

Exotic tori and complete integrability

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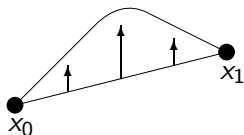
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Hamiltonian mechanics

Principle of Least Action

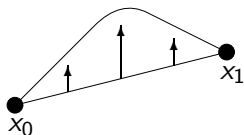
$$\int_a^b L(t, x, \dot{x}) dt \rightarrow \min$$



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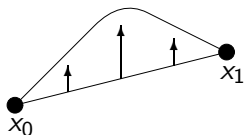
Stationary curves solve Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

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Equivalent to Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial x}$$

Poisson bracket $\{, \}$:

$$\{F, H\} = \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial p} - \frac{\partial H}{\partial x} \cdot \frac{\partial F}{\partial p}$$

which implies Hamilton's equations

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial p}, \quad \dot{p} = \{p, H\} = -\frac{\partial H}{\partial x}$$

on

$$T^*M = \{(x, p) : x \text{ is a point on } M, p \text{ is a momentum vector}\}.$$

Integrability

F and H Poisson commute if $\{F, H\} \equiv 0$.

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Theorem (Liouville-Arnold-Mineur)

Let $H = F_1, \dots, F_n$ be n Poisson commuting, independent functions on T^*M^n . If

$$T_c = F_1^{-1}(c_1) \cap \dots \cap F_n^{-1}(c_n)$$

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is a compact, connected, non-empty, regular level set, then

- T_c is diffeomorphic to the n -torus \mathbf{T}^n ;
- there is an open neighbourhood of T_c diffeomorphic to $\mathbf{T}^n \times \mathbf{R}^n$;
- there are coordinates (θ, I) such that
 - each $F_j = F_j(I)$;
 - each vector field is linear

$$\dot{\theta} = \frac{\partial F_j(I)}{\partial I} \quad \dot{I} = 0.$$

What accounts for these integrals?

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Must the dynamics be 'uncomplicated'?

M is a compact, boundaryless, n -dimensional smooth manifold.

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Definition (Convexity)

*A smooth hamiltonian $H : T^*M \rightarrow \mathbf{R}$ is convex if $H|_{T_x^*M}$ is a convex function for each $x \in M$.*

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Question

Which manifolds can be the configuration space of an integrable convex hamiltonian?

Results

Some results are known in 2 and 3 degrees of freedom.

Surfaces

Theorem (Uniformisation Theorem)

Each smooth surface admits a constant curvature metric which is locally isometric to one of

\mathbf{E}^2 ,

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Theorem (Kozlov)

*Let $H : T^*M^2 \rightarrow \mathbf{R}$ be a convex, real-analytic function. If H enjoys an independent real-analytic integral, then M admits one of the following geometries*

\mathbf{E}^2 , or S^2 .

General results

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \rightarrow \mathbf{R}$ is integrable with a non-degenerate (or real-analytic) integral F , then

- 1** *the regular levels of H are graph manifolds;*
- 2** *the first Betti number of a regular level is determined by the number of elliptic and hyperbolic periodic orbits.*

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Theorem (Paternain, Moser)

In addition,

- 3 the topological entropy vanishes.

What about 3 or more degrees of freedom?

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- 3 there is an injection $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

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$n = 2$: Kozlov's result.

$n = 3$: M must be finitely-covered by one of:

a homotopy S^3 ;

a homotopy $S^2 \times S^1$;

a homotopy \mathbf{T}^3 .

Theorem (Paternain)

If $H : (M^{2n}, \omega) \rightarrow \mathbf{R}$ is completely integrable with non-degenerate integrals, then its topological entropy vanishes.

1

- 2 N.T. Zung has also shown how non-degenerate integrable systems can be classified up to topological and symplectic type.

Bolsinov & Taimanov's Example

Theorem

Let Σ be the mapping torus of an Anosov automorphism of \mathbf{T}^2 . There is a natural riemannian metric on Σ whose geodesic flow is completely integrable. The geodesic flow has positive topological entropy.

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Generalises to mapping tori of maximal abelian torsion-free groups of toral automorphisms. The Toda lattices explain the complete integrability.

Geometrisation Conjecture

3 model geometries in 2 dimensions:

 E^2 S^2 H^2

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$$\mathbf{E}^2$$

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8 model geometries in 3 dimensions:

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Conjecture (Thurston)

Each compact 3-manifold admits a canonical decomposition into complete, finite-volume geometric 3-manifolds.

Definition (Semisimplicity)

Let

$$\begin{array}{ccccc}
 \mathbf{T}^{n\mathbb{C}} & \longrightarrow & L & \xrightarrow{\text{incl.}} & T^*M & \xrightarrow{\pi} & M \\
 & & \downarrow f & & & & \\
 & & B & & & &
 \end{array}$$

be a lagrangian fibration. If $\Gamma = T^*M - L$ is a closed, nowhere dense tamely-embedded polyhedron, then we say (f, L, B) is **semisimple**.

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Examples.

- 1 a real-analytic singular lagrangian fibration
- 2 a smooth singular lagrangian fibration with a real-analytic singular set
- 3 the Bolsinov-Taimanov example

Results in 3 degrees of freedom

Theorem (B. 2005)

Assume the GC. A compact 3-manifold admits a convex hamiltonian with a semisimple flow iff it admits one of five geometric structures:

 $\mathbf{E}^3,$ $S^2 \times \mathbf{E}^1,$ $S^3,$ $Nil,$ $Sol.$

▶ compare to surfaces

Exotic spheres and Tori

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Wall showed that the smooth structures on the topological n -torus is the set

$$\sum_{i=0}^n H^i(\mathbf{T}^n; \Gamma_i) \quad \Gamma_i = \begin{cases} \text{group of exotic } i\text{-spheres,} & i \neq 4, \\ 0 & i = 4. \end{cases}$$

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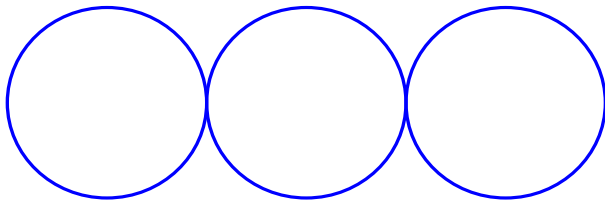
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Example.

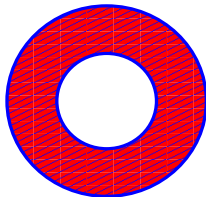
The set of smooth structures on \mathbf{T}^7 is the set of 28 smooth structures on S^7 . Each smooth structure is of the form:

$$M^7 = \mathbf{T}^7 \# \Sigma^7 \quad \Sigma^7 = \text{a homotopy 7-sphere.}$$

k-skeleton



attaching map



Exotic tori and integrability

Theorem (B. 2007)

Let Σ be a smooth n -dimensional topological torus. If T^Σ admits a real-analytically completely integrable convex hamiltonian, then Σ is diffeomorphic to the standard n -torus.*

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Remark

Real-analyticity may be weakened to (f, L, B) -semisimplicity plus $\pi_1(B)$ is finite.

Exotic tori and integrability

Theorem (B. 2007)

Let Σ be a smooth n -dimensional topological torus. If $T^*\Sigma$ admits a real-analytically completely integrable convex hamiltonian, then Σ is diffeomorphic to the standard n -torus.

Proof.

Sketch: real-analytic integrability implies there is a lagrangian torus F s.t. ρ is a diffeomorphism:

$$\begin{array}{ccc}
 F \subset & \xrightarrow{\text{incl.}} & T^*\Sigma \\
 & \searrow \rho & \downarrow \pi \\
 & & \Sigma.
 \end{array}$$

We show $\deg \rho \neq 0$, use a result of Viterbo's that $\deg \rho \neq 0$ implies ρ is a covering map, then use a cohomology argument to show ρ is 1 - 1. \square

Exotic tori and integrability

Proof.

- 1 Taimanov: \exists a lagrangian torus F s.t.

$$\mathbf{Z}^n = \pi_1(F) \xrightarrow{\rho_*} \pi_1(\Sigma) = \mathbf{Z}^n$$

has a finite index image. Therefore $\deg \rho \neq 0$.

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- 2 Wall: smooth structures on a topological (=PL) n -torus are classified by a class

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- 3 Pullback to a finite covering

$$\begin{array}{ccccc}
 F' & \hookrightarrow & T^*\Sigma' & \xrightarrow{P=\text{proj.}} & T^*\Sigma & \hookrightarrow & F \\
 \downarrow \iota_{F'}=\text{incl.} & & \downarrow \pi'=\text{proj.} & & \downarrow \pi=\text{proj.} & & \downarrow \iota_F=\text{incl.} \\
 & & \Sigma' & \xrightarrow{p} & \Sigma & & \\
 \rho'=\pi' \cdot \iota_{F'} & & & & & & \rho=\pi \cdot \iota_F
 \end{array}$$

ϕ (curved arrow from F' to F)

Maslov cycle

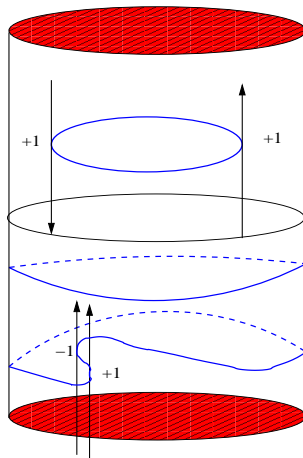


Figure: The Maslov cycle

Exotic tori and integrability

4 Theorem. (Viterbo, 1991)

Let Σ' be diffeomorphic to the standard n -torus. If F' is a lagrangian torus, and $\deg \rho' \neq 0$, then the Maslov class of F' vanishes.

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 F' \subset & \xrightarrow{\quad} & T^*\Sigma' \\
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(Bialy & Polterovich) If F' is convex, then ρ' is a covering map.

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5 $\implies F \xrightarrow{\rho} \Sigma$ is a smooth covering map. ▶ diagram

The two possibilities

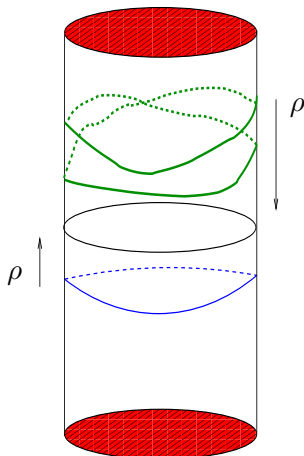


Figure: Two lagrangian tori with vanishing Maslov class

Exotic tori and integrability

Let

$$\begin{aligned}\alpha &= \lambda|_F \\ &= \alpha_0 + \alpha_1\end{aligned}$$

$$\begin{aligned}\lambda &= \text{Liouville one-form of } T^*\Sigma \\ \alpha_0 &\in \wedge^1(F)^\Delta, \alpha_1 = dh,\end{aligned}$$

where $\Delta = \text{Deck}(\rho)$.

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$$\begin{aligned}\alpha_x - \gamma^* \alpha_{\gamma(x)} &= dh_x^\gamma \\ &= (d_x \rho)^*(x - \gamma(x))\end{aligned}$$

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8 Therefore $\Delta = 1$ and ρ is a diffeomorphism.

Witten-Kreck-Stolz spaces

Let $k, l \neq 0$ be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} : |z| = 1 \right\} \subset U_2 \times U_3.$$

Let

$$M_{k,l} = S^3 \times S^5 / U \quad \text{Witten-Kreck-Stolz space.}$$

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Theorem (Kreck-Stolz)

There are 28 smooth structures on $M_{1,4}$. These structures are represented by $M_{32t+1,4}$ for $t = 0, \dots, 27 \pmod{28}$.

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Theorem (B. 2007)

The riemannian metric on $M_{k,l}$ induced by the round metrics on $S^3 \times S^5$ is real-analytically completely integrable.

Eschenburg-Krugger Spaces

Let $k, l \in \mathbf{Z}^3$ with $\sum k_i = \sum l_i$. Let

$$U_{kl} = \left\{ \begin{bmatrix} z^{k_0} & & \\ & z^{k_1} & \\ & & z^{k_2} \end{bmatrix} \oplus \begin{bmatrix} z^{l_0} & & \\ & z^{l_1} & \\ & & z^{l_2} \end{bmatrix} : |z| = 1 \right\} \subset U_3 \times U_3.$$

This group acts on SU_3 by

$$u \cdot g = u_1 g u_2^{-1}.$$

The action is free if $k - \sigma(l)$ is a primitive vector in \mathbf{Z}^3 for all permutations σ .

$$M_{k,l} = SU_3 / U_{kl}$$

Eschenburg space.

Condition C

Let $k, l \in \mathbf{Z}^3$. Define

$$A = \begin{bmatrix} k_0 - l_0 & k_0 - l_1 & k_0 - l_2 \\ k_1 - l_0 & k_1 - l_1 & k_1 - l_2 \\ k_2 - l_0 & k_2 - l_1 & k_2 - l_2 \end{bmatrix}$$

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Definition

(k, l) satisfy condition C if A has at least one column or row with pairwise coprime, non-zero entries.

Eschenburg-Krugger Spaces

Theorem (Kruggel)

Let $(k, l), (k', l')$ both satisfy condition C. $M_{k,l}$ is orientation-preserving homeomorphic to $M_{k',l'}$ iff

- 1 the ranks of H^4 are equal, the linking form is isometric, and the first Pontryagin classes are equal; and
- 2 the second Kreck-Stolz invariants $s_2 \in \mathbf{Q}/\mathbf{Z}$ are equal.

$M_{k,l}$ is orientation-preserving diffeomorphic to $M_{k',l'}$ iff, in addition,

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- 3 the first Kreck-Stolz invariants $s_1 \in \mathbf{Q}/\mathbf{Z}$ are equal.

Theorem (Paternain-Spatzier 1994, Bazaikin 2000, B. 2007)

Every Eschenburg space admits a real-analytically completely integrable geodesic flow. The topological Eschenburg-Kruggel space with $k = (-1, -1, 2)$ and $l = (-2, 0, 2)$ admits 28 distinct smooth structures each represented by an Eschenburg-Kruggel space.

28 smooth structures on $E_{-1,-1;-2,0}$

k_0	k_1	l_0	l_1	$s_1 \bmod 1$	k_0	k_1	l_0	l_1	$s_1 \bmod 1$
-29	10	-28	6	1	-21	-6	-18	-10	0.5
-38	-29	-66	22	0.96428	-5	-5	-6	-4	0.46428
-54	9	-52	4	0.92857	-13	2	-8	-6	0.42857
-17	-17	-18	-16	0.89285	-14	-5	-16	-2	0.39285
-6	-3	-8	0	0.85714	-9	-6	-12	-2	0.35714
-17	-14	-22	-8	0.82142	-38	-11	-48	8	0.32142
-14	-5	-18	2	0.78571	-22	-19	-40	12	0.28571
-1	-1	-2	0	0.75	-22	-1	-14	-12	0.25
-33	-6	-42	20	0.71428	-25	-1	-22	-6	0.21428
-46	-13	-32	-30	0.67857	-54	-9	-68	30	0.17857
-22	5	-20	0	0.64285	-39	-6	-32	-16	0.14285
-13	2	-14	6	0.60714	-29	-14	-32	-10	0.10714
-38	-11	-40	-8	0.57142	-11	1	-12	4	0.07142
-22	-1	-26	12	0.53571	-9	-9	-10	-8	0.03571

Figure: Representative Eschenburg spaces in the homeomorphism class $r = 1$, $s = 0$, $p_1 = 0$ and $s_2 = 0.25 \bmod 1$.

Questions

- 1 Are there completely integrable geodesic flows on exotic tori? Must these have positive entropy?
- 2 Is it possible to give a description of the smooth invariant which determines whether or not a topological manifold admits a semisimple geodesic flow?
- 3 Must a real-analytically integrable system in dimensions 6 and more have zero entropy?