INTEGRABLE GEODESIC FLOWS WITH WILD FIRST INTEGRALS: THE CASE OF 2-STEP NILMANIFOLDS

TO APPEAR IN ERGODIC THEORY AND DYNAMICAL SYSTEMS

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ABSTRACT. This paper has four main results: (i) it shows that left-invariant geodesic flows on a broad class of 2-step nilmanifolds – which are dubbed almost non-singular [22, 39] – are integrable in the non-commutative sense of Nehorošev; (ii) the left-invariant geodesic flows on all Heisenberg-Reiter nilmanifolds are Liouville integrable [40]; (iii) the topological entropy of a left-invariant geodesic flow on a 2-step nilmanifold vanishes; (iv) there exist 2-step nilmanifolds with non-integrable left-invariant geodesic flows. It is also shown that for each of the integrable hamiltonians investigated here, there is a C^2-open neighbourhood in $C^2(T^*M)$ such that every integrable hamiltonian vector field in this neighbourhood must have **wild** first integrals.

1. Introduction

Riemannian geometry and hamiltonian mechanics intersect in the study of the geodesic flow of a riemannian metric. The dynamics of a geodesic flow can be both complicated enough to model many aspects of even more complicated hamiltonian systems, and simple enough to understand the geodesic flow's phase portrait – or at least important aspects of it. Since the 1970s, many new **integrable** dynamical systems have been discovered, amongst which are the Euler equations of a left-invariant metric on a semi-simple Lie group [42, 46, 3, 4] and geodesic flows on

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certain quotients of compact, semisimple Lie groups [54, 31, 32, 11]. By contrast, little is known about the integrability of geodesic flows on compact quotients of nilpotent or solvable Lie groups or even the integrability of their Euler equations (see [9, 12, 14, 13, 16] however).

1.1 Integrable Geodesic Flows on a Class of Two-step Nilmanifolds: This paper studies left-invariant geodesic flows on two classes of 2-step nilpotent Lie groups and their compact quotients. The former class is called *almost non-singular* after Eberlein's [22] analogous definition of non-singular 2-step nilpotent Lie groups, and was first studied by Lee and Park in [39]. The latter class consists of the so-called Heisenberg-Reiter (HR) groups, which generalize the classical Heisenberg group. Two-step nilpotent Lie groups are the "simplest" non-abelian Lie groups and their compact quotients – two-step nilmanifolds – are also deceptively "simple." Despite this, these groups and manifolds possess geometric properties quite unlike their abelian counterparts and have been studied intensively by geometers in [36, 2, 29, 22, 23, 44, 39]. These papers have principally addressed the connection between the length spectrum of the geodesic flow and the spectral properties of the associated laplacian. This paper studies these geodesic flows from the point-of-view of the hamiltonian formalism. The first results are:

Theorem 1.1. *(i) If G is a connected, simply connected 2-step nilpotent Lie group whose Lie algebra $\mathcal{G}$ is almost non-singular and rational, then for each discrete subgroup $D \leq G$ and each left-invariant riemannian metric g on G, the geodesic flow of g is smoothly integrable on $T^*(D\backslash G)$ in the non-commutative sense of Nehorošev* [48].
(ii) If G is a connected, simply connected 2-step nilpotent Lie group whose Lie algebra $\mathcal{G}$ is rational and HR (see 2.21), then for each discrete subgroup $D \leq G$ and each left-invariant riemannian metric g on G, the geodesic flow of g is smoothly Liouville integrable on $T^*(D\backslash G)$.

The definition of non-commutative integrability and Nehorošev's theorem is recalled below. The geodesic flows in theorem 1.1 are real analytic, while the first integrals are only C^∞. The next section attempts to explain why:

1.2 Wild First Integrals: Before we explain the notion of a tame/wild map, let us recall a related notion that frequently appears in the literature on integrable systems: non-degenerate integrability.

To explain non-degenerate integrability, suppose that (M^{2n}, ω) is a symplectic manifold with the hamiltonian action of the abelian Lie group $\mathcal{A} \simeq \mathbb{R}^n$ and $F : M^{2n} \to \mathfrak{a}^*$ is the momentum map of $\mathcal{A}$'s action ($\mathfrak{a} = \mathrm{Lie}(\mathcal{A})$), which is a submersion on an open dense set. For each $m \in M$, let K_m be the linear space of hamiltonians $f = \langle F, \xi \rangle$, $\xi \in \mathfrak{a}$, such that $df_m = 0$. Let $Q_m = \{d^2 f_m \ : \ f \in K_m\}$. Finally, let $L_m = T_m\mathcal{A}.m^\omega$ be the ω-orthogonal complement to the tangent space to $\mathcal{A}$'s orbit through m. Then L_m/L_m^ω is a symplectic vector space, and Q_m induces an abelian subalgebra of linear hamiltonian vector fields on L_m/L_m^ω, call it S_m. We say that $\mathcal{A}$'s action is *non-degenerate* if $\dim S_m = \frac{1}{2} \dim L_m/L_m^\omega$ for all $m \in M$. $\dim Q_m = \dim K_m$ for all $m \in M$ [55, 20, 19, 24, 35, 49]. A hamiltonian system is non-degenerately integrable if it is Liouville integrable and its integrals generate a non-degenerate action of $\mathbb{R}^n$. Note that the non-degeneracy of $\mathcal{A}$'s action is equivalent to $\dim Q_m = \dim K_m$ for all m; this shows that the condition is really a condition on the singular set of the momentum map.

Eliasson, Dufour-Molino and Ito [24, 20, 35] demonstrate that a non-degenerately integrable system admit singular action-angle variables in a neighbourhood of so-called elliptic singular strata of the first-integral map. Paternain [49] shows that the topological entropy of a non-degnerately integrable system must vanish, and has asked if non-degenerately integrable systems are generic (in the space of integrable systems) much like Morse functions are generic.

In the the 2-degree-of-freedom setting, Fomenko and his collaborators have called non-degenerately integrable systems *Bott integrable* and an extensive classification theorem has been deduced [25, 26]. One justification for studying this restricted class of integrable hamiltonian vector fields is that most known integrable mechanical systems are Bott-integrable [27, 38, 37, 5].

Subsequent to the development of a classification theorem for Bott-integrable hamiltonian vector fields on four-dimensional symplectic manifolds, Fomenko and Matveev [45] demonstrated that the types of bifurcations or surgeries of Liouville tori encountered with *tame* first integrals is no larger than the bifurcations encountered in Bott-integrable 4-dimensional systems. In their definition, a smooth map is tame if there is a triangulation of the singular set that extends to a neighbourhood.

We will adapt this definition: a smooth map $F : M \to N$ induces a stratification of N by strata $S_k := \{F(m) \ : \ \operatorname{rank} dF_m = k\}$ and M is stratified by sets $C_k = F^{-1}(S_k)$. The map F is *tame* if $C \subset M$ is a tamely embedded polyhedron and $(S, F(M))$ are simulaneously triangulable; it is *wild* otherwise. By a theorem of Hardt on the triangulability of images of proper real-analytic maps [33, 34], a real-analytically integrable geodesic flow on a compact manifold has a tame first-integral map [52].

Let $I(T^*M) \subset C^2(T^*M;\mathbb{R})$ denote the set of C^2 integrable hamiltonians on T^*M. Let us say that $H \in I(T^*M)$ is *tamely integrable* if it has a proper first-integral map $J : T^*M \to \mathbb{R}^m$ such that J is a tame map. Then:

Theorem 1.2. *Let $\mathcal{Q}$ be the set of compact real-analytic manifolds defined in theorem 1.1. Then for each $M \in \mathcal{Q}$:*
(i) T^*M possesses an integrable metric hamiltonian with a C^∞ first-integral map;
(ii) if $H \in C^2(T^*M;\mathbb{R})$ is an integrable mechanical hamiltonian, then there is a C^2-open neighbourhood of H in $I(T^*M)$, call it U_H, such that if $F \in U_H$ then F is not tamely integrable.

In particular, we see that on the class of smooth manifolds studied here the geodesic flows are not tamely integrable, and they are not even C^2 close to a tamely integrable hamiltonian system.

It is important to stress that Theorem 1.2 does not state that there do not exist non-degenerately integrable mechanical hamiltonians on the cotangent bundles of the manifolds in question — although this is almost certainly true. Desolneux-Moulis [19] observed that the first-integral map of a non-degenerately integrable system induces a Whitney stratification of the phase space, which implies the map's singular set is tamely embedded. However, it is unclear if this is sufficient for tameness of the first-integral map. The basic difficulty in proving tameness is establishing that the induced stratificaction of the image satisfies the strong "control" hypotheses required (see [51]).

1.3 The Liouville Foliation: Given a flow $\phi_t : X \to X$, there are natural stratifications of X induced by the C^k first integrals of ϕ_t: $\mathcal{F}^k = X/\sim$ where $x \sim y$ iff for all C^k first integrals, f, of ϕ_t the point y lies in the connected component of $f^{-1}(f(x))$ containing x. If ϕ_t is an integrable system with first-integral map

$F : X \to \mathbb{R}^m$, then we call the singular foliation of X whose leaves are the connected components of the level sets of F the *Liouville foliation* of ϕ_t associated to F. In general, this foliation depends on the particular choice of first-integral map. However, this foliation does contain a great deal of information about the dynamical behaviour of ϕ_t and in the case where we consider only the regular fibres of F and ϕ_t is anisochronous – the trajectories of ϕ_t are generically dense quasi-periodic windings on the regular fibres of F – then the foliation is essentially independent of F. In section 4 the Liouville foliation of an integrable geodesic flow on quotients of the $2n+1$-dimensional Heisenberg group is studied, and it is shown that the monodromy of the Liouville foliation reflects the algebraic structure of the fundamental group quite strongly. Specifically:

Theorem 1.3. *Let $D \leq G$ be a discrete, cocompact subgroup of the $2n+1$-dimensional Heisenberg group G, let D have the presentation: $D = \langle w_1, \ldots, w_n, v_1, \ldots, v_n, z \ : \ [w_i, v_j] = z^{\delta_{ij} k_i}$ for all $i, j = 1, \ldots, n, [z, .] = 1\rangle$ where k_j are positive integers such that $k_1 | \cdots | k_n$, and let $F \lhd D$ be the normal subgroup generated by $v_1, \ldots, v_n, z$.*
Let $\Psi : S^r \to \mathbb{R} \times \mathbb{R}^n_{>0} \times \mathbb{T}^n$ be the fibration of the dense, open subset S^r of $T^*(D\backslash G)$ by the Liouville tori of an integrable, left-invariant geodesic flow on $T^*(D\backslash G)$. The bundle Ψ has the monodromy group isomorphic to $D/F \simeq \mathbb{Z}^n$. The action of w_iF on a privileged basis $[C_j]$, $j = 1, \ldots, 2n+1$, of 1 cycles of the fibres of Ψ is given by:

$$w_i F * [C_j] = [C_j] + \delta_{ij} k_i [C_{n+1}] \tag{1}$$

In particular, there do not exist global action-angle coordinates of $\Psi : S^r \to \mathbb{R} \times \mathbb{R}^n_{>0} \times \mathbb{T}^n$.

One way to interpret this result is that the monodromy in the Liouville foliation causes the geodesic flow to be integrable with smooth but not tame first integrals. Indeed, the singular fibres of the first-integral map $J : T^*(D\backslash G) \to \mathbb{R}^{2n+1}$ (see section 4) consist of three types of fibres: the first type have a neighbourhood diffeomorphic to $\mathbb{T}^l[\theta] \times \mathbb{D}^{2k}[p] \times \mathbb{D}^l[I]$, where $l + k = 2n+1$, and the first integral map $\Psi(\theta, p, I) = (p_1^2 + p_2^2, \ldots, p_{2k-1}^2 + p_{2k}^2, I)$, i.e. there exist singular action-angle variables in a neighbourhood of the type I singular fibres; the type II singular fibres are invariant lagrangian $2n+1$-dimensional tori; and the type III singular fibre consists of the zero set of the momentum map of the action of $Z(G)/Z(D) \simeq \mathbb{T}^1$

on $T^*(D\backslash G)$. The type III singular fibre is a $4n+1$-dimensional submanifold of $T^*(D\backslash G)$, and the type II singular fibres accumulate onto it. The type III fibre is itself fibred into invariant lagrangian $2n+1$-dimensional submanifolds each of which is diffeomorphic to $D\backslash G$. It appears that the action of the monodromy group makes it possible for the topology of the type II singular fibres to change in the limit as they accumulate on the type III fibre.

1.4 The Vanishing of Topological Entropy: A second concern in the theory of dynamical systems is the relationship between the topological entropy of a flow and its integrability. In essence, the topological entropy of a flow measures the supremum of the rate of growth of separation of initially nearby solution curves. For an integrable system, there is a dense set fibred by invariant Liouville tori on which the topological entropy vanishes. However, in [9] Bolsinov and Taĭmanov give an example of a solvmanifold with an integrable geodesic flow and show that the singular set of this flow's first-integral map contains an invariant set on which the topological entropy of the flow is positive. Loosely speaking, integrable behaviour is not incompatible with chaotic behaviour. This paper shows that left-invariant geodesic flows on all 2-step nilmanifolds have zero topological entropy. Specifically, this paper proves:

Theorem 1.4. *Let G be a connected, simply connected, rational 2-step nilpotent Lie group and $D \le G$ be a discrete, cocompact subgroup of G. If g is a left-invariant metric on G and Φ_t is the geodesic flow induced by g on $T^*(D\backslash G)$ then*

$$h_{top}(\Phi) = 0.$$

1.5 Non-integrable Geodesic Flows on a Two-step Nilmanifold: The class of two-step nilmanifolds is rich in another important way: not only do some manifolds admit integrable left-invariant geodesic flows, but some do not, also:

Theorem 1.5. *Let $\mathcal{G}_3$ be the non-trivial extension of $\Lambda^2(\mathbb{R}^3)$ by $\mathbb{R}^3$ given by*

$$[x+y, x'+y'] := x \wedge x',$$

for all $x, x' \in \mathbb{R}^3$ and $y, y' \in \Lambda^2(\mathbb{R}^3)$. Let G_3 be the associated connected, simply connected 2-step nilpotent Lie group. Then for each discrete cocompact subgroup $D \le G_3$ there is a left-invariant metric g such that the geodesic flow of g on $T^*(D\backslash G_3)$ is non-integrable.

The proof of theorem (1.5) does not use the standard Poincaré-Melnikov method [30] – in light of theorem (1.4) it does not work! Instead, the periodic geodesics of g are studied directly and it is shown that these periodic geodesics carry enough algebraic structure to show that no locally trivial, flow-invariant foliation by tori can exist. It should also be noted *every* left-invariant geodesic flow is non-integrable on $T^*(D\backslash G_3)$, where D is discrete and cocompact. In fact, this is true for a wide class of 2-step nilmanifolds whose universal covering group G satisfies the algebraic condition that for $\mu \in \mathcal{G}^*$, there exists a $\mu' \in \mathcal{G}^*$ arbitrarily close to μ such that the stabilizers $\mathcal{G}_\mu$ and $\mathcal{G}_{\mu'}$ do not commute. The proof of this latter claim is more involved and will appear elsewhere (see [18]).

1.1. **Outline.** The plan of this paper is: section 2 proves theorem 1.1; section 3 proves theorem 1.2; section 4 studies the Liouville foliation of an integrable geodesic flow on $T^*(D\backslash G)$ where G is the Heisenberg group and proves theorem 1.3; section 5 demonstrates theorem 1.4; section 6 proves theorem 1.5.

1.2. **The Nehorošev Theorem.** The theorem of Nehorošev is recalled [48]:

Theorem 1.6 (Nehorošev, 1972)**.** *Let $F = (H = f_1, \ldots, f_{n-k}, g_1, \ldots, g_{2k})$ be a smooth map on the symplectic manifold (M^{2n}, Ω), $k \geq 0$, that satisfies the three conditions:*

i) *rank $dF = n + k$ on an open, dense subset of M^{2n};*
ii) *for all $a, b = 1, \ldots, n-k$ and all $c = 1, \ldots, 2k$: $\{f_a, f_b\} = \{f_a, g_c\} = 0$;*
iii) *for each regular value $c \in \mathbb{R}^{n+k}$, each connected component of $F^{-1}(c)$ is compact.*

If $c \in \mathbb{R}^{n+k}$ is a regular of F and $V \subset F^{-1}(c)$ is a connected component of the level set, then V is an embedded $n-k$-dimensional torus and there is an open neighbourhood U of V with local coordinates $f : U \to \mathbb{R}^{n-k}[I] \times \mathbb{T}^{n-k}[\theta] \times \mathbb{R}^k[p] \times \mathbb{R}^k[q]$ such that

i) *the local coordinates are canonical:*

$$\Omega|_U = f^*(\sum_{i=1}^{n-k} dI_i \wedge d\theta_i + \sum_{j=1}^{k} dp_j \wedge dq_j);$$

ii) *for $a = 1 \ldots, n-k$, $f_a = \tilde{f}_a \circ f$ and $\tilde{f}_a = \tilde{f}_a(I)$;*

iii) *The flow of X_H is conjugate to a translation-type flow on $\mathbb{T}^{n-k}$:*

$$X_{\tilde{H}} = \begin{cases} \dot{I}_i = 0, & \dot{p}_j = 0, \\ \dot{\theta}_i = \frac{\partial \tilde{H}(I)}{\partial I_i}, & \dot{q}_j = 0. \end{cases}$$

Remark 1.7. A hamiltonian H that satisfies the hypotheses of the above theorem will be referred to as *integrable in the non-commutative sense of Nehorošev* or simply *integrable*. It is clear that when $k = 0$, one gets the Liouville-Arnold theorem [1].

2. Two-Step Nilpotent Lie Groups

Let $\mathcal{G}$ be a 2-step nilpotent Lie algebra with center $\mathcal{Z} = Z(\mathcal{G})$, so that $[\mathcal{G}, \mathcal{G}] \subset Z(\mathcal{G})$, let $\langle , \rangle$ be an inner product on $\mathcal{G}$ and let

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{Z},$$

be an $\langle , \rangle$-orthogonal decomposition of $\mathcal{G}$. The Lie bracket on $\mathcal{G}$ is written as $[x + y, x' + y'] = [x, x']$ for all $x, x' \in \mathcal{H}$ and $y, y' \in \mathcal{Z}$, and so the commutator defines a skew-symmetric, bilinear form $\omega : \mathcal{H} \times \mathcal{H} \to \mathcal{Z}$ by $\omega(x, x') = [x, x']$.

The Lie algebra $\mathcal{G}$ can also be given the structure of a Lie group $(G, *)$ by $X * Y := X + Y + \frac{1}{2}[X, Y]$, so that $\mathcal{G} = \mathrm{Lie}(G)$ and the exponential map is the identity. In the sequel, elements in G will often be viewed as elements in $\mathcal{G}$ under the inverse (logarithm) map – which is the identity map in these coordinates. If D is a discrete, cocompact subgroup of G then there exists a generating set $X_1, \ldots, X_h, Y_1, \ldots, Y_z$ where $Y_1, \ldots, Y_z$ generate $Z(D)$ and the cosets $X_1 + Z(D), \ldots, X_h + Z(D)$ generate $D/Z(D)$ and $h = \dim \mathcal{H}$, $z = \dim \mathcal{Z}$ [41]. The generating set therefore determines a basis of $\mathcal{G}$ and an inner product $\langle , \rangle'$ relative to which it is an orthonormal basis.

Lemma 2.1. *Let $D \leq G$ be a discrete, cocompact subgroup and let $(,)$ be an inner product on $\mathcal{G}$. Then there exists an automorphism $f : G \to G$ and a subgroup $D' = f^{-1}(D)$ with generators $X_1, \ldots, X_h, Y_1, \ldots, Y_z$ such that $(X_i, Y_j) = 0$. In addition, if g is the left-invariant metric on G determined by $(,)$, then $(D' \backslash G, f^* g)$ is isometric to $(D \backslash G, g)$.*

Proof: Let $\mathcal{G} = \mathcal{H} \oplus \mathcal{Z}$ be the $\langle , \rangle$-orthogonal decomposition of $\mathcal{G}$. Let $a(x)$ be the $(,)$-orthogonal projection of $x \in \mathcal{H}$ onto $\mathcal{Z}$. The map $F : x + y \to x - a(x) + y$ for all $x \in \mathcal{H}$ and $y \in \mathcal{Z}$ is an automorphism of $\mathcal{G}$; let $f = \exp \circ F \circ \log$ be the map induced by F on G; f is an automorphism and by construction $F(\mathcal{H})$ is $(,)$-orthogonal to $\mathcal{Z}$. $\square$

This lemma is proven in [29] for Heisenberg groups. The importance of this lemma is that, by fixing a discrete, cocompact subgroup D with a fixed generating set, attention can be confined to those metrics that are block diagonal relative to this fixed basis of $\mathcal{G}$. Here and henceforth, $\langle , \rangle$ will be a fixed inner product on $\mathcal{G}$ relative to which $\mathcal{G} = \mathcal{H} \oplus \mathcal{Z}$, D will be a discrete, cocompact subgroup of G with $\langle , \rangle$-orthonormal generating set $X_1, \ldots, X_h, Y_1, \ldots, Y_z$, $\mathcal{H} = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_h\}$ and $\mathcal{Z} = \mathrm{span}_{\mathbb{R}}\{Y_1, \ldots, Y_z\}$ and $(,)$ will be a second inner product that is block diagonal: for all $X, X' \in \mathcal{H}$ and $Y, Y' \in \mathcal{Z}$

$$(X + Y, X' + Y') = \langle X, \mathcal{A}X \rangle + \langle Y, \mathcal{B}Y' \rangle \tag{2}$$

where $\mathcal{A}_{ij} = (X_i, X_j)$ and $\mathcal{B}_{kl} = (Y_k, Y_l)$. The metric g on G will be the left-invariant metric determined by $(,)$ or equivalently the pair $\mathcal{A}, B$.

Lemma 2.2. *Let $D \leq G$ be a discrete, cocompact subgroup with generators $X_1, \ldots, X_h$, $Y_1, \ldots, Y_z$. Let $(x, y) = (x^i X_i, y^j Y_j)$ be coordinates of a point in G. Then $X_l * (x, y) = (x + X_l, y + \frac{1}{2}[X_l, x])$ and $Y_k * (x, y) = (x, y + Y_k)$; that is: $x^i \circ X_l = x^i + \delta^i_l$, $x^i \circ Y_k = x^i$, $y^j \circ X_l = y^j + \frac{1}{2}\omega^j_{l\alpha} x^\alpha$ and $y^j \circ Y_k = y^j + \delta^j_k$.*

2.1. **Geodesic equations of motion.** Let $A : \mathcal{Z}^* \to so(\mathcal{H})$ be defined for all $x, x' \in \mathcal{H}$ and $q \in \mathcal{Z}^*$ by $\langle x, A(q)x' \rangle := q \circ [x, x']$. Let (x, y, p, q) be the coordinates of a point in $T^*G = \mathcal{H} \times \mathcal{Z} \times \mathcal{H}^* \times \mathcal{Z}^*$ via left trivialization. The hamiltonian of the metric g on T^*G is $H_g = \frac{1}{2}\langle p, Rp \rangle + \frac{1}{2}\langle q, Sq \rangle$ where $R = \mathcal{A}^{-1}$ and $S = \mathcal{B}^{-1}$. The equations of motion are

$$X_{H_g} = \begin{cases} \dot{q} \;=\; 0, & \dot{y} \;=\; Sq + \frac{1}{2}[x, Rp], \\ \dot{p} \;=\; -A(q)Rp, & \dot{x} \;=\; Rp. \end{cases} \tag{3}$$

Then q is a $\mathcal{Z}^*$-valued first integral of X_{H_g} and $F := p + A(q)x$ is an $\mathcal{H}^*$-valued first integral. Let $q_i := q(Y_i)$ and $F_j := F(X_j)$ for $i = 1, \ldots, z$ and $j = 1, \ldots, h$.

2.2. **First integrals.** Let $R^{\frac{1}{2}}$ denote the unique positive definite square root of R, and let $v = R^{\frac{1}{2}}p$ and $B(q) := R^{\frac{1}{2}}A(q)R^{\frac{1}{2}}$. Then $\dot{v} = -B(q)v$. Let κ be -1 times the Killing form on $so(\mathcal{H})$ and let $L : q \to \mathrm{ad}_{B(q)}$ be the map $\mathcal{Z}^* \to so(so(\mathcal{H}))$. If r is the maximal rank of $L(q)$, then the set of q such that rank $L(q) = r$ is open and dense in $\mathcal{Z}^*$; call this set $\mathcal{Z}^*_r$. Let $so(\mathcal{H}) = C(q) \oplus F(q)$ be the eigenspace decomposition relative to $L(q)$, where $C(q) = \ker L(q)$ and $F(q)$ is the $[,]$-orthogonal complement, which is $L(q)$-invariant.

Lemma 2.3. *There exist smooth sections $\mathcal{Z}^* \to so(\mathcal{H})$, $q \to C_i(q) \in C(q)$, $i = 1, \ldots, s$ such that $C_1(q), \ldots, C_s(q)$ is a basis of $C(q)$ for an open, dense set of $q \in \mathcal{Z}_r^*$.*

Proof: Define the "centralizer bundle" to be the bundle $C \to \mathcal{Z}^*$ with fibre $C(q)$ over $q \in \mathcal{Z}^*$; this bundle is naturally a sub-bundle of the trivial bundle $so(\mathcal{H}) \times \mathcal{Z}^* \to \mathcal{Z}^*$ and consequently there is a natural norm $|.|$ on the fibres of C induced by κ. When restricted to $\mathcal{Z}_r^*$, $C|_{\mathcal{Z}_r^*}$ is a real-analytic vector bundle of rank s, where $s = \frac{1}{2}h(h-1) - r$ is the generic dimension of $C(q)$. Consequently, there exists s real-analytic sections of $C|_{\mathcal{Z}_r^*}$ that are linearly independent over an open, dense subset of $\mathcal{Z}_r^*$; let these be denoted by $S_1, \ldots, S_s$. Let $\phi(x) := \exp(-1/x^2)$ and let $m(q)$ be the sum of all squared $r \times r$ minors of $L(q)$ and let $k(q) := \Pi_{i=1}^s\ \phi(|S_i(q)|)|S_i(q)|^{-1}$. It is clear that $\phi \circ m$ and k extend to smooth functions on $\mathcal{Z}^*$ that are non-zero on an open, dense subset. Let $C_i(q) := \phi(m(q))\, k(q)\, S_i(q)$ for $q \in \mathcal{Z}_r^*$ and 0 elsewhere. It is clear that C_i is a smooth section of the trivial bundle $so(\mathcal{H}) \times \mathcal{Z}^* \to \mathcal{Z}^*$ whose image lies in C and $C_1(q), \ldots, C_s(q)$ is a basis of $C(q)$ for an open, dense subset of $q \in \mathcal{Z}^*$. $\square$

Lemma 2.4. *There exist smooth sections $\mathcal{Z}^* \to so(\mathcal{H})$, $q \to D_i(q)$, $i = 1, \ldots, s$ such that $D_1(q), \ldots, D_s(q)$ is a basis of $C(q)$ for an open, dense set of $q \in \mathcal{Z}_r^*$ and $D_1(q), \ldots, D_n(q)$ span an abelian subalgebra for all $q \in \mathcal{Z}^*$ where $n = [\frac{h}{2}] =$ rank $so(\mathcal{H})$.*

Proof: For any $q_0 \in \mathcal{Z}_r^*$, the centralizer $C(q_0)$ of $B(q_0)$ contains an element X that is in general position. Let $q_0 \in \mathcal{Z}_r^*$ be such that the real-analytic sections $S_1, \ldots, S_s : \mathcal{Z}_r^* \to C$ evaluated at q_0 forms a basis of $C(q_0)$. Then $X = \sum_{i=1}^s x_i S_i(q_0)$ for some $x_1, \ldots, x_s \in \mathbb{R}$. Let $X(q) := \sum_{i=1}^s x_i S_i(q)$; by real-analyticity, $X(q)$ is in general position for an open, dense set of $q \in \mathcal{Z}_r^*$. Let $Y(q) := \sum_{i=1}^s x_i C_i(q)$, so that $Y(q) = \phi(m(q))\, k(q)\, X(q)$, is a smooth section of C. $Y(q)$ is in general position for an open dense set of $q \in \mathcal{Z}_r^*$. The sections $Y(q), Y(q)^3, \ldots, Y(q)^{2n-1} \in C(q)$ are therefore linearly independent for an open dense set of q and span an abelian subalgebra. Let $q_1 \in \mathcal{Z}^*$ be some such generic element; then by adding in $s - n$ additional elements from $\{C_1(q_1), \ldots, C_s(q_1)\}$ – say the final $s - n$ elements – the set $\{Y(q_1), Y(q_1)^3, \ldots, Y(q_1)^{2n-1}\}$ can be completed to a basis of $C(q_1)$. It is clear that letting $D_i = Y^{2i-1}$ for $i = 1, \ldots, n$ and $D_i = C_i$ for $i = n+1, \ldots, s$ gives the desired sections. □

Lemma 2.5. *The functions*

$$h_i(p,q) := \langle v, D_i(q)^2 v\rangle = \langle p, R^{\frac{1}{2}} D_i(q)^2 R^{\frac{1}{2}} p\rangle$$

for $i = 1,\dots,s$ *where* $s = \frac{1}{2}h(h-1) - r \geq n$*, are smooth, functionally independent first integrals of* X_{H_g}*. For all* $i = 1,\dots,s$*,* $j = 1,\dots,z$ *and* $l = 1,\dots,n$*:* $\{h_i, q_j\} = \{h_i, f_l\} = 0$*. For* $i,j = 1,\dots,n$*:* $\{h_i,h_j\} = 0$*.*

Proof: X_{h_i} on $\mathcal{G}^*$ is given by $\dot{v} = -B(q)D_i(q)^2 v$, $\dot{q} = 0$ so that $\{h_i,h_j\} = -\langle B(q)D_i(q)^2 v, D_j(q)^2 v\rangle - \langle v, D_j(q)^2 B(q) D_i(q)^2 v\rangle = 0$, because D_i, D_j are commuting sections of the centralizer bundle for B. Because h_i is left-invariant, it Poisson commutes with the right-invariant hamiltonians q_j and f_l. □

Let now $\mathcal{H}^* = K(q) \oplus F(q)$ where $K(q) = \ker B(q)$ and $F(q)$ is the $B(q)$-invariant, $\langle,\rangle$-orthogonal complement of $K(q)$. Let $K \to \mathcal{Z}^*$ be the sub-bundle of $\mathcal{H}^* \times \mathcal{Z}^* \to \mathcal{Z}^*$ whose fibre at q is $K(q)$. The previous arguments may be repeated almost verbatim to prove that:

Lemma 2.6. *Let* $k = \inf \dim K(q)$*, and suppose* $k > 0$*. Then, there exists smooth sections* $K_1,\dots,K_k : \mathcal{Z}^* \to K$ *such that* $K_1(q),\dots,K_k(q)$ *forms a basis of* $K(q)$ *for an open dense set of* $q \in \mathcal{Z}^*$*.*

Lemma 2.7. *The smooth functions*

$$k_i := \langle K_i(q), v\rangle = \langle R^{\frac{1}{2}} K_i(q), p\rangle$$

are independent, Poisson commuting first integrals of X_{H_g} *for* $i = 1,\dots,k$*.*

Proof: The functions $k_i(p,q)$ are Casimirs of the Poisson tensor on $\mathcal{G}^*$. To see this, it will be shown that for each $\mu = p + q \in \mathcal{G}^*$ the hamiltonian vector field $X_{k_i}(\mu) = \mathrm{ad}^*_{dk_i(\mu)}\mu$ vanishes. Recall that there is a canonical identification of $\mathcal{G}$ with $\mathcal{G}^{**}$ so that $dk_i(p+q) = R^{\frac{1}{2}}K_i(q) + \sum_{j=1}^{z} \langle R^{\frac{1}{2}} \frac{\partial K_i}{\partial q_j}, p\rangle Z_j$. Because $p|_{[\mathcal{G},\mathcal{G}]} = 0$:

$$\mathrm{ad}^*_{dk_i(p+q)}\mu = \mathrm{ad}^*_{R^{\frac{1}{2}}K_i(q)} q. \tag{4}$$

Therefore, for all $x \in \mathcal{G}$: $(\mathrm{ad}^*_{dk_i(p+q)} q)(x) = -q \circ [R^{\frac{1}{2}}K_i(q), x] = \langle x, A(q) R^{\frac{1}{2}} K_i(q)\rangle$, which is identically zero by hypothesis. Therefore k_i Poisson commutes with all left-invariant hamiltonians. □

2.3. **First integrals on Almost Non-Singular 2-Step Nilpotent Lie Groups.**

Definition 2.8 (Almost Non-Singular Lie Algebras)**.** *Let* $A : \mathcal{Z}^* \to so(\mathcal{H})$ *be the linear map defined by* $\langle x, A(q)x' \rangle = q \circ [x, x']$ *for all* $x, x' \in \mathcal{H}$ *and* $q \in \mathcal{Z}^*$*. The 2-step nilpotent Lie algebra* $\mathcal{G}$ *is* almost non-singular *if there exists* $q \in \mathcal{Z}^*$ *such that* $\det A(q) \neq 0$.

Remark 2.9. (i) Because the map A is linear, $\det A(q)$ is an algebraic function so that if it is non-zero at some point q, it is non-zero on an open dense subset of $\mathcal{Z}^*$. (ii) An equivalent definition of an almost non-singular 2-step nilpotent Lie algebra $\mathcal{G}$ is one for which $d\mu$ has a nullity equal to $\dim \mathcal{Z}$ for some $\mu \in \mathcal{G}^*$. The exterior derivative of $\mu \in \mathcal{G}^*$ is defined by $d\mu(x, y) := -\mu([x, y])$ for all $x, y \in \mathcal{G}$. (iii) A third, equivalent definition of an almost non-singular 2-step nilpotent Lie algebra is that for some $\mu \in \mathcal{G}^*$, the isotropy algebra $\mathcal{G}_\mu = \{x \in \mathcal{G} \ : \ \mathrm{ad}^*_x \mu = 0\}$ is equal to $Z(\mathcal{G})$. (iv) A fourth way to characterize an almost non-singular Lie algebra is that there exists a $\mu \in \mathcal{G}^*$ such that $d\mu$ induces a symplectic form on $\mathcal{G}/Z(\mathcal{G})$. (v) In [47] there is a consideration of the representation theory of nilpotent Lie groups with property (iv).

Remark 2.10. In the sequel, $\mathcal{G} = \mathcal{H} \oplus \mathcal{Z}$ will be an almost non-singular 2-step nilpotent Lie algebra, $\dim \mathcal{H} = 2n$ and $\dim \mathcal{Z} = m$ for some integers $n, m \geq 1$.

Lemma 2.11. *Let* $\phi(u) = \exp(-1/u^2)$ *for all* $u \in \mathbb{R}$*, and* $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ *be a smooth, 1-periodic function. Suppose that* $\mathcal{G}$ *is almost non-singular, and that* D *is a discrete, cocompact subgroup of* G*. Then for each* $i = 1, \ldots, 2n = \dim \mathcal{H}$*,*

$$f_i(x, p, q) := \phi(\det A(q))\psi(\langle A(q)^{-1}p + x, X_i \rangle)$$

is a smooth function on T^*G *that is invariant under the action of* D *and so descends to a smooth function on* $T^*(D\backslash G)$*.*

Proof: Left-trivialization gives $T^*G = G \times \mathcal{G}^*$ and the left action of G on T^*G becomes simply left translation by G on the first factor. The action of the generators of D on G, lemma (2.2), means that $\langle A(q)^{-1}p + x, X_i \rangle$ mod 1 is invariant. $A(q)\, \mathrm{adj}\, A(q) = \det A(q) I$, so since $A(q)$ is a linear function of q, $A(q)^{-1}$ is a rational function of q. The singularities of $A(q)^{-1}$ are the zeros of $\det A(q)$. Because ψ is smooth and 1-periodic all of its derivatives are bounded, so the product $\phi(\det A(q))\, \psi(\langle A(q)^{-1}p + x, X_i \rangle)$ vanishes to all orders along the singular set

$\det A(q) = 0$. Therefore, f_i is a C^∞ function. □

Remark 2.12. The functions $f_1, \ldots, f_{2n}$ are first integrals of all left-invariant vector fields on T^*G, in particular of X_{H_g}, because they are functions of the hamiltonians of cotangent lifts of right-invariant vector fields on G.

In the almost non-singular case, lemmas (2.4,2.5) can be strenghtened for generic left-invariant geodesic flows. By hypothesis, $B(q)$ is $\langle , \rangle$-skew symmetric and non-degenerate for almost all $q \in \mathcal{Z}^*$. The skew-symmetric matrix $B(q)$ is in general position if it possesses $2n$ distinct eigenvalues. If $B(q)$ is in general position for some $q \in \mathcal{Z}^*$, then it is in general position for an open dense subset of q; it is clear that for an open, dense subset of quadratic forms R, $B(q)$ is in general position.

Definition 2.13. *The linear map $B : \mathcal{Z}^* \to so(\mathcal{H})$ is in* general position *if for some $q \in \mathcal{Z}^*$, $B(q)$ has $2n = \dim \mathcal{H}$ distinct eigenvalues.*

Lemma 2.14. *Let $B : \mathcal{Z}^* \to so(\mathcal{H})$ be in general position. The functions*

$$h_i(p, q) := \langle v, B(q)^{2i-2} v \rangle = \langle p, R^{\frac{1}{2}} B(q)^{2i-2} R^{\frac{1}{2}} p \rangle$$

are first integrals of X_{H_g} for all $i \geq 1$; moreover $h_1, \ldots, h_n$ are functionally independent on an open, dense subset of T^*G.

Remark 2.15. The functions $h_i : \mathcal{G}^* \to \mathbb{R}$ constructed in lemmas (2.5,2.14) clearly descend to any quotient $T^*(D\backslash G) = (D\backslash G) \times \mathcal{G}^*$ as first integrals of X_{H_g}. They also Poisson commute with f_j and q_l (lemma 2.11, equation 3) for all j and l.

Proof of Theorem (1.1, i): The vector field X_{H_g} has $n + m$ Poisson commuting first integrals from lemmas (2.14,2.5) and equation (3). From lemmas (2.11,2.5), X_{H_g} has an additional $2n$ first integrals that are first integrals of the $n+m$ first integrals h_i and q_l. Functional independence is obvious. Therefore, X_{H_g} has $3n + m$ independent first integrals, and $n + m$ of these first integrals commute with all $3n + m$. Since $\dim T^*(D\backslash G) = 4n + 2m$, this proves the non-commutative integrability of X_{H_g}. □

2.4. **Liouville Integrability of Left-Invariant Geodesic Flows on HR manifolds.**

Definition 2.16. Let $\mathcal{W}, \mathcal{V}, \mathcal{Z}$ be non-trivial finite-dimensional vector spaces over $\mathbb{R}$ and let λ be a bilinear mapping $\mathcal{W} \times \mathcal{V} \to \mathcal{Z}$. Define the Lie algebra $\mathcal{G}_\lambda = \mathcal{G} := \mathcal{W} \oplus \mathcal{V} \oplus \mathcal{Z}$ with Lie bracket: $[w + v + z, w' + v' + z'] := \lambda(w, v') - \lambda(w', v)$. Such a Lie algebra will be call an HR-λ Lie algebra [40].

Lemma 2.17. *An HR-λ Lie algebra is an almost non-singular 2-step nilpotent Lie algebra iff $\exists c \in \mathcal{Z}^*$ such that $c \circ \lambda$ induces an isomorphism $\mathcal{V} \simeq \mathcal{W}^*$.*

Proof: Let $\mu = a+b+c \in \mathcal{W}^* \oplus \mathcal{V}^* \oplus \mathcal{Z}^*$ and observe that $d\mu(w+v+z, w'+v'+z') = -c \circ \lambda(w, v') + c \circ \lambda(w', v)$. Let $\langle , \rangle$ be some fixed inner product on $\mathcal{G}_\lambda$ relative to which $\mathcal{W} \oplus \mathcal{V} \oplus \mathcal{Z}$ is an orthogonal direct sum and define $\alpha : \mathcal{Z}^* \to \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ by:

$$\langle \alpha(c)v, w \rangle := c \circ \lambda(w, v) \tag{5}$$

for all $c \in \mathcal{Z}^*$, $w \in \mathcal{W}$ and $v \in \mathcal{V}$. With this convention the linear map $A : \mathcal{Z}^* \to so(\mathcal{W} \oplus \mathcal{V})$ is given by:

$$A(c) = \begin{bmatrix} 0 & \alpha(c) \\ -\alpha(c)' & 0 \end{bmatrix}, \tag{6}$$

where $\alpha(c)'$ is the transposed map. Therefore $\det A(c)^2 = \det \alpha(c)'\alpha(c) \det \alpha(c)\alpha(c)'$. This is non-zero for some c iff $\alpha(c)$ is a bijection iff $v \to \alpha(c)v$ is an isomorphism of $\mathcal{V}$ with $\mathcal{W}^*$. $\square$

Remark 2.18. The map α is linear in c, and injectiveness of $\alpha(c)$ is characterized by the non-vanishing of the sum of squared $l \times l$ minors of $\alpha(c)$ so if $\alpha(c)$ is injective for some c, then $\alpha(c)$ is injective for all c in the complement of an algebraic set.

Remark 2.19. An alternative proof of the previous lemma is this: by the characterization of remark 2.9.iv, a 2-step nilpotent Lie algebra is almost non-singular iff there exists $\mu \in \mathcal{G}^*$ such that $d\mu$ is a symplectic form on $\mathcal{G}/Z(\mathcal{G}) \simeq \mathcal{V} \oplus \mathcal{W}$. Since $\mathcal{V}$ (resp. $\mathcal{W}$) is clearly a $d\mu$-isotropic subspace, its $d\mu$-symplectic dual is contained in $\mathcal{W}$ (resp. $\mathcal{V}$). By symmetry, $\mathcal{V} \simeq \mathcal{W}^*$.

Remark 2.20. The HR Lie algebra $\mathcal{G}_\lambda$ is obviously independent of its presentation. A canonical way to fix this presentation is to take the presentation of $\mathcal{G}_\lambda$ given by shrinking $\mathcal{W}$ (resp. $\mathcal{V}$) by the left (resp. right) kernel of λ: $\mathcal{G}_\lambda \simeq \left(\mathcal{W}/\ker^L \lambda\right) \oplus \left(\mathcal{V}/\ker^R \lambda\right) \oplus \mathcal{Z}'$ where $\mathcal{Z}' = \mathcal{Z} \oplus \ker^L \lambda \oplus \ker^R \lambda$.

Definition 2.21. *Let $\mathcal{G}_\lambda$ be an HR Lie algebra with $\dim \mathcal{W} \geq \dim \mathcal{V}$. If the bilinear map $c \circ \lambda \simeq \alpha(c) : \mathcal{V} \to \mathcal{W}$ induces an injection of $\mathcal{V} \to \mathcal{W}^*$ for some $c \in \mathcal{Z}^*$ then the presentation $\mathcal{G}_\lambda = \mathcal{V} \oplus \mathcal{W} \oplus \mathcal{Z}$ will be said to be an* injective presentation.

From the previous remark, it is clear that any HR Lie algebra admits an injective presentation.

Theorem 2.22 (Theorem (1.1, ii))**.** *Let $\mathcal{G}_\lambda$ be a rational, HR-λ Lie algebra and $G = G_\lambda$ its associated Lie group. Then for all left-invariant metrics g on G, the geodesic flow of g is Liouville integrable on $T^*(D\backslash G)$ for all cocompact, discrete subgroups D.*

Proof: Let $\mathcal{G} = \mathcal{V} \oplus \mathcal{W} \oplus \mathcal{Z}$ be an injective presentation of $\mathcal{G}$. From lemma (2.1) and the subsequent discussion, a generating set of D, denoted by $w_1, \ldots, w_k$, $v_1, \ldots, v_l$ and $z_1, \ldots, z_m$ exists where w_i (resp. v_i, z_i) lie in (commutative!) subalgebras of $\mathcal{G}$ isomorphic to $\mathcal{W}$ (resp. $\mathcal{V}$, $\mathcal{Z}$). Define $\langle , \rangle$ so that this basis of $\mathcal{G}$ is $\langle , \rangle$-orthonormal and let $\mu = a + b + c \in \mathcal{W}^* \oplus \mathcal{V}^* \oplus \mathcal{Z}^* = \mathcal{G}^*$ be the coordinates of a covector relative to the induced splitting of $\mathcal{G}^*$. The left-invariant metric hamiltonian associated with the left-invariant metric g can be written as

$$2H_g = \langle a, \mathrm{A}a \rangle + 2\langle a, \mathrm{B}b \rangle + \langle b, \mathrm{C}b \rangle + \langle c, \mathrm{D}c \rangle, \tag{7}$$

where notation is abused and $\langle , \rangle$ denotes both the inner product on $\mathcal{G}$ and its various restrictions. The transformations $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D are defined as previously. The vector field X_{H_g} on $T^*G = G \times \mathcal{G}^*$ is then

$$X_{H_g} = \left\{ \begin{array}{rclrcl} \dot{a} & = & -\alpha(c)[\mathrm{B}'a + \mathrm{C}b], & \dot{w} & = & \mathrm{A}a + \mathrm{B}b, \\ \dot{b} & = & \alpha(c)'[\mathrm{A}a + \mathrm{B}b], & \dot{v} & = & \mathrm{B}'a + \mathrm{C}b, \\ \dot{c} & = & 0, & \dot{z} & = & \mathrm{D}c + \frac{1}{2}[v + w, \mathrm{A}a + \mathrm{B}b + \mathrm{B}'a + \mathrm{C}b], \end{array} \right. \tag{8}$$

where $'$ indicates the transpose. Clearly $a + \alpha(c)v$ (resp. $b - \alpha(c)'w$) is a $\mathcal{W}^*$ (resp. $\mathcal{V}^*$) -valued first integral of X_{H_g}.

Let $m(c) = \det \alpha(c)'\alpha(c)$. Then $\{m(c) = 0\}$ is precisely the set of $c \in \mathcal{Z}^*$ for which $\alpha(c)$ is not an injective map. By hypothesis, $\alpha(c)$ is injective for some $c \in \mathcal{Z}^*$, so $m(c) \not\equiv 0$. There exists a unique left inverse $L(c)$ of $\alpha(c)$ that is defined on the open, dense set $\{m(c) \neq 0\}$ as follows: the symmetric operator $s(c) = \alpha(c)'\alpha(c)$

is positive definite on the set $\{m(c) \neq 0\}$ so there exists the inverse $s(c)^{-1} = (\alpha(c)'\alpha(c))^{-1}$ on this set; then $L(c) := (\alpha(c)'\alpha(c))^{-1}\alpha(c)'$. It is clear that on the set $\{m(c) \neq 0\}$, $L(c)$ is a real-analytic function (rational, even) in c. Extend L to a function $L : \mathcal{Z}^* \to \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ by setting $L(c) = 0$ on the set $\{m(c) = 0\}$. With $\phi(.) = \exp(-1/(.)^2)$ the functions $f_i := \phi(m(c)) \sin 2\pi\langle L(c)a + v, v_i\rangle$, $i = 1, \ldots, l$, are seen to be smooth functions on $T^*(D\backslash G)$.

From lemma (2.5) and equation (8), the functions $c_1, \ldots, c_m$, $h_1, \ldots, h_l$ and the functions $f_1, \ldots, f_l$, form a commutative Poisson algebra of independent first integrals of X_{H_g}. From lemma (2.6) there exist $k - l$ smooth sections from $\mathcal{Z}^*$ to the kernel of $A(c)$ but since $\alpha(c)$ is injective almost everywhere, these sections are into the kernel of $\alpha(c)'$. These sections provide an additional $k - l$ first integrals that are Casimirs of the Poisson bracket on $\mathcal{G}^*$, and so they are in involution with all other first integrals (see lemma (2.7)). This gives $m + k + l = \frac{1}{2} \dim T^*(D\backslash G)$ independent, involutive first integrals of X_{H_g}. $\square$

Remark 2.23. The simplest case of theorem (2.22) occurs when $\mathcal{V}, \mathcal{W}, \mathcal{Z} = \mathbb{R}$ and $\lambda = 1$, which gives the classical 3-d Heisenberg group. The $2n + 1$-dimensional Heisenberg group appears when $\mathcal{V}, \mathcal{W} = \mathbb{R}^n$, $\mathcal{Z} = \mathbb{R}$ and λ is the standard inner product on $\mathbb{R}^n$. The case where $\mathcal{V} = \mathbb{R}$, $\mathcal{W}, \mathcal{Z} = \mathbb{R}^n$ and λ is scalar multiplication of $\mathcal{V}$ on $\mathcal{W}$ is studied in [12, 14] where it is shown that for $n \geq 2$ the geodesic flows are Liouville integrable and generically quasiperiodic and non-degenerate in the sense of KAM theory.

3. Wild First Integrals

In this section we will prove theorem 1.2.ii: that if $H \in C^2(T^*M)$ is C^2 close to an integrable geodesic flow constructed in the previous sections, and H is integrable, then the first-integral map for H must be wild.

The proof relies on an important fact from convex geometry: if $\mathcal{K}$ is a compact strictly convex subset of finite-dimensional vector space V, and $0 \in \mathcal{K}$, then there is a compact strictly convex set $\mathcal{K}^* \subset V^*$ containing 0 that is naturally "dual" to $\mathcal{K}$ — and $\mathcal{K}^*$ is as smooth as $\mathcal{K}$. The duality of $\mathcal{K}$ and $\mathcal{K}^*$ is, in fact, simply a reflection of the Legendre transformation and it is involutive: $\mathcal{K}^{**} = \mathcal{K}$ (see [28], section 3.2).

On the other hand, if a function $f : V \to \mathbb{R}$ is C^2, then f is a strictly convex function iff for all $x \in V$, $d^2 f_x$ is a positive definite quadratic form. Clearly, if

$g \in C^2(V)$ is C^2-sufficiently close to f on a compact, convex set K, then $g|K$ is also a strictly convex function.

The idea of our proof is: if F is C^2 close to a metric (or a mechanical) hamiltonian, then the sublevel sets of F are also fibre-wise compact strictly convex sets. That is, the sets $\{F \leq c\}$ intersect each fibre $T^*_m M$ in a compact strictly convex set. Thus, the convex duals of F's sublevels (which lie in TM) are also compact and strictly convex, so this allows us to define a C^2 lagrangian on TM which is proper and strictly convex. Compactness implies the Euler-Lagrange flow is complete, and strict convexity implies that the Euler-Lagrange flow satisfies the Hopf-Rinow property. A theorem due to Taĭmanov is adapted here to deduce that if F is tamely integrable, then $\pi_1(M)$ must have an abelian subgroup of finite index. This will prove that M cannot be a compact 2-step nilmanifold.

All objects (maps, flows, manifolds, etc.) in this section will be C^2 unless stated otherwise.

3.1. **Geometric Simplicity and Hopf-Rinow.**

Let $\pi : E \to M$ be a fibre bundle and $\phi_t : E \to E$ a complete flow.

Definition 3.1. *If, for each $m \in M$ and each non-trivial $[c] \in \pi_1(M; m)$ there exists $p \in \pi^{-1}(m)$ and a $T > 0$ such that $\gamma(t) := \pi\phi_{tT}(p)$, $0 \leq t \leq 1$, is a closed curve homotopic to c, then we say ϕ_t is a* Hopf-Rinow *flow.*

We will say that a vector field is Hopf-Rinow if its flow is. Observe that if two flows are orbitally equivalent and one is Hopf-Rinow, then so is the other.

Let us now state a result which we will use below. We have adapted the definition of *geometric simplicity* that Taĭmanov uses in [52, 53]:

Definition 3.2 (c.f. [52, 53])**.** *Let M be a C^1 manifold, E a compact fibre bundle over M, $\phi_t : E \to E$ a complete flow, and suppose that $E = \Gamma \amalg L$ such that:*
(GS1) Γ is closed, ϕ_t invariant and nowhere dense;
(GS2) for each $p \in E$ and open neighbourhood $U \ni p$, there is an open neighbourhood W of p, $W \subseteq U$, such that $L \cap W$ has finitely many path-connected components;
(GS3) $L = \amalg_{i=1}^{k} L_i$ and each L_i is an open path-connected component of L and is homeomorphic to $\mathbb{T}^l \times \mathbb{D}^m$ $(l + m = \dim E)$.
Then we will say that ϕ_t is geometrically simple.

Theorem 3.3 (c.f. Taĭmanov [52, 53])**.** *Let E be a compact fibre bundle over M. If $\phi_t : E \to E$ is Hopf-Rinow and geometrically simple, then $\pi_1(M)$ has a finite-index abelian subgroup.*

In [52, 53], Taĭmanov assumes that ϕ_t is a geodesic flow on the unit tangent bundle, but only the Hopf-Rinow property and geometric simplicity are used to prove the theorem; the theorem stated here is an immediate consequence of his proof. Note also that the fibre bundle E may have a boundary; the example we have in mind is the unit disk bundle in TM.

3.2. **Tame Integrability.** Let's recall the notion of *tameness* that was mentioned in the introduction. We will say that a topological space is a polyhedron if it is homeomorphic to a locally compact simplicial complex; in this case we will also say that the space is triangulable. If $K \subset L$ and L admits a triangulation that extends a triangulation of K, then we will say that the pair (K, L) is triangulable. A subset $K \subset M$ is said to be a *tamely embedded polyhedron* if there is a neighbourhood $L \subset M$ of K, such that (K, L) is triangulable. In other words, there is a triangulation of K that is extendable to a neighbourhood of K in M.

Definition 3.4. *Let $F : M \to N$ be a C^1 map, $S \subset N$ the critical-value set of F and $C = F^{-1}(S)$. F is* tame *if (T1) C is a tamely embedded polyhedron in M; and (T2) $(S, F(M))$ is triangulable. If F is not tame, we say F is* wild.

We will say a hamiltonian flow is tamely integrable if it has a proper first-integral map which is a tame map; otherwise, we say it is wildly integrable.

If M is compact, M and N are real-analytic manifolds (possibly with boundary) and F is a real-analytic map, then C (resp. S) is a compact subanalytic subset of M (resp. S). A theorem of [33, 34] asserts that both (C, M) and $(S, F(M))$ are triangulable (see [50, 51] for further references).

Lemma 3.5. *If $\phi_t : T^*M \to T^*M$ is tamely integrable, $\xi \subset T^*M$ is a compact, ϕ_t-invariant disk sub-bundle and $\phi_t|\xi$ is Hopf-Rinow, then there is a compact disk sub-bundle E containing ξ such that $\phi_t|E$ is Hopf-Rinow and geometrically simple.*

The following is inspired by a similar proof in [52].
Proof: Clearly, if $\xi \subset E$, E is invariant and $\phi_t|\xi$ is Hopf-Rinow, then $\phi_t|E$ is Hopf-Rinow.

Let $J : T^*M \to \mathbb{R}^m$ be a tame first-integral map for ϕ_t. Let C be the critical-point set for J, $S = J(C)$. Thus, C is a tamely embedded polyhedron in T^*M and $(S, \operatorname{im} J)$ is triangulable.

Let N be a compact polyhedral neighbourhood in $\operatorname{im} J$ that contains $J(\xi)$. Since ξ is compact, the neighbourhood N exists. Let $E = J^{-1}(N)$. Since J is proper, E is compact.

Let $S(N)$ be the $m-1$ skeleton of N and let $\Gamma = J^{-1}(S(N))$, and $L = E - \Gamma$. The invariance of Γ is obvious. Because J is tame and has at least one regular value, the set $S(N)$ contains all critical values of $J|E$ and Γ contains all critical points of $J|E$. Since J is continuous, Γ is closed. If $\operatorname{int}\Gamma \neq \emptyset$ then Γ would contain an open set of regular points for J so $J(\Gamma)$ would contain an open set, contradicting the fact that $S(N) = J(\Gamma)$ is nowhere dense. Hence Γ satisfies (GS1).

Since C has a polyhedral neighbourhood in T^*M, by taking barycentric subdivisions, (GS2) is easily seen to be satisfied for any point $p \in C \cap \Gamma$. If $p \in \Gamma - C$, then p is a regular point for J, and J is a submersion on any sufficiently small neighbourhood of p. Thus, any neighbourhood V of p contains a neighbourhood W homeomorphic to $A \times B$ where $B \subset \mathbb{R}^m$ is a small open disk about $J(p)$, and A is a small open disk about p in the fibre $J^{-1}(J(p))$. By taking barycentric subdivisions of N, we may assume that B is the interior of a small complex containing p. Then $B - S(N) \cap B$ contains finitely many path-connected components and so $L \cap W$ has finitely many path-connected components, which proves (GS2).

Let $D \subset N$ be the interior of a simplex in N. Since D contains only regular values of J, $J|J^{-1}(D) \to D$ is a proper submersion with a contractible image. Hence, it is a trivial fibration. Compactness of E implies the number of connected components in $J^{-1}(D)$ is finite so $J^{-1}(D)$ is homeomorphic to a finite union of $\mathbb{T}^l \times \mathbb{D}^m$. Since N is a compact polyhderon, this proves that L is a finite, disjoint union of path-connected sets L_i such that $L_i \simeq \mathbb{T}^l \times \mathbb{D}^m$. Thus (GS3) is true. □

3.3. **Proof of theorem (1.2, ii).** Let us make the following observation:

Lemma 3.6. *If $D < G$ is a discrete, cocompact subgroup of a connected, simply-connected 2-step nilpotent Lie group G, then D does not contain an abelian subgroup of finite index.*

Proof: [Thanks to Satya Mohit] From the remarks at the beginning of section 2, there exists $x_1, x_2 \in D$ such that $[x_1, x_2] \neq 1$. Let $z = [x_1, x_2]$. Because G is

2-step nilpotent, $[x_1^k, x_2^k] = z^{k^2}$ for all $k \in \mathbb{Z}$; because G is connected and simply connected, D is torsion free so $z^k \neq 1$ for all $k \neq 0$. Assume now that $A < D$ is a finite-index abelian subgroup. Then there exists a $k \neq 0$ such that $x_1^k, x_2^k \in A$. Then $1 = [x_1^k, x_2^k] = z^{k^2} \neq 1$. Absurd. □

Let's now turn to the main result of this section.

Remark 3.7. $C^k(T^*M;\mathbb{R})$ is equipped with the topology of uniform convergence of all derivatives up to order k on compact sets. A C^2 open neighbourhood of $H \in C^2(T^*M;\mathbb{R})$ can be described as follows: let g be a complete metric on M with Levi-Civita connection ∇ and let $|.|$ denote the extension of the norm induced by g to all tensors on M; let $\bar{\nabla}$ denote the Levi-Civita connection of the Sasaki metric induced by g on T^*M. Let $\operatorname{hess} H(X,Y) := \bar{\nabla}_X \bar{\nabla}_Y H - dH(\bar{\nabla}_X Y)$ for $H \in C^2$ and smooth vector fields X, Y on T^*M. Given a compact set $K \subset T^*M$ and $\epsilon > 0$ a C^2 open neighbourhood of H then consists of all C^2 functions h such that $\sup_{p \in K} \{|H(p) - h(p)|, |dH - dh|_p, |\operatorname{hess} H - \operatorname{hess} h|_p\} < \epsilon$.

Proof of theorem (1.2, ii): Let $\mathcal{Q}$ be the set of compact 2-step nilmanifolds from section 2 and let $M \in \mathcal{Q}$. Suppose that $H = T + V$ is a C^2 mechanical hamiltonian on T^*M, with $T(p) = \frac{1}{2} g^{-1}(p,p)$ the kinetic term and $V = V(m)$ the potential energy. Let $h > h_0 = \sup_{m \in M} V(m)$. For each $h > h_0$ the sublevel set $H^{-1}((-\infty, h])$ is a compact, C^2, fibre-wise strictly convex submanifold-with-boundary of T^*M that contains the zero section. The boundary $H^{-1}(h)$ is a regular level set for H.

Fix some $h > h_0$, let $K = H^{-1}((-\infty, 2h])$ and let $0 < \epsilon < \frac{1}{2}(h - h_0)$. Let U_H be the C^2 open neighbourhood of H determined by K and ϵ.

For each $l \in \mathbb{R}$ let $\mathcal{K}_l := F^{-1}((-\infty, l]) \cap K$. If ϵ is sufficiently small, then for all $F \in U_H$, $\partial \mathcal{K}_h$ is a regular level for $F|K$, and $\mathcal{K}_h$ is a C^2 submanifold-with-boundary of T^*M that is C^2 close to $H^{-1}((-\infty, h])$. In particular, the zero section of T^*M lies in $\mathcal{K}_h$ and $\mathcal{K}_h$ is a compact, fibre-wise strictly convex set. Since strict convexity is a C^2-open property, it follows that for all l sufficiently close to h $\mathcal{K}_l$ is a compact fibre-wise strictly convex set that contains the zero section. By compactness, the fibre-wise strict convexity of $\mathcal{K}_l$ and the fact that it contains the zero section, for each $p \in T_m^*M$, $p \neq 0$, there is a unique $\lambda > 0$ such that $\lambda p \in \partial \mathcal{K}_l$. Define:

$$\mathbf{F}_l(m,p) := \lambda^{-1} \tag{9}$$

for all $m \in M$ and non-zero $p \in T^*_m M$. Because $\mathcal{K}_l$ is a compact fibre-wise strictly convex C^2 submanifold of T^*M, $\mathbf{F}_l$ is C^2 off the zero section and extends as a C^0 function to all of T^*M. In addition, $\mathbf{F}_l$ is positively homogeneous of degree 1. [See [28], section 3.2; $\mathbf{F}_l$ is analogous to the gauge function defined there.]

Because $\mathbf{F}_l^{-1}(1) = \partial\mathcal{K}_l = F^{-1}(l) \cap K$, and $\partial\mathcal{K}_l$ is a regular level for both hamiltonians, the flow of $X_{\mathbf{F}_l}|\partial\mathcal{K}_l$ is a time change of $X_F|\partial\mathcal{K}_l$.

Let $\mathbf{Q}_l = \frac{1}{2}\mathbf{F}_l^2$, which is C^2 off the zero-section and C^1 everywhere. The function $\mathbf{Q}_l$ is fibre-wise strictly convex, so we perform a Legendre transform with respect to $\mathbf{Q}_l$. Let $\mathbf{G}_l : TM \to \mathbb{R}$ be the Legendre transform of $\mathbf{Q}_l$; it is non-negative, C^2 off the zero-section, C^1 everywhere, fibre-wise strictly convex and positively homogeneous of degree 2. The function $\mathbf{L}_l := \sqrt{2\mathbf{G}_l}$ therefore determines a Finsler metric on M. [See [28], section 3.2; $\mathbf{L}_l$ is analogous to the support function defined there.]

By the Hopf-Rinow theorem for Finsler metrics (see [28], theorem 2, section 4.2), the Finsler metric induced by $\mathbf{L}_l$ is complete. Hence, the Euler-Lagrange flow of $\mathbf{G}_l$ is Hopf-Rinow. Since the Euler-Lagrange flow of $\mathbf{G}_l$ is conjugate to the hamiltonian flow of $\mathbf{Q}_l$, the latter is also Hopf-Rinow. Therefore, the flow of $X_{\mathbf{Q}_l}$ and hence $X_{\mathbf{F}_l}$ is Hopf-Rinow. Since the flow of $X_{\mathbf{F}_l}|{\mathbf{F}_l}^{-1}(c)$ is orbitally equivalent to that of $X_{\mathbf{F}_l}|\partial\mathcal{K}_l$ for any $c > 0$, it follows that $X_{\mathbf{F}_l}|\partial\mathcal{K}_l$ is Hopf-Rinow. Hence, $X_F|\partial\mathcal{K}_l$ is Hopf-Rinow.

Since the above arguments hold for $l < h$, l sufficiently close to h, it follows that $X_F|\mathcal{K}_h$ is Hopf-Rinow. Recall that $\mathcal{K}_h$ is a compact disk bundle over M.

If X_F is tamely integrable on T^*M, then lemma 3.5 implies that $X_F|E$ is geometrically simple for some compact, invariant disk bundle E containing $\mathcal{K}_h$. Theorem 3.3 implies that $\pi_1(M)$ is almost abelian. By hypothesis, the manifold M has a 2-step nilpotent fundamental group. Absurd. □

Remark 3.8. (i) The first integral map constructed in section 2 are wild; theorem 1.2.ii shows that this wildness is rooted in the topological complexity of the 2-step nilmanifold M. (ii) This proof also demonstrates that the integrable geodesic flows exhibited on the 2-step solvmanifolds in [9] and the n-step nilmanifolds in [13] also possess a C^2 open neighbourhood which is devoid of tamely integrable hamiltonian systems.

4. Monodromy of the Liouville Foliation

This section studies the bifurcations of the Liouville tori and the monodromy of the Liouville foliaton induced by the Liouville-integrable vector field X_{H_g} on $T^*(D\backslash G)$ where G is the $2n+1$-dimensional Heisenberg group. In [29] it is proven that if D is a discrete cocompact subgroup of the $2n+1$-dimensional Heisenberg group, then there exists positive integers $1 \leq k_1 | \cdots | k_n$ and generators $w_1, \ldots, w_n$, $v_1, \ldots, v_n, z_1$ such that $D = \langle w_1, \ldots, v_n, z_1 \ : \ [w_i, v_i] = z_1^{k_i}$ for all $i = 1, \ldots, n$ and all other commutators are trivial $\rangle$. We identify $\mathcal{W}$ (resp. $\mathcal{V}, \mathcal{Z}$) with the span of the w_i (resp. v_i,z_1). In the notation of the previous section:

(10)
$$X_{H_g} = \begin{cases} \dot{a} = -c[\mathrm{B}'a + \mathrm{C}b], & \dot{w} = \mathrm{A}a + \mathrm{B}b, \\ \dot{b} = c[\mathrm{A}a + \mathrm{B}b], & \dot{v} = \mathrm{B}'a + \mathrm{C}b, \\ \dot{c} = 0, & \dot{z} = \mathrm{D}c + \frac{1}{2}[w+v, \mathrm{A}a + \mathrm{B}b + \mathrm{B}'a + \mathrm{C}b]. \end{cases}$$

Some obvious first integrals of X_{H_g} are given by: c, $f_i = \phi(c) \sin 2\pi(\frac{b_i}{c} - w^i)$. There is a unique symplectic linear transformation $(a, b) \to (r, s)$ that block diagonalizes $2H_g = \langle a, \mathrm{A}a \rangle + 2\langle a, \mathrm{B}b \rangle + \langle b, \mathrm{C}b \rangle + \mathrm{D}c^2 = \sum_{i=1}^n \mu_i (r_i^2 + s_i^2) + \mathrm{D}c^2$. This transformation preserves the Poisson bracket on $\mathcal{G}^*$. Then $h_i = \frac{1}{2} r_i^2 + \frac{1}{2} s_i^2$ for $i = 1, \ldots, n$ are first integrals for X_{H_g}. The family $c, f_1, \ldots, h_n$ is a complete, involutive, independent family of first integrals for X_{H_g}.

Remark 4.1. (i) The functions $g_i = \phi(c) \sin 2\pi(\frac{a_i}{c} + v^i)$ are additional, independent first integrals that are not in involution with the family f_i. (ii) The constants μ_i may be made periodic functions of v^i, an operation that preserves the Liouville integrability of X_{H_g}. There is therefore an explicit, infinite-dimensional parameterized family of Liouville integrable geodesic flows on $T^*(D\backslash G)$.

The following lemmas are clear. The singular fibres of type I, II, and III (see introduction) are the singular sets $\cup_{i=1}^n H_i$, $\cup_{i=1}^n F_i$ and O respectively:

Lemma 4.2. *Let $J := (c, h_1, \ldots, h_n, f_1, \ldots, f_n) : T^*(D\backslash G) \to \mathbb{R} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n$ be the first integral mapping. Let $H_i := \{h_i = 0\}$, $F_i := \{f_i = \pm\phi(c)\}$ and $O := \{c = 0\}$. Then*

$$\mathrm{crit}(J) = O \cup (\cup_{i=1}^n H_i) \cup (\cup_{i=1}^n F_i)\,. \tag{11}$$

Lemma 4.3. *Let Σ denote the critical-value set of J, R denote the regular-value set and let* $\operatorname{im} J = \Sigma \cup R$. *Then:*

$$\operatorname{im} J = \{(\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n | -\phi(\alpha) \leq \gamma_i \leq \phi(\alpha),\ i = 1, \dots, n\}, \tag{12}$$

$$\Sigma = (\mathcal{O} \cap \operatorname{im} J) \cup (\cup_{i=1}^n \mathcal{H}_i \cap \operatorname{im} J) \cup (\cup_{i=1}^n \mathcal{F}_i \cap \operatorname{im} J) \tag{13}$$

where $\mathcal{O}$ is hyperplane defined by $\alpha = 0$, $\mathcal{H}_i = \{\beta_i = 0\}$ and $\mathcal{F}_i = \{(\alpha, \beta, \gamma) | \gamma_i = \pm\phi(\alpha)\}$.

Let $\Sigma^r := O \cup (\cup_{i=1}^n H_i)$, and define the map $\Psi := (c, h_1, \dots, h_n, \theta_1, \dots, \theta_n)$ where $\theta_j := \frac{b_j}{c} - w^j \mod 1$. Then $\Psi : T^*(D\backslash G) - \Sigma^r \to \mathbb{R} \times \mathbb{R}^n_{>0} \times \mathbb{T}^n$ is a proper, real-analytic submersion with lagrangian tori as fibres, hence Ψ is a real-analytic lagrangian fibration. The monodromy of the bundle is determined by the action of the fundamental group of the base $B = \mathbb{R} \times \mathbb{R}^n_{>0} \times \mathbb{T}^n$ on the fibres. The most straightforward way to see this action is to lift the lagrangian fibration Ψ to a lagrangian fibration $\tilde{\Psi} : \tilde{S}^r \to \tilde{B} = \mathbb{R} \times \mathbb{R}^n_{>0} \times \mathbb{R}^n$. The following diagram realizes this lifting:

$$\begin{array}{cccccc} & \tilde{S}^r & \xrightarrow{\Pi} & S^r & \\ \tilde{\Psi} & \downarrow & & \downarrow & \Psi \\ & \tilde{B} & \xrightarrow{\pi} & B & , \end{array}$$

where $S^r = T^*(D\backslash G) - \Sigma^r$. The covering $\Pi : \tilde{S}^r \to S^r$ is obtained by taking the abelian subgroup $F = \langle v_1, \dots, v_n, z \rangle$ of D and forming the covering $\Pi : T^*(F\backslash G) \to T^*(D\backslash G)$. Then one takes $\tilde{S}^r = \Pi^{-1}(S^r)$ and observes that the map $\tilde{\Psi} = (c, h_1, \dots, h_n, \Theta_1, \dots, \Theta_n)$ with $\Theta_i = \frac{b_i}{c} - w^i$ is a proper, real-analytic lagrangian fibration. Since the image of $\tilde{\Psi}$ is contractible, $\tilde{S}^r$ is a trivial $\mathbb{T}^{2n+1}$ bundle. The covering map $\pi : \tilde{B} \to B$ is the map $(\alpha, \beta, \gamma) \to (\alpha, \beta, \gamma \mod \mathbb{Z}^n)$.

The action of $\pi_1(B)$ on the bundle Ψ is obtained by identifying the fibres of $\tilde{\Psi}$ under the action of D on $\tilde{S}^r$. Because $F \lhd D$ is normal in D, D acts on the left on $F\backslash G$ by $d * Fg := Fdg$ for all $d \in D$ and $g \in G$. The action of F is clearly trivial, and so we need only consider the action of $D/F \simeq \langle w_1, \dots, w_n \rangle$ on $T^*(F\backslash G)$. It is clear that $\pi_1(B)$ is naturally identified with D/F.

Let us now fix a basis of 1-cycles for the fibres of the map $\tilde{\Psi}$ in $\tilde{S}^r$ as follows. Let $\sigma = (\alpha, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n)$ be the coordinates on $\mathbb{R} \times \mathbb{R}^n_{>0} \times \mathbb{R}^n$ and define the section of the bundle $\tilde{\Psi}$ by:

$$\xi(\sigma) = \begin{cases} c &= \alpha, & z &= 0 + \mathbb{Z}, \\ r_i &= 0, & s_i &= \sqrt{2\beta_i}, \\ w^i &= \frac{b_i(r,s)}{c} - \gamma_i, & v^i &= 0 + \mathbb{Z}. \end{cases} \tag{14}$$

Let $g = (w, v, z)$ and $P = (r, s, c)$ and:

$$\begin{aligned} c_i(t, Fg, P) &:= ((tv_i) * Fg, P), \\ c_{n+1}(t, Fg, P) &:= ((tz_1) * Fg, P), \\ c_{n+i+1}(t, Fg, P) &:= (Fg, (r + e_i(r_i(\cos 2\pi t - 1) + s_i \sin 2\pi t), \\ &\qquad s + e_i(r_i \sin 2\pi t + s_i(\cos 2\pi - 1)t), c)), \end{aligned} \tag{15}$$

for $i = 1, \ldots, n$; e_i is the i-th standard basis vector of $\mathbb{R}^n$. Note that $0, v_i, z_1 \in F$ so the c_i do define closed loops in $T^*(F\backslash G)$.

Let $C_j(t, \sigma) := c_j(t, \xi(\sigma))$, $t \in \mathbb{R}/\mathbb{Z}$, which define a basis of $\pi_1(\tilde{\Psi}^{-1}(\sigma); \xi(\sigma))$ that smoothly varies with σ. We will let $[C_j](\sigma)$ denote the homotopy class in $\pi_1(\tilde{\Psi}^{-1}(\sigma); \xi(\sigma))$ of $C_j(t, \sigma)$. The action of $w_i F$ on $[C_j](\sigma)$ is given by left translation. The only component of $\tilde{\Psi}$ altered by translation by $w_i F$ is Θ_i: it is decreased by 1. Thus, the translated cycle lies in $\pi_1(\tilde{\Psi}^{-1}(\alpha, \beta, \gamma - e_i); \xi(\alpha, \beta, \gamma - e_i))$. A simple calculation using the multiplication structure on G shows that:

$$w_i F * [C_j](\alpha, \beta, \gamma) = [C_j](\alpha, \beta, \gamma - e_i) + \delta_{ij} k_i [C_{n+1}](\alpha, \beta, \gamma - e_i) \tag{16}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, 2n + 1$. This proves

Theorem 4.4 (Theorem 1.3)**.** *The bundle Ψ has the monodromy group isomorphic to $\mathbb{Z}^n \simeq D/F$. In particular, there do not exist global action-angle coordinates of $\Psi : T^*(D\backslash G) - \Sigma^r \to \mathbb{R} \times \mathbb{R}^n_{>0} \times \mathbb{T}^n$ [21].*

Remark 4.5. Lemma 4.2 shows that the Liouville foliations of any two left-invariant metric hamiltonians are isomorphic. That is, if H and H' are two left-invariant metric hamiltonians and J (J') is the first-integral map for H (H'), then there exists a diffeomorphism $\phi : T^*(D\backslash G) \to T^*(D\backslash G)$ such that $J' = J \circ \phi$. On the other hand, a calculation shows that if D is normalized to 1 and the constants μ_i of H (μ'_i of H') satisfy $\sum_{i=1}^n \mu_i \neq \sum_{i=1}^n \mu'_i$, then there does not exist a homeomorphism $\varphi : T^*(D\backslash G) \to T^*(D\backslash G)$ such that φ maps the trajectories of X_H onto those of $X_{H'}$. That is, there is no orbital equivalence of these geodesic flows. This compares with the situation observed by Bolsinov and Fomenko, who show that the geodesic flow on ellipsoids E, $E' \subset \mathbb{R}^3$ are orbitally equivalent iff the ellipsoids are similar while the Liouville foliations of the geodesic flows on all ellipsoids are isomorphic [8].

5. $h_{top}(\Phi) = 0$

In this section, theorem (1.4) is proven. The proof of this theorem follows the idea in [12]. An incorrect proof of the vanishing of the topological entropy for a left-invariant geodesic flow on a nilmanifold occurs in [43]. The author there makes the assumption that the metric's exponential map coincides with the group's, which is incorrect. In [17], the present author constructs examples of 3-step nilmanifolds with positive entropy, left-invariant geodesic flows. To prove theorem (1.4), most of the work is done by the following two theorems due to Bowen:

Theorem 5.1 ([10]). *Let $T : X \to X$ be continuous endomorphism of the compact metric space X and suppose that $f : X \to Y$ is a continuous endomorphism of compact metric spaces that is T-invariant. Then:*

$$h_{top}(T) = \sup_{y \in Y} h_{top}(T|f^{-1}(y)).$$

Theorem 5.2 ([10]). *Let $T : X \to X$ be a continuous endomorphism of the compact metric space X, and let G be a compact topological group that acts freely as a group of automorophisms of X. Let $\pi : X \to Y$ be the orbit map. If T is G-invariant and $S : Y \to Y$ is the endomorphism of Y induced by T, $S \circ \pi = \pi \circ T$, then*

$$h_{top}(T|X) = h_{top}(S|Y).$$

It is recalled that the topological entropy of a geodesic flow $\phi_t : T^*M \to T^*M$ is the topological entropy of the time-1 map of the geodesic flow restricted to the unit cotangent bundle. In this section, notation will be abused and the topological entropy of a (complete) vector field will be understood to mean the topological entropy of the time-1 map of its flow.

The equations of motion for the left-invariant hamiltonian $H : T^*G \to \mathbb{R}$ are given by (equation 3):

$$X_H(x, y, p, q) = \begin{cases} \dot{x} \;=\; R^{\frac{1}{2}} v, & \dot{v} \;=\; -B(q)v, \\ \dot{y} \;=\; Sq + \frac{1}{2}[x, R^{\frac{1}{2}} v], & \dot{q} \;=\; 0, \end{cases} \tag{17}$$

where $v = R^{\frac{1}{2}} p$. The vector field X_H on $T^*(D\backslash G)$ restricts to the unit cotangent bundle $\{H = \frac{1}{2}\}$, which is compact. By theorem (5.1), it suffices to consider the restriction of X_H to $E_q := \{H = \frac{1}{2}, q = cst.\}$ to determine the topological entropy of the geodesic flow. The vector field $X_H|_{E_q}$ is invariant under the action of the compact symmetry group $Z(D)\backslash Z(G) \simeq \mathbb{T}^z$ where $z = \dim Z(G)$. This symmetry group acts freely on E_q – because it is the cotangent lift of the right action of $Z(G)$ on $D\backslash G$ –, so the space $M_q := E_q/\mathbb{T}^z$ is a manifold. Indeed, $M_q = \mathbb{T}^h \times S_r^{z-1}$ in the case where $r^2 = 1 - \langle q, Sq\rangle > 0$, S_r^k is the k-dimensional sphere of radius $r > 0$ and $M_q = \mathbb{T}^h$ in the case $r^2 = 0$. The induced vector field on M_q in the first case is

$$Y_{H,q}(x,p) = \left\{ \begin{array}{rclrcl} \dot{x} & = & R^{\frac{1}{2}}v, & \dot{v} & = & -B(q)v, \end{array} \right. \tag{18}$$

where notation is abused and the coordinates (x, v) are employed on this reduced space; in the second case, the induced vector field $Y_{H,q} \equiv 0$. It is clear that only the first case where $r > 0$ is relevant. In this case, the vector field $Y_{H,q}$ is invariant under the free action of the torus $\mathbb{T}^h$ on M_q which acts by $\theta : (x, v) \to (x + \theta, v)$. The manifold M_q can be reduced by this action to obtain $M_q/\mathbb{T}^h = S_r^{z-1}$. The vector field $Y_{H,q}$ descends to

$$Z_{H,q}(p) = \left\{ \begin{array}{rcl} \dot{v} & = & -B(q)v. \end{array} \right. \tag{19}$$

Applying Bowen's theorem (5.2) twice yields that the topological entropy of the time-1 map of $X_H|_{E_q}$ equals the topological entropy of the time-1 map of $Z_{H,q}$. Because $B(q)$ is skew-symmetric, the time-1 map of $Z_{H,q}$ is an isometry, so its topological entropy is zero.

That is:

$$h_{top}(X_H|S^*(D\backslash G)) = \sup\, h_{top}(X_H|E_q) = \sup\, h_{top}(Y_{H,q}) = \sup\, h_{top}(Z_{H,q}) = 0.$$

□

6. Non-integrable geodesic flows on G_3

6.1. A remark on non-integrability. This section offers a generalized definition of integrability inspired by that of Bogoyavlenskij [6, 7]. A criterion is developed for manifolds with a non-commutative fundamental group that allows one to demonstrate the complete absence of flow-invariant toroidal neighbourhoods.

Definition 6.1. *Let $\phi_t : M \to M$ be a 1-parameter group of homeomorphisms of M (a flow). Then ϕ_t is* locally integrable *at $m \in M$ if there exists a neighbourhood*

U of m and a homeomorphism $h : U \hookrightarrow \mathbb{D}^s \times \mathbb{T}^r$ such that $h \circ \phi_t \circ h^{-1} = T_t$ where $T_t(x, \theta) = (x, \theta + t\omega(x))$ and $\omega : \mathbb{D}^s \to \mathbb{R}^r$ is a continuous map.

In the sequel $[c]$ will denote the homotopy class of a curve; $\bar{c}$ will denote its free homotopy class.

Definition 6.2. *Let $\mathcal{F}(M)$ denote the set of free homotopy classes of curves in M, M an arc-wise connected space. Then $\bar{c}, \bar{c}' \in \mathcal{F}(M)$* commute *if for some $m \in M$, there exists $[c], [c'] \in \pi_1(M; m)$ such that $[c] * [c'] = [c'] * [c]$ and $[c] \in \bar{c}, [c'] \in \bar{c}'$. Let $C(\bar{c}; M)$ denote the set of free homotopy classes in M that commute with $\bar{c}$.*

One notes that the commutativity of free homotopy classes is well-defined: if $n \in M$ is a second point then $\pi_1(M; n)$ is isomorphic to $\pi_1(M; m)$, and the isomorphism preserves free homotopy classes.

Lemma 6.3. *Let $\phi_t : M \to M$ be a 1-parameter group of homeomorphisms of M that is locally integrable at $m \in M$. Then there is an open, ϕ_t-invariant neighbourhood U of m such that if $n, n' \in U$ are periodic points of ϕ_t then the free homotopy classes of these orbits commute.*

Proof: The neighbourhood U is homeomorphic to $\mathbb{D}^s \times \mathbb{T}^r$, so $\pi_1(U; m) \simeq \mathbb{Z}^r$ and any closed curve $c : \mathbb{T}^1 \to U$ is freely homotopic to a closed curve based at m. Hence, $C(\bar{c}; U) = \mathcal{F}(U)$. Therefore, the free homotopy classes of ϕ_t's periodic orbits in U all commute. $\square$

Remark 6.4. The contrapositive of 6.3 says simply that if $m \in M$ is a periodic point of the flow ϕ_t and in any neighbourhood U of m, there exists a periodic point m' such that the free homotopy classes of the periodic orbits through m and m' do not commute, then ϕ_t is not locally integrable at m.

6.2. **An example: non-integrable geodesic flows on 2-step nilmanifolds.** Let $G = G_3 = (\mathbb{R}^3 \times \Lambda^2(\mathbb{R}^3), *)$ with multiplication on G defined by

$$(x, y) * (x', y') := (x + x', y + y' + \frac{1}{2} x \wedge x'),$$

where $\wedge$ is the exterior product in $\mathbb{R}^3$. By choosing the standard basis in $\mathbb{R}^3$, $\Lambda^2(\mathbb{R}^3)$ may be identified with $\mathbb{R}^3$ and $\wedge$ may be identified with the cross product. G may be viewed also as the extension $0 \to \Lambda^2(\mathbb{R}^3) \to G \to \mathbb{R}^3 \to 0$.

6.2.1. *discrete, cocompact subgroups of G.*

Lemma 6.5. *Let D be a cocompact, discrete subgroup of G. Then for some $k \in \mathbb{Z}^{3+}$, $k_1|k_2|k_3$, there is an automorphism $\phi : G \to G$ such that $\phi(D) = D(k)$ where*

$$D(k) := \langle a_1, a_2, a_3, b_1, b_2, b_3 \ : \ [a_1, a_2] = b_3^{k_3}, [a_2, a_3] = b_1^{k_1}, [a_3, a_1] = b_2^{k_2}, [b_i, .] = 1\rangle.$$

The generators of $D(k)$ are $a_i := (e_i, 0)$ and $b_i := (0, k_i^{-1} e_i)$, where e_i are the standard basis vectors of $\mathbb{R}^3$.

The proof of this lemma is straightforward. Most important for our purposes is that it can be assumed that $D = D(k)$ contains the subset $\mathbb{Z}^3 \times \mathbb{Z}^3$.

6.2.2. *A family of left-invariant geodesic flows.* Left-trivialization of the cotangent bundle T^*G produces the identification $T^*G \simeq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ with coordinates (x, y, p, q). A riemannian metric g on G naturally induces a hamiltonian on T^*G via the Legendre transform: $F : V \in T_xG \to P = g_x(V, \cdot) \in T_x^*G$. The solutions of Hamilton's equations on T^*G for the hamiltonian $2H(x, P) = g_x(F^{-1}P, F^{-1}P)$ project to g-geodesics on G. On T^*G a left-invariant metric has the hamiltonian $2H = \langle p, \mathrm{A}p\rangle + 2\langle p, \mathrm{B}q\rangle + \langle q, \mathrm{C}q\rangle$ where $\langle, \rangle$ is the euclidean inner product on $\mathbb{R}^3$ and $\mathrm{A}, \mathrm{B}, \mathrm{C}$ are symmetric 3×3 matrices. The left-invariant hamiltonian

$$2H := |p|^2 + \frac{1}{2\pi}\mu|q|^2, \tag{20}$$

where $\mu \in \mathbb{Q}$ and $|.|$ is the euclidean norm on $\mathbb{R}^3$, yields the vector field:

$$X_H := \begin{cases} \dot{p} \ = \ A(q)p, & \dot{q} \ = \ 0, \\ \dot{x} \ = \ p, & \dot{y} \ = \ \frac{1}{2\pi}\mu q + \frac{1}{2} x \wedge p. \end{cases} \tag{21}$$

The matrix $A(q)$ is defined by $A(q)p = q \wedge p$. Let ϕ_t denote the flow of X_H on T^*G and let Φ_t denote the induced flow on $T^*(D(k)\backslash G)$.

6.2.3. *Proof of non-integrability of Φ_t.*

Lemma 6.6. *Let $0 \neq l \in \mathbb{Z}^3$, $|l| \in \mathbb{Z}$. Let Q_l be the set of $(x, y, p, q) \in T^*G$ for which there exists $t \in \mathbb{R}$ and $m \in \mathbb{Z}^3$ such that $\phi_t(x, y, p, q) = (x + l, y + m + \frac{1}{2} l \wedge x, p, q)$. Then $P_l := \cup_{a \in \mathbb{Z}} Q_{al}$ is dense in $T^*G_l := \{(x, y, p, q) \ : \ q \in \mathrm{span}_{\mathbb{R}}\{l\}\}$.*

Corollary 6.7. *Let $P := \cup_{l \in \mathbb{Z}^3, \ |l| \in \mathbb{Z}} P_l$. Then P is a dense subset of T^*G.*

Proof of corollary 6.7: Assuming lemma (6.6), it is only necesary to show that $\cup_{l \in \mathbb{Z}^3, \ |l| \in \mathbb{Z}} T^*G_l$ is dense in T^*G. This is equivalent to the density of rational points on the unit sphere in $\mathbb{R}^3$. By stereographic projection, this is clear. $\square$

We now prove the following theorem, which clearly implies theorem (1.5):

Theorem 6.8. *The flow Φ_t is non-integrable in the sense of definition (6.1) on any open subset U of $T^*(D(k)\backslash G)$.*

Proof: For a path-connected topological space X, the free homotopy classes of maps $C^0(\mathbb{T}^1, X)$ are in one-to-one correspondence with the conjugacy classes of the fundamental group $\pi_1(X; x)$ for an arbitrary point $x \in X$. In the case of $T^*(D(k)\backslash G)$, its fundamental group $\pi_1(T^*(D(k)\backslash G); (D(k)e, P)) \simeq D(k)$ in the natural way; it follows that the free homotopy classes of maps $C^0(\mathbb{T}^1; T^*(D(k)\backslash G))$ can naturally be identified with the conjugacy classes of $D(k)$. This identification can be made explicit as follows: let $c \in \bar{c}$ be a loop in the free homotopy class $\bar{c}$ that is based at $(D(k)e, 0)$; let $\tilde{c} : [0,1] \to T^*G$ be the unique lift of c such that $\tilde{c}(0) = (e, 0) \in T^*G$; because $c(1) = c(0)$, $\tilde{c}(1) = d \in D(k)$. The free homotopy class $\bar{c}$ is then identified with the conjugacy class of d: $\bar{c} \equiv \{gdg^{-1} : g \in D(k)\}$. This identification is used in the proof.

Assume lemma (6.6) and corollary (6.7). Then, if U is an open subset of $T^*(D(k)\backslash G)$, there exists $n, n' \in U$ such that the flow Φ_t is periodic through each point and the free homotopy classes of these periodic orbits (call them $\bar{c}$ and $\bar{c}'$) are $\bar{c} = \{(l, m+h) \ : \ (0,h) \in [D(k), D(k)]\}$ and $\bar{c}' = \{(l', m'+h) \ : \ (0,h) \in [D(k), D(k)]\}$ for some $l, l' \in \mathbb{Z}^3$ such that $l \wedge l' \neq 0$. Therefore, the free homotopy classes $\bar{c}$ and $\bar{c}'$ do not commute; now apply theorem (6.3). $\square$

Remark 6.9. Because the geodesic flow on $H^{-1}(a)$ for $a > 0$ is a time reparameterizations of the geodesic flow on $H^{-1}(\frac{1}{2})$, this proves the absence of any open sets $U \subset S^*(D(k)\backslash G)$ that are fibred by invariant tori.

Proof of lemma (6.6): Fix $0 \neq l \in \mathbb{Z}^3$ such that $|l| \in \mathbb{Z}$. It may be assumed that the vertical momentum $q \neq 0$, since this is a dense subset of T^*G. Let $p = u + v$ be the orthogonal decomposition of p relative to q: $u = u(p,q) = \frac{\langle p,q\rangle}{\langle q,q\rangle} q$ and $v = v(p,q) = p - u$ [Recall: we have identified $\mathbb{R}^3 \equiv \Lambda^2(\mathbb{R}^3)$ via the euclidean inner product]. The set of p such that $u, v \neq 0$ is a dense subset of T^*G, so it will be assumed that $u, v \neq 0$. In order that $\phi_t(x, y, p, q) = (x + l, y + m + \frac{1}{2} l \wedge x, p, q)$

it is necessary that there exist $t, c \in \mathbb{R}$ such that:

$$\exp A(q)t = 1, \tag{22}$$

$$tu = l, \tag{23}$$

$$q = cl. \tag{24}$$

The skew-symmetric matrix $A(q)$ is not invertible on $\mathbb{R}^3$; nonetheless, its restriction to $\mathrm{span}_{\mathbb{R}}\{q\}^{\perp}$, the orthogonal complement to the subspace spanned by q, is invertible. In the sequel, $A(q)^{-1}$ will denote the inverse on $\mathrm{span}_{\mathbb{R}}\{q\}^{\perp}$. Then,

$$m = x \wedge l + l \wedge A(q)^{-1}v + \frac{1}{2}tA(q)^{-1}v \wedge v + t\frac{\mu}{2\pi}cl. \tag{25}$$

There is a redundancy in the parameters due to the fact that the flow ϕ_t on different energy levels is simply a reparameterization of the flow on S^*G. For this reason, it can be assumed that $|q| = 1$; then $t = 2\pi n$ for some $n \in \mathbb{Z}$, $c = |l|^{-1}$ and $A(q)^{-1} = -A(q) = -|l|^{-1}A(l)$. Therefore,

$$m = \left[x + |l|^{-1}A(l)v\right] \wedge l + \pi n|l|^{-1}v \wedge A(l)v + n\mu|l|^{-1}l. \tag{26}$$

The former term lies in $\mathrm{span}_{\mathbb{R}}\{l\}^{\perp}$ while the latter two terms lie in $\mathrm{span}_{\mathbb{R}}\{l\}$. Let $L :=\mathrm{span}_{\mathbb{Q}}\{l\}$ be the rational span of l, and $L^{\perp}$ be the rational subspace that is orthogonal to L. The set of $p = u + v$ such that $u \in \frac{1}{\pi}L$ and $v \in \frac{1}{\sqrt{\pi}}L^{\perp}$ is dense in $\mathbb{R}^3$. Then $\pi n|l|^{-1}v \wedge A(l)v \in L$ for p in a dense set, and for v fixed, there is a dense set of x such that $\left[x + |l|^{-1}A(l)v\right] \wedge l \in L^{\perp}$.

Therefore, in a neighbourhood of any point $(x', y', p', q = |l|^{-1}l)$, there exists an $(x, y, p, q = |l|^{-1}l)$ such that $m \in L \oplus L^{\perp} = \mathbb{Q}^3$. By taking some multiple al of l for $a \in \mathbb{Z}$ and taking $n \in \mathbb{Z}$ large enough (without altering the starting point $(x, y, p, q = |l|^{-1}l)$), the two components of m can therefore be made to lie in $\mathbb{Z}^3$ and so $m \in \mathbb{Z}^3$. $\square$

Remark 6.10. (i) This proof uses only two properties of G: (i) for $\mu, \mu' \in \mathcal{G}^*$ in general position, the sum of the stabilizer subalgebras $\mathcal{G}_\mu + \mathcal{G}_{\mu'}$ is not a commutative subalgebra; (ii) the periodic points of Φ_t are dense in $T^*(D(k)\backslash G)$. Eberlein, Lee and Park, and Mast [22, 39, 44] have studied a question connected with (ii): given a left-invariant metric g on G, when are the periodic points of the quotient geodesic flow on $T^*(\Gamma\backslash G)$ dense for *all* discrete, cocompact subgroups Γ? Let us note that the metric defined in equation 20 does not satisfy this property: Let $\alpha = \mathrm{diag}(u_1, u_2, u_3)$ satisfy $\det\alpha = 1$; let $\Lambda^2\alpha$ denote the linear map on $\Lambda^2(\mathbb{R}^3)$ induced by α; and let $\phi =$

$\operatorname{diag}(\alpha, \Lambda^2\alpha)$. The linear map ϕ is an automorphism of G_3 for all such α. Assume that $u_1, u_2 \in \mathbb{R}$ are chosen so that $\mathbb{Q} < \mathbb{Q}(u_1) < \mathbb{Q}(u_1, u_2)$ are transcendental field extensions, and let $\Gamma = \phi(D(k))$ for any k. Then the set of periodic points in $T^*(\Gamma\backslash G_3)$ of the geodesic flow of equation 20 is nowhere dense. Note the contrast with the almost non-singular nilpotent Lie groups: a left-invariant geodesic flow on one of these Lie groups has a dense set of periodic points on one compact quotient iff the periodic points are dense on all compact quotients [22, 39]. We believe that this *uniformity* across compact quotients is equivalent to integrability in the 2-step case.

7. Concluding Comments

In this paper, we have seen that two wide classes of two-step nilmanifolds admit integrable geodesic flows. The integrability of these left-invariant geodesic flows has been seen to depend crucially on the geometric properties of the coadjoint orbits of the covering Lie groups, G. Specifically, both almost non-singular and HR Lie groups have the property that if $\mu, \mu' \in \mathcal{G}^*$ are in generic position, then $\mathcal{G}_\mu + \mathcal{G}_{\mu'}$ is a commutative subalgebra of $\mathcal{G}$. That is, there is an abelian subgroup A such that for $\mu \in \mathcal{G}^*$ in general position, $G_\mu \subset A$. A is necessarily normal in G.

If $H : T^*G \to \mathbb{R}$ is a metric hamiltonian, then how many first integrals can be found for H that push down to a quotient $T^*(D\backslash G)$? Clearly, if i is the index of $\mathcal{G}$, then the $i = \dim \mathcal{G} - \dim \mathcal{O}(\mu)$ Casimirs of $\{,\}$ on $\mathcal{G}^*$ push down. When H is quadratic, an additional $n - \frac{1}{2}i$ quadratic integrals push down where $n = \dim G$. If we take the momentum map $\psi(g,\mu) = \mathrm{Ad}_g^*\mu$ of the left action of G on T^*G, then ψ is a first integral of a left-invariant hamiltonian H. The question then becomes: to what extent can ψ be "pushed down" to $T^*(D\backslash G)$? Let us note that we want to find functions $f : \mathcal{G}^* \to \mathbb{R}$ such that $f \circ \psi$ is D invariant. Because $\psi(Dg,\mu) = \{\mathrm{Ad}_{dg}^*\mu \ : \ d \in D\}$ this is equivalent to studying the action of D on $\mathcal{G}^*$. Now, $\mathcal{O}(\mu) \simeq G/G_\mu \simeq H/H_\mu$ where $H = G/Z(G)$ and $H_\mu = G_\mu/Z(G)$, so $D\backslash\mathcal{O}(\mu) \simeq E\backslash H/H_\mu$ where $E = D/Z(D)$. In our case, $E\backslash H \simeq \mathbb{T}^p$ and H_μ acts on this torus by translation, so we can form the projection $E\backslash H/H_\mu \to E\backslash H/\mathbf{H}_\mu \simeq \mathbb{T}^{l_\mu}$ where $\mathbf{H}_\mu$ is the closure of $E\backslash H_\mu$ in $E\backslash H$ and $l_\mu = \dim \mathbf{H}_\mu$.

In general, one expects that l_μ will jump around as μ varies and the rationality properties of H_μ relative to E change. There is an exceptional case, however: when $G_\mu \leq A$ for $\mu \in \mathcal{G}^*$ in general position and A is an abelian subgroup. Then $\mathbf{H}_\mu \subset \mathbf{B}$ where $\mathbf{B}$ is the closure of $E\backslash B$ in the torus $E\backslash H$ and $B = A/Z(G)$. One has the projection $\mathbb{T}^p \simeq E\backslash H/H_\mu \to E\backslash H/\mathbf{H}_\mu \to E\backslash H/\mathbf{B} \simeq \mathbb{T}^l$. Observe that l is a geometric-algebraic invariant of the pair (D, G) and that $E\backslash H/\mathbf{B} \simeq \mathbb{T}^l$ is a Poisson manifold, the rank of which is constant for μ in general position.

In this special case, then, one can construct an algebra of integrals on $T^*(D\backslash G)$ whose dimension is $\frac{1}{2}(n+i)+p-l$ where $p = \dim H$ (section 2). The dimension of the centre of this algebra is $\frac{1}{2}(n+i)+s$ where s is dominated by the dimension of a maximal abelian subalgebra of $C^\infty(E\backslash H/\mathbf{B})$. Letting $i = q+j$ where $q = \dim Z(G)$, we get the condition that

$$s = l - j \tag{27}$$

in order for H to be integrable in the non-commutative sense. In section 2 we studied the special case where $l = j = 0$. In order for H to be Liouville integrable, one has the condition that

$$p + j = 2l. \tag{28}$$

In section 2.4, we studied this case.

All of these considerations suggest that the study of integrable left-invariant hamiltonians on $T^*(D\backslash G)$ for G a simply connected Lie group and D a discrete subgroup of G, reduces to a simulaneous investigation of the integrability of the Euler equations on $\mathcal{G}^*$ and the coadjoint action of D on $\mathcal{G}^*$. With this idea we are able to prove:

Theorem 7.1. *Let $C : T^*\mathrm{SL}(2;\mathbb{R}) \to \mathbb{R}$ denote the Casimir, $C(g,p) = \mathrm{trace}\ p^2$, $D \leq \mathrm{SL}(2;\mathbb{R})$ be a discrete subgroup, and $H : T^*\mathrm{SL}(2;\mathbb{R}) \to \mathbb{R}$ be a smooth, left-invariant function. Then H is both Liouville and non-commutatively integrable on the open submanifold $\{C > 0\}$ in $T^*(D\backslash\mathrm{SL}(2;\mathbb{R}))$.*
In addition, if D is a lattice subgroup and μ is a left-invariant probability measure on $S^*(D\backslash\mathrm{SL}(2;\mathbb{R}))$, then for any $\epsilon > 0$ there exists metric hamiltonians $H_\pm : T^*(D\backslash\mathrm{SL}(2;\mathbb{R})) \to \mathbb{R}$ such that the μ-measure of the set of $X_{H_\pm}$-invariant Liouville tori on $S^*(D\backslash\mathrm{SL}(2;\mathbb{R}))$ is $\geq 1-\epsilon$ (resp. $\leq \epsilon$).

Details of this will appear elsewhere.

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