2 \\ k_1 - l_0 & k_1 - l_1 & k_1 - l_2 \\ k_2 - l_0 & k_2 - l_1 & k_2 - l_2 \end{bmatrix}$$

## Definition

$(k, l)$  satisfy condition C if  $A$  has at least one column or row with pairwise coprime, non-zero entries.

# Eschenburg-Kruggel Spaces

## Theorem (Kruggel)

Let  $(k, l), (k', l')$  both satisfy condition C.  $M_{k,l}$  is orientation-preserving homeomorphic to  $M_{k',l'}$  iff

- 1 the ranks of  $H^4$  are equal, the linking form is isometric, and the first Pontryagin classes are equal; and
- 2 the second Kreck-Stolz invariants  $s_2 \in \mathbf{Q}/\mathbf{Z}$  are equal.

$M_{k,l}$  is orientation-preserving diffeomorphic to  $M_{k',l'}$  iff, in addition,

- 3 the first Kreck-Stolz invariants  $s_1 \in \mathbf{Q}/\mathbf{Z}$  are equal.

# Eschenburg-Kruggel Spaces

## Theorem (Kruggel)

Let  $(k, l), (k', l')$  both satisfy condition C.  $M_{k,l}$  is orientation-preserving homeomorphic to  $M_{k',l'}$  iff

- 1 the ranks of  $H^4$  are equal, the linking form is isometric, and the first Pontryagin classes are equal; and
- 2 the second Kreck-Stolz invariants  $s_2 \in \mathbf{Q}/\mathbf{Z}$  are equal.

$M_{k,l}$  is orientation-preserving diffeomorphic to  $M_{k',l'}$  iff, in addition,

- 3 the first Kreck-Stolz invariants  $s_1 \in \mathbf{Q}/\mathbf{Z}$  are equal.

## Theorem (Paternain-Spatzier 1994, Bazaikin 2000, B. 2007)

Every Eschenburg space admits a real-analytically completely integrable geodesic flow. The topological Eschenburg-Kruggel space with  $k = (-1, -1, 2)$  and  $l = (-2, 0, 2)$  admits 28 distinct smooth structures each represented by an Eschenburg-Kruggel space.

28 smooth structures on  $E_{-1,-1;-2,0}$ 

$k_0$	$k_1$	$l_0$	$l_1$	$s_1 \text{ mod } 1$	$k_0$	$k_1$	$l_0$	$l_1$	$s_1 \text{ mod } 1$
-29	10	-28	6	1	-21	-6	-18	-10	0.5
-38	-29	-66	22	0.96428	-5	-5	-6	-4	0.46428
-54	9	-52	4	0.92857	-13	2	-8	-6	0.42857
-17	-17	-18	-16	0.89285	-14	-5	-16	-2	0.39285
-6	-3	-8	0	0.85714	-9	-6	-12	-2	0.35714
-17	-14	-22	-8	0.82142	-38	-11	-48	8	0.32142
-14	-5	-18	2	0.78571	-22	-19	-40	12	0.28571
-1	-1	-2	0	0.75	-22	-1	-14	-12	0.25
-33	-6	-42	20	0.71428	-25	-1	-22	-6	0.21428
-46	-13	-32	-30	0.67857	-54	-9	-68	30	0.17857
-22	5	-20	0	0.64285	-39	-6	-32	-16	0.14285
-13	2	-14	6	0.60714	-29	-14	-32	-10	0.10714
-38	-11	-40	-8	0.57142	-11	1	-12	4	0.07142
-22	-1	-26	12	0.53571	-9	-9	-10	-8	0.03571

Figure: Representative Eschenburg spaces in the homeomorphism class  $r = 1$ ,  $s = 0$ ,  $p_1 = 0$  and  $s_2 = 0.25 \text{ mod } 1$ .

# Smooth structures and integrability

The existence of a completely integrable convex hamiltonian is an invariant of the configuration space's smooth structure.

This work has used symplectic topology to establish this result.

# Questions

- 1 Are there completely integrable geodesic flows on exotic tori? Must these have positive entropy?
- 2 Is it possible to give a description of the smooth invariant which determines whether or not a topological manifold admits a semisimple geodesic flow?
- 3 Must a real-analytically integrable system in dimensions 6 and more have zero entropy?