A WEAK LIOUVILLE-ARNOL′D THEOREM

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Abstract. This paper studies properties of Tonelli Hamiltonian systems that possess n independent but not necessarily involutive constants of motion. We obtain results reminiscent of the Liouville-Arnol′d theorem under a suitable hypothesis on the regular set of these constants of motion.

1. Introduction

In the study of Hamiltonian systems, a special role is played by *integrable systems*. These systems appear naturally in geometry and physics, where they frequently have a variational character. Sometimes they are identified by the possibility of writing their solutions explicitly, *i.e.* as *exactly solvable* models. For the purposes of this note, an integrable system is a Hamiltonian system that is *completely integrable* if it satisfies the hypotheses of the Liouville-Arnol′d theorem (below). This theorem states that an integrable system is tangent to a singular foliation, whose regular leaves are Lagrangian tori and on which the system is conjugate to a rigid rotation.

Let us explain this in a more precise way. The cotangent bundle, T^*M, of a smooth manifold M is equipped with a canonical Poisson structure $\{\cdot,\cdot\}$ that makes the algebra of smooth functions on T^*M into a Lie algebra of derivations, *i.e.* a Lie algebra of smooth vector fields. Given a smooth function H, the vector field $X_H = \{H,\}$ is a *Hamiltonian system* with Hamiltonian H. The skew-symmetry of $\{\cdot,\cdot\}$ implies that if $\{H,F\} \equiv 0$, then the vector field X_H is tangent to the level sets of F and it commutes with X_F. In such a situation, these Hamiltonians are said to *Poisson-commute*, or be *in involution*, and F is said to be a *constant of motion*, or *first integral*. The Liouville-Arnol′d theorem, in the more general setting of symplectic manifolds, is

Theorem (Liouville-Arnol′d). *Let (V,ω) be a symplectic manifold with $\dim V = 2n$ and let $H : V \longrightarrow \mathbb{R}$ be a proper Hamiltonian. Suppose that there exists n integrals of motion $F_1,\ldots,F_n : V \longrightarrow \mathbb{R}$ such that:*

i) *$F_1, \ldots, F_n$ are independent almost everywhere on V, i.e. their differentials $dF_1,\ldots,dF_n$ are linearly independent as vectors;*
ii) *$F_1,\ldots,F_n$ are pairwise in involution, i.e. $\{F_i,F_j\} = 0$ for all $i,j = 1,\ldots n$.*

Suppose the non-empty regular level set $\Lambda_a := \{F_1 = a_1,\ldots,F_n = a_n\}$ is connected. Then there is a neighbourhood W of Λ_a and a symplectic system of coordinates $(I,\theta) : W \longrightarrow \mathbb{R}^n \times \mathbb{T}^n$ such that $I^{-1}(0) = \Lambda_a$ and $F_i = F_i(I)$. In particular, Λ_a is a Lagrangian torus and the Hamiltonian flow of H is conjugate to a rigid rotation on $\{0\} \times \mathbb{T}^n$.

Date: November 1, 2010.

2000 *Mathematics Subject Classification.* 37J50 (primary); 37J35, 53D12, 70H08 (secondary).

L. B. thanks K. F. Siburg for helpful discussions and the MFO for its hospitality. A. S. would like to acknowledge the support of the *Herchel-Smith foundation* and the French ANR project *Hamilton-Jacobi et théorie KAM faible*.

Remark. This theorem requires only that the integrals F_i are C^2. There are numerous proofs of this theorem in its modern formulation, see *inter alia* [23, 3, 5, 12, 20]. The map $F := (F_1, \ldots, F_n)$ is referred to as an *integral map*, *first-integral map* and a *momentum map*. The algebra generated by $F^*C^\infty(\mathbb{R}^n)$ under the Poisson bracket is an algebra of first integrals of H.

Complete integrability is a very strong assumption with significant implications for the dynamics of the system. The invariance of the level set Λ_a simply follows from F being an integral of motion; the fact that it is a Lagrangian torus and that the Hamiltonian flow is conjugate to a rigid rotation, strongly relies on these integrals being pairwise in involution and independent.

In this work, continuing the work of Sorrentino [30], we would like to address the following question:

Question I. *Without the involutivity hypothesis, what remains of the Liouville-Arnol′d theorem?*

To address this question, let us introduce the notion of a *weakly integrable* system.

1.1. **Definition *(Weak integrability).*** Let $H \in C^2(T^*M)$. If there is a C^2 map $F : T^*M^n \longrightarrow \mathbb{R}^n$ whose singular set is nowhere dense, and F Poisson-commutes with H, then we say that H is *weakly integrable*.

Remark. Both *complete* and *non-commutative integrability* imply weak integrability, but as the name suggests, weak integrability is distinctly weaker. In [8], Butler and Paternain show that many left-invariant, fibrewise quadratic Hamiltonians $H : T^*G \longrightarrow \mathbb{R}$, where G is a compact semi-simple Lie group of rank 2 or more, have positive topological entropy and are not completely integrable. However these Hamiltonians are weakly integrable: the first-integral map in this case is the momentum map $F : T^*G \longrightarrow \mathfrak{g}^*$ of the right-action of G on itself.

1.1. **Results.** Recall that a Hamiltonian $H \in C^2(T^*M)$ is *Tonelli* if it is fibrewise strictly convex and enjoys fibrewise superlinear growth. We use the variational properties of Tonelli Hamiltonians, in particular the Aubry and Mather sets (see section 2), to prove the following.

1.1. **Theorem (Weak Liouville-Arnol′d).** *Let M be a closed manifold of dimension n and $H : T^*M \longrightarrow \mathbb{R}$ a weakly integrable Tonelli Hamiltonian with integral map $F : T^*M \longrightarrow \mathbb{R}^n$. If for some cohomology class $c \in H^1(M;\mathbb{R})$ the corresponding Aubry set $\mathcal{A}^*_c \subset \operatorname{Reg} F$, then there exists an open neighborhood $\mathcal{O}$ of c in $H^1(M;\mathbb{R})$ such that the following holds.*

i) *For each $c' \in \mathcal{O}$ there exists a smooth invariant Lagrangian graph $\Lambda_{c'}$ of cohomology class c', which admits the structure of a smooth $\mathbb{T}^d$-bundle over a base B^{n-d} that is parallelisable, for some $d > 0$.*
ii) *The motion on each $\Lambda_{c'}$ is Schwartzman strictly ergodic (see [15]), i.e. all invariant probability measures have the same rotation vector and the union of their supports equals $\Lambda_{c'}$. In particular, all orbits are conjugate by a smooth diffeomorphism isotopic to the identity.*
iii) *Mather's α-function (or effective Hamiltonian) $\alpha_H : H^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is differentiable at all $c' \in \mathcal{O}$ and its convex conjugate $\beta_H : H_1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is differentiable at all rotation vectors $h \in \partial\alpha_H(\mathcal{O})$, where $\partial\alpha_H(\mathcal{O})$ denotes the set of subderivatives of α_H at some element of $\mathcal{O}$.*
iv) *If $\dim M = 2$, then M is diffeomorphic to $\mathbb{T}^2$. If $\dim M = 3$, M is diffeomorphic to either $\mathbb{T}^3$ or a non-trivial principal $\mathbb{T}^1$-bundle over $\mathbb{T}^2$.*
v) *If $\dim H^1(M;\mathbb{R}) \geq \dim M$, then $\dim H^1(M;\mathbb{R}) = \dim M$ and M is diffeomorphic to $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.*

Remark. (i) In (v) we conclude, in fact, that a neighborhood of Λ_c is foliated by invariant Lagrangian tori on which the motion is conjugate to a rotation of rotation vector $h_{c'} = \partial\alpha_H(c')$, where $\partial\alpha_H(c')$ is the derivative of α_H at c'. (ii) The theorem remains true if one replaces the hypothesis $\mathcal{A}_c^* \subset \operatorname{Reg} F$ with $\mathcal{M}_c^* \subset \operatorname{Reg} F$. (iii) We conjecture that weak integrability implies that $\dim H^1(M;\mathbb{R}) \leq \dim M$ with equality if and only if M is a torus even without the a priori assumption $\mathcal{A}_c^* \subset \operatorname{Reg} F$.

Theorem 1.1 can be sharpened. Recall that a smooth manifold is *irreducible* if, when written as a connect sum, one of the summands is a standard sphere. In 3-manifold topology, a central role is played by those closed 3-manifolds which contain a non-separating *incompressible* surface, or dually, which have non-vanishing first Betti number. Such manifolds are called *Haken*; it is an outstanding conjecture that every irreducible 3-manifold with infinite fundamental group has a finite covering that is Haken [17, Questions 1.1–1.3]. This conjecture is implied by the virtually fibred conjecture [1]. Given the proof of the geometrisation conjecture, the virtual Haken conjecture is proven for all cases but hyperbolic 3-manifolds. Thurston and Dunfield have shown there is good reason to believe the conjecture is true in this case [13].

1.1. **Corollary.** *Assume the hypotheses of Theorem 1.1. Then M is diffeomorphic to a trivial $\mathbb{T}^d$-bundle over a parallelisable base B such that all finite covering spaces of B have zero first Betti number. Therefore*

i) $\dim M = 3$ *implies that M is diffeomorphic to $\mathbb{T}^3$;*
ii) $\dim M = 4$ *implies, assuming the virtual Haken conjecture, that M is diffeomorphic to either $\mathbb{T}^4$ or $\mathbb{T}^1 \times E$, where E is an orientable 3-manifold finitely covered by S^3.*

Finally, we investigate weakly integrable Tonelli Hamiltonians that are locally homogeneous. In particular, we consider the case of *amenable* homogeneous space and see how much different the situation is from the generic case. Recall that a topological group is *amenable* if it admits a left-invariant, finitely additive, Borel probability measure. Due to the Levi decomposition, an amenable Lie group is a semi-direct product of its solvable radical and a compact subgroup. A solvable Lie group is said to be *exponential* or *type (E)* if the exponential map of the Lie algebra is surjective; we will say an amenable Lie group is of type (E) if its radical is of type (E).

1.2. **Theorem.** *Let G be a simply-connected amenable Lie group of type (E) and let $\Gamma \lhd G$ be a lattice subgroup, $M = \Gamma\backslash G$ and H be induced by a left-invariant Tonelli Hamiltonian on T^*G. If $c \in H^1(M;\mathbb{R})$, then there is a closed, bi-invariant 1-form ϕ on G such that the Mather set $\mathcal{M}_c^*(H) = \operatorname{graph}(\phi)$. If H is weakly integrable and there is a C^1 Lagrangian graph $\Lambda \subset H^{-1}(h)$ and $\Lambda \cap \operatorname{Reg} F \neq \emptyset$, then M is finitely covered by a compact reductive Lie group with a non-trivial centre.*

For the proof of this theorem we need to introduce a generalised notion of rotation vector and a novel averaging procedure (see section 4), which are likely to be of independent interest.

1.2. **Methodological remarks.** The reason why Question **I** is particularly hard to tackle and cope with, is that the involution condition is essential for any reasonable theorem *à la* Liouville to be proven. Without such an ingredient it is impossible to deduce any property of these level sets, apart from their being invariant and smooth (smoothness simply follows from the independence of the integrals of motion). Therefore, in order to deduce any further geometric, topological and dynamical property, one needs to recover the involution hypothesis or find a suitable replacement.

The main idea that we shall pursue consists in combining classical methods with the *action-minimizing methods* – generally known as *Aubry-Mather theory* – that have revealed quite powerful in the study of convex and superlinear systems. Following the ideas outlined in [30], we shall study the relationship between the existence of integrals of motion and the structure of the invariant sets obtained by *action-minimizing methods*, the *Mather, Aubry* and *Mañé sets*, and use their intrinsic Lagrangian structure to make up for the lack of involution.

To give a naïve description of the difference between our method and the classical one used to prove Liouville-Arnol′d theorem, we could say that while the latter follows an inward direction, we rather move in the outward one. More specifically, in the classical proof of Liouville theorem, what one does is restrict to a regular level set of the integral map and prove, using the involution hypothesis, that this possesses the desired properties. Contrarily, we consider the action-minimizing sets – the Mather and Aubry sets – which lie in the regular level sets of the integral map (their existence follows from Mather's theory and it is independent of the integrals of motion) and prove, using the properties of the integral map, that they must be sufficiently large, namely they must be smooth n-dimensional Lagrangian graphs. Observe that these graphs being Lagrangian translates into a local Poisson-commutation of the integrals of motion, that will be therefore deduced from the intrinsic symplectic structure of these sets and not asked a priori!

2. Action-minimizing sets and integrals of motion

In the study of weakly integrable systems, or more generally of convex and superlinear Hamiltonian systems, the main idea behind dropping the hypothesis on the involution of the integrals of motion consists in studying the relationship between the existence of integrals of motion and the structure of some invariant sets obtained by *action-minimizing methods*, which are generally called *Mather, Aubry* and *Mañé sets*.

In this section we want to provide a brief description of this theory, originally developed by John Mather, and the main properties of these sets. We refer the reader to [14, 21, 22, 19, 31] for more exhaustive presentations of this material. Roughly speaking these action-minimizing sets represent a generalization of invariant Lagrangian graphs, in the sense that, although they are not necessarily submanifolds, nor even connected, they still enjoy many similar properties. What is crucial for our study of weakly integrable systems is that these sets have an intrinsic Lagrangian structure, which implies many of their symplectic properties, including a forced *local involution* of the integrals of motion, as noticed in [30].

More specifically, we are interested in studying the existence of *action-minimizing invariant probability measures* and *action-minimizing orbits* in the following setting.

Let $H : T^*M \to \mathbb{R}$ be a C^2 Hamiltonian, which is strictly convex and uniformly superlinear in the fibres. H is called a *Tonelli Hamiltonian*. This Hamiltonian defines a vector field on T^*M, known as *Hamiltonian vector field*, that can be defined as the unique vector field X_H such that $\omega(X_H, \cdot) = dH$, where ω is the canonical symplectic form on T^*M. We call the associated flow *Hamiltonian flow* and denote it by Φ^t_H.

To any Tonelli Hamiltonian system one can also associate an equivalent dynamical system in the tangent bundle TM, called *Lagrangian system*. Let us consider the associated *Tonelli Lagrangian* $L : TM \to \mathbb{R}$, defined as $L(x, v) := \max_{p\in T^*_x M} (\langle p, v\rangle - H(x, p))$. It is possible to check that L is also strictly convex and uniformly superlinear in the fibres. In particular this Lagrangian defines a flow on TM, known as *Euler-Lagrange flow* and denoted by Φ_L^t, which can be obtained by integrating the so-called *Euler-Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,v) = \frac{\partial L}{\partial x}(x,v).$$

The Hamiltonian and Lagrangian flows are totally equivalent from a dynamical system point of view, in the sense that there exists a conjugation between the two. In other words, there exists a diffeomorphism $\mathcal{L}_L : TM \longrightarrow T^*M$, called *Legendre transform*, defined by $\mathcal{L}_L(x,v) = (x, \frac{\partial L}{\partial v}(x,v))$, such that $\Phi_H^t = \mathcal{L} \circ \Phi_L^t \circ \mathcal{L}^{-1}$.

In classical mechanics, a special role in the study of Hamiltonian dynamics is represented by invariant Lagrangian graphs, *i.e.* graphs of the form $\Lambda := \{(x, \eta(x) : x \in M\}$ that are Lagrangian (*i.e.* $\omega\big|_\Lambda \equiv 0$) and invariant under the Hamiltonian flow Φ_H^t. Recall that being a Lagrangian graph in T^*M is equivalent to say that η is a closed 1-form ([9, Section 3.2]). These graphs satisfy many interesting properties, but unfortunately they are quite rare. The theory that we are going to describe aims to provide a generalization of these graphs; namely, we shall construct several compact invariant subsets of the phase space, which are not necessarily submanifolds, but that are contained in Lipschitz Lagrangian graphs and enjoy similar interesting properties.

Let us start by recalling that the Euler-Lagrange flow Φ_L^t can be also characterised in a more variational way, introducing the so-called *Lagrangian action*. Given an absolutely continuous curve $\gamma : [a,b] \longrightarrow M$, we define its action as $A_L(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))\,dt$. It is a classical result that a curve $\gamma : [a,b] \longrightarrow M$ is a solution of the Euler-Lagrange equations if and only if it is a critical point of A_L, restricted to the set of all curves connecting $\gamma(a)$ to $\gamma(b)$ in time $b-a$. However, in general, these extrema are not minima (except if their time-length $b-a$ is very small). Whence the idea of considering minimizing objects and seeing if - whenever they exist - they enjoy special properties or possess a more distinguished structure.

Mather's approach is indeed based this idea and is concerned with the study of invariant probability measures and orbits that minimize the Lagrangian action (by action of a measure, we mean the collective average action of the orbits in its support, *i.e.* the integral of the Lagrangian against the measure). It is quite easy to prove (see [15, Lemma 3.1] and [31, Section 3]) that invariant probability measures (resp. Hamiltonian orbits) contained in an invariant Lagrangian graph Λ (actually its pull-back using $\mathcal{L}$) minimize the Lagrangian action of $L-\eta$, which we shall denote $A_{L-\eta}$, over the set $\mathfrak{M}(L)$ of all invariant probability measures for Φ_L^t (resp. over the set of all curves with the same end-points and defined for the same time interval). This idea of changing Lagrangian (which is at the same time a necessity) plays an important role as it allows one to magnify some motions rather than others. For instance, consider the case of an integrable system: one cannot expect to recover all these motions (which foliate the whole phase space) by just minimizing the same Lagrangian action! What is important to point out is that even if we modify L, because of the closedness of η we do not change the associated Euler-Lagrange flow, *i.e.* $L-\eta$ has the same Euler-Lagrange flow as L (see [21, p. 177] or [31, Lemma 4.6]). This is a crucial step in Mather's approach in [21]: consider a family of modified Tonelli Lagrangians given by $L_\eta(x,v) = L(x,v) - \langle \eta(x), v\rangle$, where η is a closed 1-form on M. These Lagrangians have the same Euler-Lagrange flow as L, but different action-minimizing orbits and measures. Moreover, these action-minimizing objects depend only on the cohomology class of η [21, Lemma p.176].

Hence, for each $c \in \mathrm{H}^1(M;\mathbb{R})$, if we choose η_c to be any smooth closed 1-form on M with cohomology class $[\eta_c] = c$, we can study action-minimizing invariant probability measures (or orbits) for $L_{\eta_c} := L - \eta_c$. In particular, this allows one to define several compact invariant subsets of TM:

- $\widetilde{\mathcal{M}}_c(L)$, the *Mather set of cohomology class* c, given by the union of the supports of all invariant probability measures that minimize the action of L_{η_c} (c-*action minimizing measure* or *Mather's measures of cohomology class* c). See [21].
- $\widetilde{\mathcal{N}}_c(L)$, the *Mañé set of cohomology class* c, given by the union of all orbits that minimize the action of L_{η_c} on the finite time interval $[a, b]$, for any $a < b$. These orbits are called c- *global minimizers* or c-*semi static curves.* [21, 22, 19].
- $\widetilde{\mathcal{A}}_c(L)$, the *Aubry set of cohomology class* c, given by the union of the so called $c-$*regular minimizers* of L_{η_c} (or c-*static curves*). These are special kind of c-global minimizers that, roughly speaking, do not only minimize the Lagrangian action to go from the starting point to the end-point, but that - up to a change of sign - also minimize the action to go backwards, *i.e.* from the end-point to the starting one. A precise definition would require a longer discussion. Since we are not using this definition in the following, we refer the interested reader to [22, 19, 31].

2.1. *Remark.* i) These sets are non-empty, compact, invariant and moreover they satisfy the following inclusions:

$$\widetilde{\mathcal{M}}_c(L) \subseteq \widetilde{\mathcal{A}}_c(L) \subseteq \widetilde{\mathcal{N}}_c(L) \subseteq TM\,.$$

ii) The most important feature of the Mather set and the Aubry set is the so-called *graph property*, namely they are contained in Lipschitz graphs over M (*Mather's graph theorem* [21, Theorem 2]). More specifically, if $\pi : TM \to M$ denotes the canonical projection along the fibres, then $\pi|\widetilde{\mathcal{A}}_c(L)$ is injective and its inverse $\big(\pi|\widetilde{\mathcal{A}}_c(L)\big)^{-1}\colon \pi\big(\widetilde{\mathcal{A}}_c(L)\big) \longrightarrow \widetilde{\mathcal{A}}_c(L)$ is Lipschitz. The same is true for the Mather set (it follows from the above inclusion). Observe that in general the Mañé set does not necessarily satisfy the graph property.

iii) As we have mentioned above, when there is an invariant Lagrangian graph Λ of cohomology class c (*i.e.* it is the graph of a closed 1-form of cohomology class c), then $\widetilde{\mathcal{N}}_c(L) = \mathcal{L}^{-1}(\Lambda)$. A priori $\widetilde{\mathcal{A}}_c(L) \subseteq \mathcal{L}_L^{-1}(\Lambda)$ and $\widetilde{\mathcal{M}}_c(L) \subseteq \mathcal{L}_L^{-1}(\Lambda)$. In particular $\widetilde{\mathcal{M}}_c(L) = \mathcal{L}_L^{-1}(\Lambda)$ if and only if the whole Lagrangian graph is the support of an invariant probability measure (*i.e.* the motion on it is recurrent).

iv) Similarly to what happens for invariant Lagrangian graphs, the energy $E(x, v) = \big\langle \frac{\partial L}{\partial v}(x, v), v\big\rangle - L(x, v)$ (*i.e.* the pull-back of the Hamiltonian to TM using the Legendre transform) is constant on these sets, *i.e.* for any $c \in H^1(M;\mathbb{R})$ the corresponding sets lie in the same energy level $\alpha_H(c)$. Moreover, Carneiro [10] proved a characterization of this energy value in terms of the minimal Lagrangian action of $L - \eta_c$. More specifically:

$$\alpha_H(c) = -\min_{\mu\in\mathfrak{M}(L)} A_{L-\eta_c}(\mu).$$

This defines a function $\alpha_H : H^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ that is generally called *Mather's* α-*function* or *effective Hamiltonian* (see also [21, p. 177]).

v) It is possible to show that Mather's α-function is convex and superlinear [21, Theorem 1]. In particular, one can consider its convex conjugate, using Fenchel duality, which is a function on the dual space $(H^1(M;\mathbb{R}))^* \simeq H_1(M;\mathbb{R})$ and is given by:

$$\begin{aligned} \beta_H : H_1(M;\mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \max_{c\in H^1(M;\mathbb{R})} \left(\langle c, h\rangle - \alpha_H(c)\right). \end{aligned}$$

This function is also convex and superlinear and is usually called *Mather's* β-*function*, or *effective Lagrangian.* It has also a meaning in terms of the minimal Lagrangian action. In fact, one can interpret elements in $H_1(M;\mathbb{R})$ as rotation vectors of invariant probability measures [21, p. 177] (or 'Schwartzman asymptotic cycles' [28]). In particular $\beta_H(h)$ represents the minimal Lagrangian action of L over the set of all invariant probability measures with rotation vector h. Observe that in this case we do not need to modify the Lagrangian, since the constraint on the rotation vector will play somehow the role of the previous modification (it is in some sense the same idea as with Lagrange multipliers and constrained extrema of a function). We refer the reader to [21, 31] for a more detailed discussion on the relation between these two different kinds of action-minimizing processes.

Using the duality between Lagrangian and Hamiltonian, via the Legendre transform introduced above, one can define the analogue of the Mather, Aubry and Mañé sets in the cotangent bundle, simply considering

$$\mathcal{M}^*_c(H) = \mathcal{L}_L\big(\widetilde{\mathcal{M}}_c(L)\big), \qquad \mathcal{A}^*_c(H) = \mathcal{L}_L\big(\widetilde{\mathcal{A}}_c(L)\big) \qquad \text{and} \qquad \mathcal{N}^*_c(H) = \mathcal{L}_L\big(\widetilde{\mathcal{N}}_c(L)\big).$$

These sets continue to satisfy the properties mentioned above, including the graph theorem. Moreover, it follows from Carneiro's result [10], that they are contained in the energy level $\{H(x,p) = \alpha_H(c)\}$. However, one could try to define these objects directly in the cotangent bundle. For any cohomology class c, let us fix a representative η_c. Observe that if $\Lambda := \{(x,\eta(x) : \ x \in M\}$ is an invariant Lagrangian graph of cohomology class c, *i.e.* $\eta = \eta_c + du$ for some $u : M \to \mathbb{R}$, then $H(x, \eta_c + du(x)) = \text{const}$. Therefore, the Lagrangian graph is a solution (and of course a subsolution) of Hamilton-Jacobi equation $H(x, \eta_c + du(x)) = k$, for some $k \in \mathbb{R}$. In general solutions of this equation, in the classical sense, do not exist. However Albert Fathi proved that it is always possible to find *weak solutions*, in the *viscosity sense*, and use them to recover the above results. This theory, that can be considered as the analytic counterpart of the variational approach discussed above, is nowadays called *weak KAM theory.* We refer the reader to [14] for a more complete and precise presentation.

It turns out that for a given cohomology class c these weak solutions can exist only in a specific energy level, that - quite surprisingly - coincides with Mather's value $\alpha_H(c)$. This is also the least energy value for which Hamilton-Jacobi equation can have subsolutions:

$$H(x, \eta_c + du(x)) \leq k \tag{1}$$

where $u \in C^1(M)$. Observe that the existence of C^1-subsolutions corresponding to $k = \alpha_H(c)$ is a non-trivial result due to Fathi and Siconolfi [16]. Moreover they proved that these subsolutions are dense in the set of Lipschitz subsolutions. We shall call these subsolutions, η_c*-critical subsolutions.* Patrick Bernard [6] improved this result proving the existence and the denseness of $C^{1,1}$ η_c-critical subsolutions, which is the best result that one can generally expect to find. The main problem in fact is represented by the Aubry set itself, that plays the role of a *non-removable intersection* (see also [25]). More specifically, for any η_c-critical subsolution u, the value of $\eta_c + d_x u$ is prescribed on $\pi(\mathcal{A}^*_c(H))$, where $\pi : T^*M \longrightarrow M$ is the canonical projection. Therefore, if the Aubry set is not sufficiently smooth (it is at least Lipschitz), then these subsolutions cannot be smoother. However, on the other hand this obstacle provides a new characterization of the Aubry set in terms of these subsolutions. Namely, if one denotes by $\mathcal{S}_{\eta_c}$ the set of $C^{1,1}$ η_c-critical subsolutions, then:

$$\mathcal{A}^*_c(H) = \bigcap_{u \in \mathcal{S}_{\eta_c}} \{(x, \eta_c + d_x u) : \ x \in M\}. \tag{2}$$

As we have already recalled, in T^*M, with the standard symplectic form, there is a 1-1 correspondence between Lagrangian graphs and closed 1-forms (see for instance [9, Section 3.2]). Therefore, we could interpret the graphs of the differentials of these critical subsolutions as Lipschitz Lagrangian graphs in T^*M. Therefore the Aubry set can be seen as the intersection of these distinguished Lagrangian graphs and it is exactly this property that provides to this set the intrinsic Lagrangian structure mentioned above and that will play a crucial role in our proof.

In [30], in fact, Sorrentino used this characterization to study the relation between the existence of integrals of motion and the size of the above action-minimizing sets. Let H be a Tonelli Hamiltonian on T^*M and let F be an integral of motion of H. If we denote by Φ_H and Φ_F the respective flows, then:

2.1. **Proposition** (see Lemma 1 in [30]). *The Mather set $\mathcal{M}^*_c(H)$ and the Aubry set $\mathcal{A}^*_c(H)$ are invariant under the action of Φ^t_F, for each $t \in \mathbb{R}$ and for each $c \in H^1(M;\mathbb{R})$.*

Moreover one can study the implications of the existence of *independent* integrals of motion, *i.e.* integrals of motion whose differentials are linearly independent, as vectors, at each point of these sets. It follows from the above proposition that this relates to the size of the Mather and Aubry sets of H. In order to make clear what we mean by the 'size' of these sets, let us introduce some notion of tangent space. We call *generalized tangent space* to $\mathcal{M}^*_c(H)$ (resp. $\mathcal{A}^*_c(H)$) at a point (x,p), the set of all vectors that are tangent to curves in $\mathcal{M}^*_c(H)$ (resp. $\mathcal{A}^*_c(H)$) at (x,p). We denote it by $T^G_{(x,p)}\mathcal{M}^*_c(H)$ (resp. $T^G_{(x,p)}\mathcal{A}^*_c(H)$) and define its *rank* to be the largest number of linearly independent vectors that it contains. Then:

2.2. **Proposition** (See Proposition 1 in [30]). *Let H be a Tonelli Hamiltonian on T^*M and suppose that there exist k independent integrals of motion on $\mathcal{M}^*_c(H)$ (resp. $\mathcal{A}^*_c(H)$). Then,* $\operatorname{rank} T^G_{(x,p)}\mathcal{M}^*_c(H) \geq k$ *(resp.* $\operatorname{rank} T^G_{(x,p)}\mathcal{A}^*_c(H) \geq k$*) at all points $(x,p) \in \mathcal{M}^*_c(H)$ (resp. $(x,p) \in \mathcal{A}^*_c(H)$).*

2.2. *Remark.* In particular, the existence of the maximum possible number of integrals of motion (*i.e.* $k = n$) implies that these sets are invariant smooth Lagrangian graphs (see [30, Lemma 2 and Lemma 3]).

However the most important peculiarity of these action-minimizing sets observed in [30], at least as far as we are concerned, is that they force the integrals of motion to Poisson-commute on them. In fact, using the characterization of the Aubry set in terms of critical subsolutions of Hamilton-Jacobi and its symplectic interpretation given above (see (2) and the subsequent comment), one can recover the involution property of the integrals of motion, at least locally.

2.3. **Proposition** (See Proposition 2 in [30]). *Let H be a Tonelli Hamiltonian on T^*M and let F_1 and F_2 be two integrals of motion. Then for each $c \in H^1(M;\mathbb{R})$ we have that $\{F_1, F_2\}(x, \hat{\pi}_c^{-1}(x)) = 0$ for all $x \in \overline{\operatorname{Int}(\mathcal{A}_c(H))}$, where $\hat{\pi}_c = \pi|\mathcal{A}^*_c(H)$ and $\mathcal{A}_c(H) = \pi(\mathcal{A}^*_c(H))$.*

2.3. *Remark.* Observe that the above set $\overline{\operatorname{Int}(\mathcal{A}_c(H))}$ may be empty. What the proposition says is that whenever it is non-empty, the integrals of motion are forced to Poisson-commute on it. In the cases that we shall be considering hereafter, $\mathcal{A}_c(H) = M$ and therefore it is not empty.

3. Proof of Theorem 1.1

3.1. **Proposition.** *Let $\Lambda \subset H^{-1}(h)$ be a C^1 Lagrangian graph. If H is a weakly integrable Tonelli Hamiltonian and $\Lambda \subset \operatorname{Reg} F$, then M admits the structure of a smooth $\mathbb{T}^d$-bundle over a parallelisable base B^{n-d} for some $d > 0$.*

Proof (Proposition 3.1). Since Λ is a C^1 Lagrangian graph that lies in an energy surface of H, Λ is the graph of a C^1 closed 1-form λ with cohomology class c. It follows that λ solves the Hamilton-Jacobi equation and from (2) that $\mathcal{A}^*_c(H) \subseteq \Lambda$ (see also [30, Section 3]). Moreover, Proposition 2.2 and Remark 2.2 allow us to conclude that $\mathcal{A}^*_c(H) = \Lambda$. Therefore, Proposition 2.1 implies that each vector field $X_{F_i}, i = 1, \ldots, n$ is tangent to Λ. Let $Y = X_H|\Lambda$ and $Y_i = X_{F_i}|\Lambda$. Since $\Lambda \subset \operatorname{Reg} F$, $\{Y_i\}$ is a framing of $T\Lambda$.

Let ϕ^i (resp. ϕ) be the flow of Y_i (resp. Y). Let Γ be the group of diffeomorphisms generated by the flows ϕ^i and ϕ. The Stefan-Sussman orbit theorem implies that Λ is the orbit of Γ: $\Lambda = \left\{ \prod_{j=1}^m \phi^{i_j}_{t_j}(p) \ : \ t_j \in \mathbb{R}, m \in \mathbb{N} \right\}$ for any $p \in \Lambda$ [34, 33, 32]. Since H Poisson-commutes with each of the F_i, the vector field Y commutes with Y_i for all i. Therefore, the flow ϕ of Y commutes with each ϕ^i, *i.e.* ϕ lies in the centre Z of Γ.

Let $p \in \Lambda$ be a given point and $q \in \Lambda$ a second point. Let $\Phi = \prod_{j=1}^m \phi^{i_j}_{t_j}$ be an element in Γ satisfying $\Phi(p) = q$. If φ_t is a 1-parameter subgroup of Z, then $\varphi_t(q) = \Phi(\varphi_t(p))$ for all $t \in \mathbb{R}$. Therefore, each orbit of φ is conjugate by a smooth conjugacy isotopic to the identity. We have seen that $\phi_t \in Z$ for all t, and the above shows that each orbit of ϕ_t (indeed, of Z) is conjugate.

Define a smooth Riemannian metric g on Λ by defining $\{Y_i\}$ to be an orthonormal framing of $T\Lambda$. Then, we see that each element in Z preserves g. Therefore Z is a group of isometries of a compact Riemannian manifold. The closure of Z in the group of C^1 diffeomorphisms of Λ, $\bar{Z}$, is therefore a compact connected abelian Lie group by the Montgomery-Zippin theorem [24]. Therefore, $\bar{Z}$ is a d-dimensional torus for some $d > 0$ (since it contains the 1-parameter group ϕ_t).

Since Z centralises Γ, so does its closure $\bar{Z}$. Therefore, each orbit of $\bar{Z}$ is conjugate. It follows that $\bar{Z}$ acts freely on Λ. This gives Λ the structure of a principal $\mathbb{T}^d$-bundle.

Finally, let $p \in \Lambda$ be given. Possibly after a linear change of basis, we can suppose that Y_i, $i = 1, \ldots, d$, is a basis of the tangent space to the $\mathbb{T}^d$-orbit through p, and Y_i, $i = d+1, \ldots, n$ is a basis of the orthogonal complement. Therefore, Y_i, $i = d+1, \ldots, n$ is a basis of the orthogonal complement to the fibre at all points on Λ. Since each vector field Y_i is $\mathbb{T}^d$-invariant, it descends to $B = \Lambda/\mathbb{T}^d$. Therefore, the vector fields on B induced by Y_i, $i = d+1, \ldots, n$ frame TB. □

3.1. *Remark.* A few remarks are in order. First, there is a $\xi \in \mathfrak{t} = \operatorname{Lie} \mathbb{T}^d$ such that $\exp(t\xi) \cdot p = \phi_t(p)$ for all $t \in \mathbb{R}$ and $p \in \Lambda$. This follows from the fact that $\{\phi_t\} \subset \bar{Z}$ is a 1-parameter subgroup. Therefore, there is a torus T of dimension $c \leq d$ which is the closure of $\{\exp(t\xi)\}$ in $\mathbb{T}^d$ such that each orbit closure of ϕ is the orbit of T. Second, for almost all constants $(\alpha_i) \in \mathbb{R}^d$, the vector field $Y_\alpha = Y + \sum_{i=1}^d \alpha_i Y_i$ will have dense orbits in each $\mathbb{T}^d$ orbit. In particular, by means of bump functions $\alpha_i = \alpha_i(F)$, we can perturb H in a neighbourhood of Λ to a Tonelli Hamiltonian H_α that is weakly integrable with the same integrals F but $Y_\alpha = X_{H_\alpha}|\Lambda$ is in general position. Third, since each orbit of ϕ is conjugate by a diffeomorphism isotopic to 1, the asymptotic homology of Λ is unique (see [15, Proposition A.1]). Finally, if, as in Theorem 1.1, one has an upper semicontinuous family of such Lagrangian graphs $\Lambda_{c'}$, then the dimension d' of the torus is an upper semicontinuous function of c'.

Proof (Theorem 1.1). Since $\mathcal{A}^*_c$ is contained in the set of regular points of F, it follows from Proposition 2.2 and Remark 2.2 that the Aubry set $\mathcal{A}^*_c$ is a C^1 invariant Lagrangian graph Λ_c of cohomology class c and that it coincides with the Mather set $\mathcal{M}^*_c$ (see also [30, Lemmas 2 & 3]). Therefore, Λ_c supports an invariant probability measure of full support. In particular, since all c-critical subsolutions of the

Hamilton-Jacobi equation (1), with $k = \alpha_H(c)$, have the same differential on the (projected) Aubry set [14, Theorem 4.11.5], it follows that, up to constants, there exists a unique c-critical subsolution, which is indeed a solution. It follows then that the Mañé set $\mathcal{N}^*_c = \mathcal{A}^*_c$ (see [14, Definition 5.2.5]). We can use the upper semicontinuity of the Mañé set (see for instance [2, Proposition 13]) to deduce that the Mañé set corresponding to nearby cohomology classes must also lie in $\operatorname{Reg} F$ (note in fact that in general the Aubry set is not upper semicontinuous [7]). Hence, there exists an open neighborhood $\mathcal{O}$ of c in $H^1(M;\mathbb{R})$ such that $\mathcal{A}^*_{c'} \subseteq \mathcal{N}^*_{c'} \subset \operatorname{Reg} F$ for all $c' \in \mathcal{O}$ and applying the same argument as above, we can conclude that each $\mathcal{A}^*_{c'}$ is a smooth invariant Lagrangian graph of cohomology class c' and that it coincides with the Mather set $\mathcal{M}^*_{c'}$.

At this point (i) and (ii) follow from Proposition 3.1 and Remark 3.1.

The proof of (iii) is the same as in [30, Corollary 4], but in this case we also know that these graphs are Schwartzman uniquely ergodic, *i.e.* all invariant probability measures on $\Lambda_{c'}$ have the same rotation vector $h_{c'} \in H_1(M;\mathbb{R})$ (see Remark 3.1). The differentiability of α_H follows then from [15, Corollary 3.6]. The differentiability of β_H follows the disjointness of these graphs (see for instance [15, Theorem 3.3] or [31, Remark 4.26 (ii)]).

Let us now prove (iv). If $\dim M = 2$, then it follows from (i) that M is orientable and has genus 0, therefore it must be $\mathbb{T}^2$. If $\dim M = 3$, we have several cases: $(d = 1)$ we have an orientable Seifert manifold over a compact parallelisable surface, hence a principal $\mathbb{T}^1$ bundle over $\mathbb{T}^2$; $(d = 2)$ we have an orientable principal $\mathbb{T}^2$ bundle over $\mathbb{T}$, hence $\mathbb{T}^3$; $(d = 3)$ we obtain $\mathbb{T}^3$. This completes the proof of (iv).

As for (v), let us denote $\Lambda_{c'} = \{(x, \lambda_{c'}(x)) :\ x \in M\}$. Observe that the map:

$$\begin{array}{rcl} \Psi : \mathcal{O} \times M & \longrightarrow & T^*M \\ (c', x) & \longmapsto & \lambda_{c'}(x) \end{array}$$

is continuous. It is sufficient to show that if $c_n \to c'$ in $\mathcal{O}$, then λ_{c_n} converge uniformly to $\lambda_{c'}$. In fact, the sequence $\{\lambda_{c_n}\}_n$ is equilipschitz (it follows from Mather's graph theorem [21, Theorem 2]) and equibounded, therefore applying Ascoli-Arzelà theorem we can conclude that - up to selecting a subsequence - λ_{c_n} converge uniformly to $\tilde{\lambda} = \eta_{c'} + du$, for some $u \in C^1(M)$. Observe that since $H(x, \lambda_{c_n}(x)) = \alpha_H(c_n)$ for all $x \in M$ and all n, and α_H is continuous, then $H(x, \tilde{\lambda}(x)) = \alpha_H(c')$ for all x. Therefore, u is a solution of Hamilton-Jacobi equation $H(x, \eta_{c'} + du) = \alpha_H(c')$. As we have observed in the beginning of this proof, for each $c' \in \mathcal{O}$ there is a unique solution of this equation, hence $\tilde{\lambda} = \lambda_{c'}$. This concludes the proof of the continuity of Ψ. Notice that this could be also deduced from the fact that Ψ is injective and semicontinuous.

The continuity of Ψ implies that these Lagrangian graphs $\Lambda_{c'}$ foliate an open neighborhood of Λ_c. It follows from Proposition 2.3 that the components of F commute in this open region. Therefore, each $\Lambda_{c'}$ is an n-dimensional manifold which is invariant under the action of n commutating vector fields, which are linearly independent at each point. It is a classical result that $\Lambda_{c'}$ is then diffeomorphic to an n-dimensional torus and that the motion on it is conjugate to a rotation (see for instance [3]). $\square$

Proof (Corollary 1.1). Let d be the largest dimension of the torus fibre of Λ_c for $c \in \mathcal{O}$. The upper semicontinuity of this dimension implies that there is an open set on which the dimension of the fibre equals d; without loss of generality, it can be supposed that this open set is $\mathcal{O}$. By (iii) of Theorem 1.1, Mather's α-function is differentiable on $\mathcal{O}$. Since α_H is a locally Lipschitz function, it is continuously differentiable on $\mathcal{O}$. Therefore, the map

$$c \longmapsto h = \partial\alpha_H(c), \qquad\qquad \mathcal{O} \xrightarrow{\partial\alpha_H} H_1(M;\mathbb{R})$$

is continuous and one-to-one (by [15, Theorem 3.3]) and hence a homeomorphism onto its image.

Let $b_1(M) = \dim H_1(M;\mathbb{R})$ be the first Betti number of M. By remark 3.1, we can assume that, for a residual set of $c \in \mathcal{O}$, the orbits of the Tonelli Hamiltonian are dense in the torus fibres of Λ_c, *i.e.* in the notation of the proof of Proposition 3.1, the 1-parameter group ϕ_t is dense in the torus $\bar{Z}$. It follows that if $d > b_1(M)$, then there exists $c, c' \in \mathcal{O}$ such that the rotation vectors $\rho(\Lambda_c)$ and $\rho(\Lambda_{c'})$ coincide. This contradicts the injectivity of $\partial\alpha_H$. Therefore $d \leq b_1(M)$ and so $d = b_1(M)$.

Let $\kappa : \check{M} \longrightarrow M$ be a finite covering. It is claimed that $b_1(\check{M}) = b_1(M)$.

(3)
$$\begin{tikzcd}
\check{\Lambda}_c \arrow[r, hook] \arrow[d, "\mathcal{K}|\check{\Lambda}_c"'] & T^*\check{M} \arrow[r, two heads] \arrow[d, "\mathcal{K}"] & \check{M} \arrow[d, "\kappa"] \arrow[l, bend right, "\check{\eta}_c=\kappa^*\eta_c"'] \\
\Lambda_c \arrow[r, hook] & T^*M \arrow[r, two heads] & M \arrow[l, bend left, "\eta_c"]
\end{tikzcd}$$

Since the cotangent lift of κ, $\mathcal{K}$, is a local symplectomorphism, the Tonelli Hamiltonian $\check{H} = \mathcal{K}^*H$ is weakly integrable with the first-integral map $\check{F} = \mathcal{K}^*F$. Let $c \in \mathcal{O}$ be a cohomology class and η_c a solution to the Hamilton-Jacobi equation for H whose graph Λ_c equals the Mather set $\mathcal{M}_c^*$ (diagram (3)). The pullback $\check{\eta}_c = \kappa^*\eta_c$ solves the Hamilton-Jacobi equation for $\check{H}$ and its graph $\check{\Lambda}_c$ is an invariant C^1 Lagrangian graph. By Proposition 3.1, there is a $\check{d} > 0$ such that $\check{\Lambda}_c$ admits the structure of a principal $\mathbb{T}^{\check{d}}$-bundle. This torus action is defined by $\check{d}$ commuting vector fields $\check{Y}_i = X_{\check{F}_i}|\check{\Lambda}_c$, $i = 1, \ldots, \check{d}$ induced by the first-integral map $\check{F}$. Since $\mathcal{K}$ is a local symplectomorphism, $\mathcal{K}|\check{\Lambda}_c$ is a local diffeomorphism. This shows that the dimension $\check{d}$ equals d. By the previous paragraph, weak integrability implies that $\check{d} = b_1(\check{M})$ so $b_1(\check{M}) = b_1(M)$.

When $\dim B \leq 2$, B has the homotopy type of a point, hence it is a point. Assume that $\dim B = 3$. If $\pi_1(B)$ is a free product of irreducible finitely-presented groups G_i $(i = 0, \ldots, g)$, then Kneser's theorem [18] implies that $B = B_0 \# \cdots \# B_g$ where B_i is a closed 3-manifold with $\pi_1(B_i) = G_i$. Since $H_1(B) = \bigoplus_i H_1(B_i)$, each homology group $H_1(B_i)$ is finite. According to [27, Proposition 2.1], if $H_1(B)$ is finite and $\pi_1(B_i)$ is not perfect for some i, then the universal abelian covering $\hat{B}$, or a 2-fold cover thereof, is a finite cover of B which has first Betti number at least 1. Thus, the only case to be resolved is that when $\pi_1(B_i)$ is perfect for all $i = 0, \ldots, g$. By [27, Remark at bottom of p. 570], Stallings' theorem implies that $G_i = [G_i, G_i]$ is isomorphic to π_1 of the Klein bottle – which is absurd. This proves that B is an irreducible 3-manifold. If $\pi_1(B)$ is infinite, then the virtual Haken conjecture implies that B has a finite covering with non-zero first Betti number. Therefore, $\pi_1(B)$ is finite and so by the proof of the Poincaré conjecture, B is finitely covered by S^3.

Let us prove that M is a trivial principal $\mathbb{T}^d$-bundle. This argument is indebted to that of Sepe [29]. A principal $\mathbb{T}^d$-bundle is classified up to isomorphism by a classifying map

(4)
$$\begin{tikzcd}
 & \mathbb{T}^d \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \\
M = f^*E\mathbb{T}^d \arrow[r] \arrow[d, "\pi_f"'] & E\mathbb{T}^d \arrow[r, equal] \arrow[d, "\pi"'] & \prod_{i=1}^d S^\infty \arrow[d, "\text{Hopf fib.}"] \\
B \arrow[r, "f"] & B\mathbb{T}^d \arrow[r, equal] & \prod_{i=1}^d \mathbb{C}P^\infty.
\end{tikzcd}$$

The classifying map f is null homotopic if and only if the pullback bundle is trivial. Classical obstruction theory shows that the single obstruction to a null homotopy of f is a cohomology class – the *Chern class* – with the following description. The trivial section $* \mapsto * \times 0$ of $E\mathbb{T}^d$ restricted to its 0-skeleton extends over the 1-skeleton. The obstruction to extending this section over the 2-skeleton defines a cohomology class $\eta \in H^2(B\mathbb{T}^d; \pi_1(\mathbb{T}^d)) = H^2(B\mathbb{T}^d; H_1(\mathbb{T}^d))$. By naturality, the obstruction to extending the trivial section of $f^*E\mathbb{T}^d$ over the 2-skeleton is the cohomology class $\eta_f = f^*\eta \in H^2(B; H_1(\mathbb{T}^d))$ – called the *Chern class*.

In terms of the E_2 page of the Leray-Serre spectral sequence with $\mathbb{Z}$-coefficients for the bundle $\mathbb{T}^d \hookrightarrow M \longrightarrow B$, one has the differential $d_2^{0,1} : E_2^{0,1} = H^1(\mathbb{T}^d) \longrightarrow E_2^{2,0} = H^2(B)$. It has been shown above that the inclusion map $\mathbb{T}^d \hookrightarrow M$ is injective on H_1, hence surjective on H^1. Since a class in $E_2^{0,1}$ survives to a class in E_∞ if and only if it is in the kernel of $d_2^{0,1}$, the differential $d_2^{0,1}$ must therefore vanish. Since the differential $d_2^{2,0}$ vanishes, it follows that $H^2(B)$ survives to E_∞.

On the other hand, for any cohomology class $\phi \in H^1(\mathbb{T}^d)$, the class $\eta \cup \phi = \langle \eta, \phi \rangle$ is a class in $H^2(B\mathbb{T}^d)$ which satisfies $\pi^*(\eta \cup \phi) = 0$ in $H^2(E\mathbb{T}^d)$. By naturality, the class $\eta_f \cup \phi \in H^2(B)$. This class, if non-zero, survives to E_∞. On the other hand, $\pi_f^*(\eta_f \cup \phi) = 0$ in $H^2(M)$. This shows that $\eta_f \cup \phi = 0$ in $H^2(B)$. Since the class ϕ was arbitrary, it follows that η_f vanishes. Therefore $M = f^*E\mathbb{T}^d$ is a trivial principal $\mathbb{T}^d$-bundle. Finally, since $M \simeq \mathbb{T}^d \times B$ and $b_1(M) = d$, the first Betti number of B vanishes.

Let us now prove (i–ii).

When $\dim M = 3$, one cannot have $d < 3$, since there are no parallelisable $(3-d)$-dimensional manifolds with trivial first Betti number. Therefore, $d = 3$ and $M = \mathbb{T}^3$.

When $\dim M = 4$, if $d = 1$, then the base B is a compact orientable 3-manifold. The proof now proceeds in the same way as in the proof of Corollary 1.1.

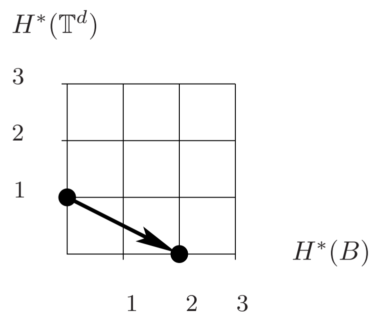

FIGURE 1. E_2 page of the spectral sequence.

□

4. Amenable groups, measures and rotation vectors

In this section it is assumed that X is a compact, path-connected, locally simply-connected metrizable space and (G, m_G) is a locally compact, simply-connected, metrizable, amenable topological group with Haar measure m_G. We will use d to denote a metric on both spaces; it will be assumed that the metric on G is *right*-invariant, without loss of generality. The space of m_G-essentially bounded measurable functions on G is denoted by $L^\infty(G)$. $L^\infty(G)^*$ has a distinguished subspace of functionals invariant under G's left (resp. right) action; this subspace will be denoted by $L^\infty(G)^*_{G_-}$ (resp. $L^\infty(G)^*_{G_+}$). A functional $\nu \in L^\infty(G)^*$ which satisfies $\nu(1) = 1$ is called a *mean.* The set of left-invariant (resp. right-invariant) means is denoted by $\mathfrak{m}(G)^{G_-}$ (resp. $\mathfrak{m}(G)^{G_+}$); amenability of G implies that both $\mathfrak{m}(G)^{G_\pm}$ is non-empty, as is the intersection $\mathfrak{m}(G)$.

Let $\hat\pi : \hat X \longrightarrow X$ be the universal abelian covering space of X, *i.e.* the regular covering space whose fundamental group is $[\pi_1 X, \pi_1 X]$ and on which $H_1(X;\mathbb{Z})$ (singular homology) acts as the group of deck transformations of $\hat\pi$.

Let $\phi : G \longrightarrow X$ be a uniformly continuous map (it is not assumed that there is an action of G on X). The simple-connectedness of G implies that there is a lift $\hat\phi$ of ϕ to $\hat X$. It is well-known that the first singular cohomology group of X is naturally isomorphic to the group of homotopy classes of maps from X to S^1, denoted by $[X, S^1]$. For each $f \in [X, S^1]$, let us construct the following commutative diagram

(5)
$$\begin{tikzcd}
 & \hat X \arrow[r, "\hat f"] \arrow[d, "\hat\pi"] & \mathbb{R} \arrow[d, "p"] \\
G \arrow[ur, "\hat\phi"] \arrow[r, "\phi"] \arrow[urr, bend left, "\hat g"] \arrow[rr, bend right, "g"'] & X \arrow[r, "f"] \arrow[ur, dotted, "\hat f"] & S^1
\end{tikzcd}$$

where $p(x) = x \bmod 1$ and $\hat f$ is a lift of f to $\hat X$ — the dotted diagonal line exists if and only if f is null-homotopic. Define the map

(6)
$$G \times G \xrightarrow{\ \zeta\ } \mathbb{R}^1 \qquad\qquad (s,t) \xmapsto{\ \zeta\ } g(st) - g(t)\,.$$

A priori, ζ is a map into S^1, but the simple-connectedness of G implies there is a unique lift of the map in (6) that is identically zero when $s = 1$ (the lift is trivially $\hat g(st) - \hat g(t)$). For a fixed $s \in G$, let $\zeta_s(t) = \zeta(s,t)$.

4.1. **Lemma.** *For each $s \in G$, $\zeta_s \in L^\infty(G)$.*

Proof. Since X is compact, f is uniformly continuous. Since ϕ is assumed to be uniformly continuous, g and therefore $\hat g$ is uniformly continuous. Therefore, there is a $\delta > 0$ such that if $a, b \in G$ and $d(a,b) < \delta$ then $|\hat g(a) - \hat g(b)| < 1$. Let N be an integer exceeding $d(s,1)/\delta$. Then the right-invariance of the metric d implies that for all $t \in G$, $d(st,t) = d(s,1) < N\delta$, so by the triangle inequality, one concludes $|\hat g(st) - \hat g(t)| < N$. Thus $|\zeta_s(t)| < N$ for all $t \in G$. □

4.2. **Lemma.** *Let $\nu \in \mathfrak{m}(G)^{G_-}$ be a left-invariant mean on G. If $g \in L^\infty(G)$, then $\langle \nu, \zeta_s \rangle = 0$ for all $s \in G$. In particular, if*

(1) *f is null-homotopic; or*
(2) *$\operatorname{Im}\hat\phi$ is contained in a compact set,*

then $\langle \nu, \zeta_s \rangle$ vanishes for all $s \in G$.

Proof. If $g \in L^\infty(G)$, then $\langle \nu, \zeta_s \rangle = \langle s_*\nu, g \rangle - \langle \nu, g \rangle = 0$ by left-invariance of ν. If f is null-homotopic, then the image of $\hat f$ is a compact subset of $\mathbb{R}$, so $g \in L^\infty(G)$; likewise, if $\operatorname{Im}\hat\phi$ has compact closure. □

4.3. **Lemma.** *Let $\phi, \phi' : G \longrightarrow X$ be uniformly continuous maps. If there is a $K > 0$ such that their lifts $\hat{\phi}, \hat{\phi}' : G \longrightarrow \hat{X}$ satisfy $d(\hat{\phi}(s), \hat{\phi}'(s)) < K$ for all $s \in G$, then $\langle \nu, \zeta_s - \zeta'_s \rangle$ vanishes for all $s \in G$ and $\nu \in \mathfrak{m}(G)^{G_-}$.*

Proof. The proof of this lemma mirrors the preceding. By the assumption that $d(\hat{\phi}(t), \hat{\phi}'(t)) < K$ for all $t \in G$, one has that $\hat{h}(t) := \hat{f}\hat{\phi}(t) - \hat{f}\hat{\phi}'(t)$ lies in $L^\infty(G)$. Therefore, $\langle \nu, \zeta_s - \zeta'_s \rangle = \langle s_* \nu, \hat{h} \rangle - \langle \nu, \hat{h} \rangle = 0$ by left-invariance of the mean ν. □

4.4. **Lemma.** *Let $\nu \in \mathfrak{m}(G)^{G_-}$ be a left-invariant mean and $\phi : G \longrightarrow X$ a uniformly continuous map. For each $s \in G$, the map*

$$f \longmapsto \langle \nu, \zeta_s \rangle \tag{7}$$

(see (6)) induces a linear function $\rho_s(\nu) : H^1(X;\mathbb{R}) \longrightarrow \mathbb{R}$. The function $\rho_s : \mathfrak{m}(G)^{G_-} \longrightarrow H_1(X;\mathbb{R})$ is affine and continuous in the weak- topology on $L^\infty(G)^{**}$.*

Proof. It suffices to show that this map is additive on $H^1(X;\mathbb{Z}) = [X, S^1]$, since it is extended by multiplicativity to a map on $H^1(X;\mathbb{R})$. First, let us show the map is well-defined on homotopy classes. Let f, f' be representatives of the homotopy class $[f]$. By compactness of $X \times [0,1]$, there is an $N > 0$ such that $|\hat{f}(x) - \hat{f}'(x)| < N$ for all $x \in \hat{X}$. Therefore, $|g(st) - g'(st)| < N$ and $|g(t) - g'(t)| < N$ for all $t \in G$ (using the obvious notation), so both $s^*(g - g')$ and $g - g'$ are in $L^\infty(G)$. Thus, $\langle \nu, \zeta_s - \zeta'_s \rangle = \langle s_* \nu, g - g' \rangle - \langle \nu, g - g' \rangle = 0$ by left-invariance of ν. This proves the map (7) is well-defined on $[X, S^1]$.

To prove that the map (7) is additive, let $f, h : X \longrightarrow S^1$ be representatives of the homotopy classes $[f], [h]$. The homotopy class $[f] + [h]$ is represented by $[f + h]$. From the diagram (5), it is clear that $\zeta^{f+h} = \zeta^f + \zeta^h$ where $\zeta^\bullet$ denotes ζ constructed with $\bullet$. This suffices to prove additivity, and that suffices to show that $\rho_s(\nu)$ is a linear form on $H^1(X;\mathbb{R})$.

Since the pairing defining $\rho_s(\nu)$ is the bilinear pairing between $L^\infty(G)^*$ and $L^\infty(G)$, it follows that ρ_s is an affine map that is continuous in the weak-* topology on linear maps $\mathrm{Hom}(L^\infty(G)^*; H^1(X;\mathbb{R})^*)$. □

4.1. **Definition.** Let $s \in G$. The set

$$R_s = \rho_s \left(\mathfrak{m}(G)^{G_-} \right) \tag{8}$$

is the rotation set of the left translation s.

4.1. **Theorem.** *The map $\rho : G \longrightarrow \mathrm{Hom}(\mathfrak{m}(G)^{G_-}; H_1(X;\mathbb{R}))$ is continuous. For each $s \in G$, the rotation set R_s is a compact, convex subset of $H_1(X;\mathbb{R})$. The rotation-set map*

$$s \longmapsto R_s \tag{9}$$

is an upper semi-continuous set function.

Proof. If $s_n \to s$ in G, then for a fixed $f : X \longrightarrow S^1$, one sees that $\zeta_{s_n} \to \zeta_s$ in $L^\infty(G) \cap C^0(G;\mathbb{R})$. Therefore, for any $\nu \in \mathfrak{m}(G)^{G_-}$, $\langle \rho_{s_n}(\nu), [f] \rangle \longrightarrow \langle \rho_s(\nu), [f] \rangle$. This proves ρ is continuous in the weak-* topology.

Clearly $\mathfrak{m}(G)^{G_-}$ is convex. Since $\mathfrak{m}(G)^{G_-} \subset L^\infty(G)^*$ is a closed subset of the unit ball in $L^\infty(G)^*$, it is a compact set in the weak-* topology. Since ρ_s is continuous and affine, its image is compact and convex. □

4.1. **Examples.** Let us compute some rotation sets.

4.1.1. *Translations on tori.* Let $X = \mathbb{T}^n$ and let $G = \mathbb{R}^n = \tilde{X}$ be the universal covering group acting in the tautological manner; the map ϕ is the orbit map of $\theta_0 \in \mathbb{T}^n$. A cohomology class $f \in [X, S^1]$ has a canonical representative, *viz.* $f(\theta) = \langle v, \theta \rangle \bmod 1$ where $v \in \mathrm{Hom}(\mathbb{Z}^n; \mathbb{Z})$. One arrives at the map $\tilde{g}(t) = \langle v, t + \tilde{\theta}_0 \rangle$ and $\zeta_s(t) = \langle v, s \rangle$ – which is independent of $t \in G$ –, whence the mean of ζ_s equals $\langle v, s \rangle$ for any mean $\nu \in \mathfrak{m}(G)^G$. If one employs the tautological isomorphism between the real homology (resp. cohomology) group of $\mathbb{T}^n$ and $\mathbb{R}^n$ (resp. $\mathrm{Hom}(\mathbb{R}^n; \mathbb{R})$), one obtains

$$\rho_s(\nu) = s$$

for all $s \in G$, $\nu \in \mathfrak{m}(G)^G$.

We note that this calculation computes the rotation vector/set of a subgroup, given a mean on the whole group. Lemma 4.6 below shows that there is no loss of generality.

4.1.2. *Translations on quotients of contractible amenable Lie groups of type (E).* Let G be a contractible, amenable Lie group of type (E) (hence a solvable Lie group of type (E)), $\Gamma \lhd G$ be a co-compact subgroup and $X = \Gamma \backslash G$. Let $g, g' \in G$ and let $\phi : G \longrightarrow X$ be the map $\phi(t) = \Gamma g t^{-1} g'$. Let N be the commutator subgroup of G; it is known that $\Gamma \cap N$ is a lattice in N, that the commutator subgroup of Γ is of finite index in $\Gamma \cap N$ and therefore ΓN is closed subgroup of G [11, Lemma 3]. The map $F : X \longrightarrow N \backslash X$ is therefore a submersion onto a torus whose dimension is the codimension of N in G. From the fact that the derived subgroup of Γ is of finite index in $\Gamma \cap N$, one sees that $[X, S^1] = F^*[N \backslash X, S^1]$.

Therefore, we have reduced the problem to the case of a translation on a torus, whence $\rho_s(\nu) = -Ns$ in the simply connected abelian Lie group $N \backslash G$, $\nu \in \mathfrak{m}(G)^{G_-}$.

4.1.3. *Translations on quotients of amenable Lie groups of type (E).* The situation with simply-connected amenable Lie groups of type (E) is somewhat more complicated than the previous example, as exemplified by [11, Examples 1 & 2]. These examples show how the first Bieberbach theorem may fail, but in these examples the Levi decomposition is trivial: the groups themselves are solvable and one might be lead to believe that this is the only way that such pathological examples can arise.

Example. Let us give an example where the Levi decomposition is non-trivial and the first Bieberbach theorem fails. That is, let us give an example where $G = SK$ is a simply-connected amenable Lie group of type (E) where S is its solvable radical and K is a maximal compact subgroup, and $\Gamma < G$ is a lattice subgroup such that $\Gamma \cap S$ is not a lattice subgroup of S.

Let $k > 2$ be integers and let N be the nilpotent Lie group whose multiplication is defined by

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 \otimes y_2 - x_2 \otimes y_1)) \tag{10}$$
$$\text{where } x_i, y_i \in \mathbb{R}^k, z_i \in \mathbb{R}^k \otimes \mathbb{R}^k.$$

The cyclic group generated by

$$a = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

acts as a group of automorphisms of N, and this group is a discrete subgroup of a 1-parameter group of automorphisms A. Let $S = NA$, a solvable group of type

(E). On the other hand, let K be the universal covering group of $\mathrm{SO}_k \times \mathrm{SO}_k$ (since $k > 2$, K is compact) and let K act on N via

$$\kappa \cdot g = (u \cdot x, v \cdot y, (u \otimes v) \cdot z) \\ \text{where } g = (x, y, z) \in N, \kappa \in K \longmapsto (u, v) \in \mathrm{SO}_k \times \mathrm{SO}_k. \tag{12}$$

This is an action by automorphisms of N and this action commutes with the action of A, so this action induces a natural action of K on S. This suffices to describe the group $G = SK$, an amenable Lie group of type (E).

The lattice subgroup Γ is described as follows. Let

$$N_{\mathbb{Z}} = \left\{ g = (x, y, z) \in N \,:\, x, y \in \mathbb{Z}^k, 2z \in \mathbb{Z}^k \otimes \mathbb{Z}^k \right\}$$

and observe that a preserves $N_{\mathbb{Z}}$. Let $b \in K$ and let $\gamma = ab$. The group Γ generated by γ and $N_{\mathbb{Z}}$ is discrete and co-compact in G for any choice of b. If b is of infinite order, then the intersection of Γ with S is just $N_{\mathbb{Z}}$ and is *not* a lattice in S. The projection of Γ to $K = S\backslash G$ is the group generated by b; if b is chosen in general position, then the identity component of the closure is a maximal torus.

This example shows how the first Bieberbach theorem can fail for type (E) amenable Lie groups. However, the representation of K as a group of automorphisms of S is almost faithful, and this implies many of the nice properties mentioned in the previous paragraph. On the other hand, if one takes the amenable Lie group $G = \mathbb{C}^n \times \mathrm{SU}_n$ with the lattice subgroup Γ generated by the set $\{(e_j, \rho_j), (ie_j, \rho_j) \,:\, j = 1, \ldots, n\}$ where each ρ_j is a generic element in the maximal torus of diagonal matrices, then one sees that the intersection of Γ with S is trivial and the projection of Γ onto SU_n is dense in the maximal torus.

Let $G = SK$ be a simply-connected, amenable Lie group where S is its radical and K its maximal compact subgroup, and let $\Gamma < G$ be a lattice subgroup. Let us consider two cases in successive generality:

K is virtually a subgroup of $Aut(S)$. In this case, we suppose that the action of K on S by conjugation has a finite kernel. In this case, the machinery of [11, 4] is applicable.

Let S^* be the identity component of the closure of ΓS in G and let $\Gamma^* = S^* \cap \Gamma$. By [11, Lemma 3] and [4], one knows that S^* is a solvable subgroup containing S, Γ^* is of finite index in Γ, $S\backslash S^*$ is a torus subgroup, T, of K, the nilradical of S^* equals the nilradical R of S, $\Gamma \cap R$ is a lattice subgroup of R. Likewise, the derived subgroup of S, $N = [S, S] = [S^*, S^*]$, intersects Γ in a lattice subgroup of N. This information is summarised in the commutative diagram (13), where $\mathrm{B} = \Gamma \cap N$, $F = \mathrm{B}\backslash N$, $Z = \mathrm{B}\backslash G$, $T^* = (\mathrm{B}\backslash N)\backslash(N\backslash S^*)$ (a torus) and $A = S\backslash S^*$.

(13)

$$\begin{tikzcd}
 & & & \mathrm{B} \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \mathrm{B} \arrow[dr] \arrow[ddd] & & & \\
 & & & & N \arrow[rrr] \arrow[dr] \arrow[ddd] & & & G \arrow[rr, two heads] \arrow[dr] \arrow[ddd] & & N\backslash G = AK \arrow[ddd, two heads] \\
 & & & & & F \arrow[rrr] \arrow[ddd] & & & Z \arrow[ur, two heads] \arrow[ddd] & \\
\mathrm{B}\backslash\Gamma^* \arrow[dr] & & & \Gamma^* \arrow[lll, two heads] \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \Gamma^* \arrow[dr] \arrow[ddd] & & & \\
 & N\backslash S^* \arrow[dr] & & & S^* \arrow[lll, two heads] \arrow[rrr] \arrow[dr] \arrow[ddd] & & & G \arrow[rr, two heads] \arrow[dr] \arrow[ddd] & & S^*\backslash G = T\backslash K \arrow[ddd, two heads] \\
 & & T^* & & & Y^* \arrow[lll, two heads] \arrow[rrr] \arrow[ddd] & & & X^* \arrow[ur, two heads] \arrow[ddd] & \\
 & & & \Gamma \arrow[rrr] \arrow[dr] & & & \Gamma \arrow[dr] & & & \\
 & & & & \overline{\Gamma S} \arrow[rrr] \arrow[dr] & & & G \arrow[rr, two heads] \arrow[dr] & & \overline{\Gamma S}\backslash G = WT\backslash K \\
 & & & & & Y \arrow[rrr] & & & X \arrow[ur, two heads] &
\end{tikzcd}$$

In diagram (13), all southeast sequences are fibrations with discrete fibre (covering spaces), all eastern sequences are fibrations, as are the backwards L sequences. In particular, X^* is a finite regular covering space of X which is fibred by the solvmanifold Y^* over the K-homogeneous space $T\backslash K$; the solvmanifold Y^* is itself fibred by the nilmanifold F over the torus T^*. Since S^* is the identity component of $\overline{\Gamma S}$, the group $W = \Gamma^*\backslash\Gamma$ permutes the components of $\overline{\Gamma S}$, which shows that $Y^* = Y$, so X is fibred by solvmanifolds, also.

Since $WT\backslash K$ has finite fundamental group, its first cohomology group over $\mathbb{Z}$ vanishes. Therefore, the Leray-Serre spectral sequence for the fibring of X by Y shows that the restriction to a fibre induces an injection of $H^1(X;\mathbb{R})$ into $H^1(Y;\mathbb{R})$ (the image is the kernel of $d_2^{0,1}$ in the figure 1). The fibring of Y by the nilmanifold F over the torus T^* is exactly as described in the previous example. In particular, the projection map induces an isomorphism of $H^1(Y;\mathbb{R})$ and $H^1(T^*;\mathbb{R})$. Since $S^* = ST$, we see that $N\backslash S^* = AT$ where $A = N\backslash S$. Since T is contractible in G, one sees that the first real homology group of X^* is naturally identified with A; or Z is visibly the universal abelian covering space of X^*. It follows that $H^1(X;\mathbb{R})$ is naturally identified with A^W, the fixed-point set of W acting on A.

Let $\phi : G \longrightarrow X$ be defined by $\phi(t) = \Gamma g t^{-1} g'$ for some $g, g' \in G$. A few applications of Lemma 4.3 imply that one can suppose, without changing the rotation map, that $\phi(t) = \Gamma a\kappa\alpha^{-1}\kappa^{-1}b$ where $a, b \in S$, $\kappa \in K$ and $t = \beta\alpha$ is the decomposition into $\beta \in K$ and $\alpha \in S$. Let $\hat{F} : Z \longrightarrow A = N\backslash G/K$ be the map that induces the isomorphism of $[X^*, S^1] \otimes \mathbb{R}$ with A. Concretely, if $Nt \in Z$, let $t = \beta_t\alpha_t$ be the decomposition of t into $\beta_t \in K$, $\alpha_t \in S$; then $\hat{F}(Nt) = K\alpha_t N$. One computes that

$$\zeta_s(t) = -K(\kappa\beta_t^{-1}) \cdot \alpha_s \cdot (\kappa\beta_t^{-1})^{-1} N \qquad s, t \in G. \tag{14}$$

It is clear that ζ_s is S-invariant since $t \mapsto \beta_t$ is the projection $G \longrightarrow K$. Since the restriction of any mean on a compact Lie group to its continuous functions is the Haar probability measure [26], one sees that for any $\nu \in \mathfrak{m}(G)^{G_-}$, $\rho_s(\nu) = -\bar{\alpha}_s N$ is the projection of $\alpha_s N$ onto the subspace of K-invariant vectors.

Note that if one restricts ϕ to S, then the rotation vector of $s \in S$ with respect to the mean $\nu \in \mathfrak{m}(S)^{S_-}$ is the projection of $-\kappa s\kappa^{-1}N$ onto the subspace of W-invariant vectors.

When K is not a virtual subgroup of $Aut(S)$. Let us now examine the case where the kernel of representation $K \longrightarrow \mathrm{Aut}(S)$ is not finite. Let $K_1 \lhd K$ be the identity component of this kernel. Since K is compact and simply-connected, K is semi-simple and so $K = K_0 \oplus K_1$ is a sum of semi-simple factors, and the representation of $K_0 \longrightarrow \mathrm{Aut}(S)$ has finite kernel. By construction, K_1 is a normal subgroup of G and the lattice Γ intersects K_1 in a compact set, hence $\Gamma \cap K_1$ is a finite, normal subgroup of Γ. We obtain the fibration

$$\Gamma \cap K_1 \backslash K_1 \longrightarrow \Gamma\backslash G \overset{\rho}{\twoheadrightarrow} \bar{\Gamma}\backslash\bar{G} \qquad = (\Gamma \cap K_1\backslash\Gamma)\backslash(K_1\backslash G)\,. \tag{15}$$

The quotient $\bar{G} = SK_0$ has the property that K_0 is a virtual subgroup of $\mathrm{Aut}(S)$. The fibre $\Gamma \cap K_1\backslash K_1$ has a finite fundamental group. It follows that the map $\rho^* : H^1(\bar{\Gamma}\backslash\bar{G};\mathbb{R}) \longrightarrow H^1(\Gamma\backslash G;\mathbb{R})$ is an isomorphism. From this, one concludes that the preceding computations of the ζ-map (14) and the rotation vectors of a mean remain correct in this enlarged setting.

4.1.4. *Quotients of amenable Lie groups of type (E) – II.* Let us continue with the notations of the previous example. Let $H = G\times G'$ be a product of simply connected amenable Lie groups (in applications, $G' = \mathbb{R}$, but what follows is perfectly general). Let $\varphi : G' \longrightarrow X$ be a uniformly continuous map and let

$$\phi : H \longrightarrow X \qquad \phi(h) = \Gamma g^{-1}\varphi(g'), \text{ where } h = (g, g') \in H. \tag{16}$$

Similar to that above, one computes that with $s = (1, b)$ and $t = (g, a)$, one has

$$\zeta_s(t) = -K\delta_b(a)N \qquad \delta_b(a) = \text{the projection of } \varphi(ba)^{-1}\cdot\varphi(a) \text{ onto } S, \tag{17}$$

using the factorisation of an element in G as in the previous example. In particular, this implies that ζ_s is independent of g when $s = (1, b)$. This implies that if $\nu \in \mathfrak{m}(H)^{H-}$ is a mean on H, then the rotation vector $\rho_s(\nu)$ ($s = (1, b)$) equals the rotation vector $\rho_b(\bar{\nu})$ for the map φ and the projected mean $\bar{\nu} \in \mathfrak{m}(G')^{G'_-}$.

In the next section we show how this result can be interpreted in terms of the rotation vector of two measures with different sized supports.

4.2. **Relation to Schwartzman cycles.** Let us suppose that $\Phi : G \times X \longrightarrow G$ is a left-action of G on X. For each $x \in X$, one has the orbit map $\phi_x(t) := \Phi(t, x)$. The action will also be denoted by $\Phi(t, x) = t \cdot x$.

4.5. **Lemma.** *The orbit map $\phi_x : G \longrightarrow X$ is uniformly continuous for all $x \in X$.*

Proof. Let us define $\epsilon(\delta) = \max\{d(\Phi(1, x), \Phi(t, x)) : x \in X, d(1, t) \leq \delta\}$. By local compactness of G and compactness of X, the maximum is attained. Moreover, ϵ is a continuous increasing function of δ that vanishes at $\delta = 0$. This implies uniform continuity of the orbit map ϕ_x. □

Let $\nu \in \mathfrak{m}(G)^{G-}$ be a left-invariant mean on G. For each $x \in X$, the pull-back of $C^0(X)$ by the orbit map ϕ_x lies inside $L^\infty(G)$. Thus, $\phi_{x,*}\nu$ determines a positive, continuous linear functional on $C^0(X)$ and so by the Riesz representation theorem, $\phi_{x,*}\nu$ induces a Borel probability measure μ_x on X. It is clear that μ_x is G-invariant. The support of μ_x is clearly contained in the ω-limit set of x,

$$\omega_G(x) = \bigcap_{T>0} \overline{\{t \cdot x \,:\, d(1, t) > T\}}. \tag{18}$$

In [15, Appendix A], one finds a definition of the rotation vector of an invariant measure of a flow (an $\mathbb{R}$-action). Let μ be an invariant Borel probability measure of the flow $\varphi : \mathbb{R} \times X \longrightarrow X$ and $[f] \in [X, S^1]$ a cohomology class. The rotation vector of μ is defined as

$$\langle [f], \rho_\varphi(\mu)\rangle = \int_{x\in X} \zeta_\varphi(x)\, d\mu(x), \tag{19}$$

where $\zeta_\varphi(x) = f(\varphi_1(x)) - f(x)$ similar to (6). We have:

4.2. **Theorem.** *Let $\Phi : G \times X \longrightarrow X$ be a G-action, φ be an action of a 1-dimensional subgroup with $\varphi_1 = s$, and let $\nu \in \mathfrak{m}(G)^{G_-}$, $\mu_x = \phi_{x,*}\nu$ for some $x \in X$. Then*

$$\rho^x_s(\nu) = \rho_\varphi(\mu_x), \tag{20}$$

where ρ^x is the rotation map for the orbit map ϕ_x.

The proof is an application of change of variables.

4.3. **Averaged rotation vectors.** In this subsection, let us suppose that G fits in the exact sequence of (amenable) groups

$$H \hookrightarrow G \twoheadrightarrow F. \tag{21}$$

Let $\nu_H \in \mathfrak{m}(H)^{H_-}$ (resp. $\nu_F \in \mathfrak{m}(F)^{F_-}$) be left-invariant means. One can define an invariant mean ν_G as follows: let $f \in L^\infty(G)$ and define $f_H \in L^\infty(F)$ by averaging over H, $f_H(Ht) = \langle \nu_H, f_t\rangle$ where $f_t(x) = f(tx)$. The normality of H and left-invariance of ν_H implies that f_H is well-defined and $f_H \in L^\infty(F)$. Then, one defines the left-invariant mean ν_G by $\langle \nu_G, f\rangle := \langle \nu_F, f_H\rangle$.

4.2. **Definition.** The mean $\nu_G \in \mathfrak{m}(G)^{G_-}$ is denoted by $\nu_G = \nu_F \times \nu_H$ and called a *product mean*.

Let us suppose that H acts on X by an action φ and that there is a uniformly continuous map $\phi : G \longrightarrow X$ satisfying

$$\phi(s \cdot t) = \varphi(s) \cdot \phi(t) \qquad \forall s \in H, t \in G. \tag{22}$$

Let $t_0 \in G$, $x = \phi(t_0)$ and $\mu_{H,x} = \varphi_{x,*}\nu_H$ is the pushed forward measure on X. The measure $\mu_G = \phi_*\nu_G$ (where $\nu_G = \nu_F \times \nu_H$) is H-invariant due to the cocycle condition (22) and supp $\mu_{H,x} \subset$ supp μ_G.

The following lemma shows that under a suitable condition on the map ϕ, one can average over the group G to obtain a measure μ_G with a larger support and the same rotation set.

4.6. **Lemma.** *Suppose that the lift $\hat\phi$ (see (5)) has the property that for each $t \in G$, there is a $K > 0$ such that $d(\hat\varphi(s) \cdot \hat\phi(t_0), \hat\varphi(s) \cdot \hat\phi(t)) < K$ for all $s \in H$. Then for all $s \in H$, $\rho_s(\mu_{H,x})$ is independent of the point $x \in \operatorname{Im}\phi$. In particular,*

$$\rho_s(\mu_{H,x}) = \rho_s(\mu_G). \tag{23}$$

To be clear, ρ_s refers to the rotation map of the flow generated by the 1-parameter group through s, as in (19). The proof of this lemma follows from Lemma 4.3 and Theorem 4.2 along with an unravelling of the product mean.

Note that the example in section 4.1.3 does not contradict this lemma. In that example, the map ϕ does not satisfy the uniform boundedness condition.

5. Homogeneous structures

Let G be a connected Lie group. Define the left (resp. right) translation map by

$$L_h(g) := hg, \qquad R_h(g) := gh \tag{24}$$

for all $g, h \in G$. These two maps define a left action of $G_- = G$ (resp. $G_+ = G^{op}$) on G and therefore on T^*G by Hamiltonian symplectomorphisms. The momentum maps of these actions are

$$\begin{aligned} &\Psi_- : T^*G \longrightarrow \mathfrak{g}^*_- && \Psi_+ : T^*G \longrightarrow \mathfrak{g}^*_+ \\ &\Psi_-(g,\mu_g) := (T_1R_g)^*\mu_g && \Psi_+(g,\mu_g) := (T_1L_g)^*\mu_g, \end{aligned} \tag{25}$$

for each $g \in G$, $\mu_g \in T^*_gG$.

A co-vector field $\mu : G \longrightarrow T^*G$ is left- (resp. right-) invariant if $\mu(1) = (T_1 L_g)^* \mu(g)$ (resp. $\mu(1) = (T_1 R_g)^* \mu(g)$) for all $g \in G$. If one trivialises T^*G with respect to the left-invariant co-vectors, then the momentum maps are simply

$$\Psi_-(g,\mu) := \mathrm{Ad}^*_{g^{-1}} \mu \qquad\qquad \Psi_+(g,\mu) := \mu, \tag{26}$$

for all $g \in G, \mu \in \mathfrak{g}^* = T_1^* G$, where $\mathrm{Ad}^*_g = (T_1 L_g R_{g^{-1}})^*$.

One says that a function $H : T^*G \longrightarrow R$ is *collective* for the left-action (resp. right-action) if $H = \Psi_-^* h$ (resp. $H = \Psi_+^* h$) for some $h : \mathfrak{g}^* \longrightarrow \mathbb{R}$. If H is collective for the left-action (resp. right-action) then (25) shows it is right-invariant (resp. left-invariant). In particular, a Hamiltonian that is collective for the left-action [right-invariant] (resp. right-action [left-invariant]) Poisson-commutes with Ψ_+ (resp. Ψ_-).

Let $H : T^*G \longrightarrow \mathbb{R}$ be a smooth, left-invariant (= right collective) Tonelli Hamiltonian. Therefore, there is a smooth convex Hamiltonian $h : \mathfrak{g}^* \longrightarrow \mathbb{R}$ such that $H = \Psi_+^* h$. Moreover, since H is left-invariant, it Poisson-commutes with the momentum map of the left action Ψ_-.

Let $\Gamma \lhd G$ be a co-compact lattice subgroup and $M = \Gamma \backslash G$. It is assumed that G is simply connected, so that the universal cover of M, $\tilde{M}$, is G. Let $[\Gamma,\Gamma] = \Gamma_1$ be the commutator subgroup of Γ, which is the fundamental group of the universal abelian cover $\hat{M}$. This leads to the commuting diagram of covering maps:

$$\begin{tikzcd}
T^*G = T^*\tilde{M} \arrow[r, two heads] \arrow[d, "\hat{\Pi}"] \arrow[dd, bend right=60, "\Pi"'] & G = \tilde{M} \arrow[d, "\tilde{\pi}"] \arrow[dd, bend left=60, "\pi"] \\
T^*(\Gamma_1 \backslash G) = T^*\hat{M} \arrow[r, two heads] \arrow[d, "\hat{\Pi}"] & \Gamma_1 \backslash G = \hat{M} \arrow[d, "\hat{\pi}"] \\
T^*(\Gamma \backslash G) = T^*M \arrow[r, two heads] & \Gamma \backslash G = M .
\end{tikzcd} \tag{27}$$

We adopt the notational convention that the pull-back of x to $\hat{M}$ (resp. $\tilde{M}$) is denoted by $\hat{x}$ (resp. $\tilde{x}$).

Let $c \in H^1(M;\mathbb{R})$ be a cohomology class, let $(x,p) \in \mathcal{M}^*_c(H)$ be a recurrent point in the Mather set and let $\delta : \mathbb{R} \longrightarrow M$ be the minimizer with initial conditions $\delta(0) = x$ and $\mathcal{L}(x, \dot{\delta}(0)) = (x,p)$, where $\mathcal{L}$ denotes the associated Legendre transform (see section 2). By the arguments of [21], we can suppose that the rotation set of δ is a singleton $\{h\} \subset H_1(M;\mathbb{R})$ and any weak-* limit of uniform measures along the orbit is a minimizing measure. Fix a lift $\tilde{\delta}$ of δ to $\tilde{M}$. For each $g \in G$, let $\tilde{\delta}_g = L_{g^{-1}} \circ \tilde{\delta}$ be a left-translate of this lift. Left invariance of H implies that $\tilde{\delta}_g$ is the projection of an integral curve, which implies that the projection of $\tilde{\delta}_g$ to $\hat{M}$ and M are also projections of orbits. All of this allows the definition of a map

$$\begin{tikzcd}
G \times \mathbb{R} \arrow[r, "\phi"] \arrow[dr, "\bar{\phi}"'] & T^*M \arrow[d] \\
& M
\end{tikzcd} \qquad \begin{aligned} \phi(g,t) &= \Pi \circ (TL_{g^{-1}})^* \tilde{\varphi}_t(x,p), \\ \bar{\phi}(g,t) &= \Gamma g^{-1} \tilde{\delta}(t) = \Gamma \delta_g(t) \end{aligned} \tag{28}$$

where $\tilde{\varphi}$ is the flow of H on the universal cover T^*G. By the example in section 4.1.4, the rotation vector of the map δ_g is independent of g for any mean on $G \times \mathbb{R}$. This implies the same is true for $\phi(g,t)$.

Let $\nu_{\mathbb{R}} \in \mathfrak{m}(\mathbb{R})^{\mathbb{R}}$ be an invariant mean such that the rotation vector of $\nu_{\mathbb{R}}$ at $s = 1$ under the map δ is h. By hypothesis, there is such a mean. The preceding discussion proves the following Lemma.

5.1. **Lemma.** *Let $\nu_{\mathbb{R}} \in \mathfrak{m}(\mathbb{R})^{\mathbb{R}}$, $\nu_G \in \mathfrak{m}(G)^{G-}$ and $\mu = \phi_*(\nu_{\mathbb{R}} \times \nu_G)$. Then μ minimizes A_c - i.e. it is c-action minimizing - and the projection of* supp μ *covers M.*

Therefore, by [21, Theorem 2], we know that supp μ is a Lipschitz graph over M. Therefore, the lift to $T^*\tilde{M}$ contains the smooth manifold $\tilde{\phi}(G \times 0)$ which is a smooth graph over $\tilde{M}$. Therefore supp μ is a smooth Lagrangian graph over M, supp $\mu = \mathrm{graph}(\eta)$, and lifting this picture to $T^*\tilde{M}$ shows that $\tilde{\eta}$ is closed and left-invariant. Therefore, $\tilde{\eta}$ is a bi-invariant 1-form. This proves

5.1. **Theorem.** *Let $M = \Gamma\backslash G$ be a compact manifold, where G is a simply connected amenable Lie group and $\Gamma \lhd G$ is a lattice subgroup. If $H : T^*G \longrightarrow \mathbb{R}$ is a left-invariant Tonelli Hamiltonian and $c \in H^1(M;\mathbb{R})$, then there is a bi-invariant 1-form on G, $\tilde{\eta}$, with cohomology class c such that $\mathcal{M}_c^*(H) = \mathrm{graph}(\eta)$.*

5.1. *Remark.* Let $\Lambda \subset T^*M$ be a C^1 Lagrangian graph that is contained in an energy level of H. That is, there is a closed 1-form η on M such that $H(q, \eta(q)) = E$ is constant for all $q \in M$. The previous theorem implies that η is a closed, bi-invariant 1-form. Since H is left-invariant, it follows that

$$E = H(q,\eta(q)) = h \circ \Psi_+(q,\eta(q)) = h((T_1L_g)^*\eta(q)) = h(\eta(1)) \tag{29}$$

$$\Psi_-(q,\eta(q)) = \eta(1), \tag{30}$$

for all $q \in M$. (30) follows because η is bi-invariant, which implies that the co-adjoint orbit of $\eta(1)$ is a single point.

Hamilton's equations for the Hamiltonian H are

$$X_H(g,\mu) : \left\{ \begin{array}{rcl} \dot{g} & = & (T_1L_g) \cdot \mathrm{d}h(\mu), \\ \dot{\mu} & = & -\mathrm{ad}^*_{\mathrm{d}h(\mu)}\mu. \end{array} \right. \qquad \forall g \in G, \mu \in \mathfrak{g}^*. \tag{31}$$

In particular, if μ is a closed form, then $\mathrm{ad}^*_\xi \mu$ vanishes for all $\xi \in \mathfrak{g}$. Therefore, the orbit of (g, μ) is $\{(T_1R_{\exp(t\xi)})^*(g,\mu) = (g\exp(t\xi), \mu) \ : \ t \in \mathbb{R}\}$ where $\xi = \mathrm{d}h(\mu)$, *i.e.* it is the orbit of a 1-parameter subgroup.

Proof (Theorem 1.2). The sole remaining thing to prove is that if H is weakly integrable and $\Lambda \subset T^*M$ lies inside an iso-energy surface and intersects $\mathrm{Reg}\, F$, then M is a homogeneous space of a compact reductive Lie group. By Remark 5.1, $\Lambda = \mathrm{graph}(\eta)$ where η is a bi-invariant, closed 1-form on G. By Corollary 1.1, M is diffeomorphic to $\mathbb{T}^b \times B$ where B is a parallelisable manifold with finite coverings having zero first Betti number. Therefore, the lattice $\Gamma = \pi_1(M)$ splits as $\Gamma = \mathbb{Z}^b \oplus P$ where $P = \pi_1(B)$. From the description in (13), one knows that B and hence N must be trivial. This implies that $\dim S = b$ (we do not claim that the $\mathbb{Z}^b$ factor is a lattice in S). On the other hand, one also sees that $P = \pi_1(B)$ must be finite: since Γ is virtually polycyclic, so is $\mathbb{Z}^b\backslash\Gamma = P$, but a virtually polycyclic group is either finite or it contains a finite index subgroup that has non-zero first Betti number.[1]Additionally, since $P < G$ is a finite subgroup, it is compact and therefore a subgroup of a maximal compact subgroup; up to an inner automorphism, we can assume that $P < K$.

[1]If D is solvable, then the derived series $D_k = [D_{k-1}, D_{k-1}]$, $D_0 = D$, terminates at 1 for some k. If each quotient D_{k-1}/D_k is finite, then D is finite; if D is not finite, then there is a least k such that D_k/D_{k+1} is infinite. This D_k is therefore of finite index with non-zero first Betti number.

Therefore M is finitely covered by $\hat{M} = \mathbb{T}^b \times \tilde{B}$ and Remarks 3.1 & 5.1 show that $\mathbb{T}^b$ is the closure of the projection of a 1-parameter subgroup of S. This proves that S is abelian.

Finally, let $\Gamma_1 < \Gamma$ be a torsion-free subgroup such that $\hat{M} = \Gamma_1\backslash G$. One knows that Γ_1 is generated by elements $\epsilon_i = e_i\delta_i$ for $i = 1, \ldots, b$ where $e_i \in S$, $\delta_i \in K$. Since Γ_1 is abelian, the δ_i pairwise commute and e_i commutes with δ_j for all $i \neq j$. From the argument of section 4.1.3 one knows that there are integers $n_i > 0$ such that $\delta_i^{n_i}$ generate a torus subgroup $T < K$. It follows that there are torsion elements $c_i \in K$ and $\xi_i \in \mathrm{Lie}\, T$ such that $\delta_i = c_i \exp(\xi_i)$ and the c_i pairwise commute and commute with all δ_j. Let us define $\epsilon_{i,t} = e_i c_i \exp(t\xi_i)$ and Γ_t be the lattice subgroup of G generated by $\epsilon_{i,t}$. The identity map on G induces a diffeomorphism of $\Gamma_0\backslash G$ with $\Gamma_1\backslash G = \tilde{M}$. The lattice Γ_0 is generated by $\epsilon_{i,0} = e_i c_i$. Since Γ_0 is abelian, the c_i must fix each c_j, $j \neq i$, and c_i must send e_i to $\pm e_i$. If $c_i e_i c_i^{-1} = -e_i$, then $\epsilon_{i,0}$ is a torsion element in the free abelian group Γ_0, hence it is 1, absurd. Therefore, c_i fixes e_i, too. Since $\{e_i\}$ generates a lattice in S, each c_i commutes with S. Therefore, $c_i \in \ker(K \longrightarrow \mathrm{Aut}(S))$ for each i.

To sum up: let $\Gamma_0^t \lhd \Gamma_0$ be the sublattice generated by the pure translations in Γ_0. Then $\Gamma_0^t\backslash G$ is diffeomorphic to $\mathbb{T}^b \times K$, a reductive Lie group and it is a smooth covering space of M. □

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