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Toda lattices and positive-entropy integrable systems

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Abstract. This paper studies completely integrable hamiltonian systems on $T^*\Sigma$ where Σ is a $\mathbb{T}^{n+1}$ bundle over $\mathbb{T}^n$ with an $\mathbb{R}$-split, free abelian monodromy group. For each periodic Toda lattice there is an integrable hamiltonian system on $T^*\Sigma$ with positive topological entropy. Bolsinov and Taĭmanov's example of an integrable geodesic flow with positive topological entropy fits into this general construction with the $A_1^{(1)}$ Toda lattice. Topological entropy is used to show that the flows associated to non-dual Toda lattices are typically topologically non-conjugate via an energy-preserving homeomorphism. The remaining cases are approached via the homology spectrum. An energy-preserving conjugacy implies the congruence of two rational quadratic forms over the unit group of a number field F. When $F/\mathbb{Q}$ is normal a classification of flows is obtained. In degree 3, this results from a well-known result of Gelfond; in higher degrees, the result is conditional on the conjecture that a rationally independent set of logarithms of algebraic numbers is algebraically independent over $\mathbb{Q}$.

1. Introduction

Say that a smooth flow $\varphi_t : M \to M$ is *integrable* if there is an open dense subset $L \subset M$ such that L is covered by smooth coordinate charts $(I, \phi) : U \to \mathbf{D}^a \times \mathbb{T}^b$ and the coordinate maps conjugate φ_t to a smooth translation-type flow $T_t(I, \phi) = (I, \phi + t\xi(I))$. From this local form, it is tempting to believe that integrable flows cannot be interesting from a dynamical point-of-view. This paper constructs a family of integrable hamiltonian systems with a rich phase portrait. The topological classification of these flows relates to an outstanding conjecture in transcendental number theory. To explain:

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Let $\Gamma < \mathrm{GL}(n+1;\mathbb{Z})$ be a rank n, torsion-free, abelian group that splits over $\mathbb{R}$ and acts irreducibly on $\mathbb{Z}^{n+1}$. Γ acts uniformly discretely on $\hat{\Sigma}^{2n+1} = \mathbb{R}^n \times \mathbb{T}^{n+1}$, so let $\Sigma = \hat{\Sigma}^{2n+1}/\Gamma$ be the compact real-analytic quotient. Let Ψ be the basis of a root system of a simple Kac-Moody Lie algebra of rank n.

Theorem 1. *For each manifold Σ and root basis Ψ there are $(n+1)!$ real analytic, polynomial-in-momenta hamiltonians $H : T^*\Sigma \to \mathbb{R}$ such that:*

i. the hamiltonian flow, φ_t, of H is integrable;
ii. for each $A \in \Gamma$, the automorphism of $\mathbb{T}^{n+1}$ induced by A is a subsystem of φ_1;
iii. if $\Psi = A_n^{(1)}$ or $D_{n+2}^{(2)}$, then H is induced by a real-analytic riemannian metric on Σ.

For each root basis there is a mechanical system, called a Bogoyavenskij-Toda lattice,[1] that is completely integrable. The proof of Theorem 1 shows that there is an open real-analytic set on which the lift of φ_t to $T^*\hat{\Sigma}$ is semi-conjugate to the flow of the corresponding Bogoyavlenskij-Toda lattice.

Root bases arise in the classification of simple Kac-Moody Lie algebras [21]. It will be convenient to say that a *root basis* of rank n is a spanning set Ψ of an n-dimensional real euclidean space $(\mathfrak{h}^*, \langle\langle , \rangle\rangle)$ such that: (R0) Ψ contains $n+1$ elements; (R1) for all distinct $r, s \in \Psi$, $2\langle\langle r, s\rangle\rangle/|r|^2 \in \mathbb{Z}_{\leq 0}$; and (R2) $\min_{r\in\Psi} |r| = 1$. For each root basis Ψ there are unique positive integers ω_r such that $\gcd(\omega_r : r \in \Psi) = 1$ and $\sum_{r\in\Psi} \omega_r r = 0$. Let $\omega = \mathrm{lcm}(\omega_r : r \in \Psi)$ and $w_r = \omega_r/\omega$. Let Ψ_n be a subset of Ψ that is isometric to a root basis of a simple Lie algebra of rank n and let $\mathbf{C}$ denote the Cartan matrix of Ψ_n.

The flows in Theorem 1 enjoy an invariant set on which their topological entropy can be calculated. Σ has the structure of a $\mathbb{T}^{n+1}$ bundle over $\mathbb{T}^n$ with projection $\mathbf{p}$. Let $\mathbf{V} = \ker d\mathbf{p}$ be the subbundle of vectors tangent to the $\mathbb{T}^{n+1}$ fibres, and let $\mathbf{V}^\perp \subset T^*\Sigma$ be the annihilator of $\mathbf{V}$. Let $\mathcal{V} = \mathbf{V}^\perp \cap H^{-1}(\frac{1}{2})$. $\mathcal{V}$ is a compact invariant set, so let $\varphi_t|\mathcal{V} = \Phi_t$.

Theorem 2. *The topological entropy of Φ_1 is:*

$$h_{top}(\Phi_1) = \max_{I\subseteq\Psi} \left|\sum_{r\in I} w_r r\right| =: h^*. \tag{1}$$

Clearly these flows, although integrable, have an interesting phase portrait.

Each flow from Theorem 1 can be normalized in a way that corresponds to rescaling H so that the volume of $\mathcal{V}$ is unity. Let λ be the normalization constant, and $\mathfrak{r}(F)$ denote the regulator of an algebraic number field naturally associated with Γ. Then

[1] A Bogoyavlenskij-Toda lattice is called a periodic Toda lattice by some authors.

Corollary 1.

$$h_{top}(\Phi_\lambda) = \left[\frac{\mathfrak{r}(F)}{\Pi_{r\in\Psi_n}\ \omega_r |r|}\right]^{\frac{1}{n}} \frac{\sqrt{2}\,\omega\, h^*}{(\det \mathbf{C})^{\frac{1}{2n}}}. \tag{2}$$

Remark. When $n = 1$ and $\Psi = A_1^{(1)} = \{1, -1\}$, Σ is the 3-dimensional mapping torus of a hyperbolic toral automorphism, $\mathbf{A} \in \mathrm{GL}(2;\mathbb{Z})$, whose eigenvalues have absolute value $\xi^{\pm 1}$. The number field $F = \mathbb{Q}(\xi)$, $\mathfrak{r}(F) = |\ln \xi|$, $\Psi_1 = \{1\}$, $\omega_r = 1$ for all $r \in \Psi$, $\mathbf{C} = [2]$, $h^* = 1$ and $\lambda = |\ln \xi|$. Formula 2 gives $h_{top}(\Phi_\lambda) = |\ln \xi|$. On the other hand, $\mathcal{V}$ is diffeomorphic to $\Sigma \times \{\pm 1\}$. On $\Sigma \times \{\pm 1\}$, $\Phi_{\pm\lambda t}$ is conjugate to the suspension flow of $\mathbf{A}$. Since the suspension flow has topological entropy $|\ln \xi|$, this shows the formula of Corollary 1 is correct in this simple case. This example was studied in [5] by Bolsinov and Taĭmanov and is the first known example of an integrable system with positive topological entropy.

There is a natural involution on the set of root bases, $r \to \check{r} = r/|r|^2$. Ψ and $\check{\Psi}$ are said to be *dual* root bases; Ψ is self-dual if $\check{\Psi} = \Psi$. The root bases $B_n^{(1)}$ and $A_{2n-1}^{(2)}$, $C_n^{(1)}$ and $D_{n+1}^{(2)}$, $F_4^{(1)}$ and $E_6^{(2)}$, $G_2^{(1)}$ and $D_4^{(3)}$ are dual, and all other root bases are self-dual. Theorem 2 and Corollary 1 allow us to prove:

Theorem 3. *Let* $[[u]]$ *be the integer part of* u*. The following is a list of* $h^*(\Psi)$ *for each root basis* Ψ*:*

Ψ (rank $= n$)	h^*	$\Pi_{r\in\Psi_n} \lvert\omega_r r\rvert$	ω	$\det \mathbf{C}$	$h_{top}(\Phi_\lambda)$
$A_n^{(1)},\ n \ge 1$	$\sqrt{[[\frac{n+1}{2}]]}$	1	1	$n+1$	$\mathfrak{r}(\Gamma)^{\frac{1}{n}} \sqrt{2\frac{[[\frac{n+1}{2}]]}{(n+1)^{\frac{1}{n}}}}$
$B_n^{(1)},\ n \ge 3$	$\sqrt{n-1}$	$2^{\frac{3}{2}(n-1)}$	2	2	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2^{\frac{1}{n}}\ \sqrt{n-1}$
$C_n^{(1)},\ n \ge 2$	$\sqrt{\frac{n}{2}}$	$2^{n-\frac{1}{2}}$	2	2	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ \sqrt{n}$
$D_n^{(1)},\ n \ge 4$	$\sqrt{\frac{n-2}{2}}$	2^{n-3}	2	4	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2^{-\frac{2}{n}}\ \sqrt{n-2}$
$A_{2n}^{(2)},\ n \ge 2$	$\sqrt{n}$	$2^{\frac{1}{2}(3n-1)}$	2	2	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ \sqrt{n}$
$A_{2n-1}^{(2)},\ n \ge 3$	$\sqrt{\frac{n-1}{2}}$	$2^{n-\frac{3}{2}}$	2	2	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2^{\frac{1}{n}}\ \sqrt{n-1}$
$D_{n+1}^{(2)},\ n \ge 2$	$\sqrt{n}$	$2^{\frac{1}{2}(n-1)}$	1	2	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ \sqrt{n}$
$G_2^{(1)},\ (n = 2)$	$1/\sqrt{3}$	$6\sqrt{3}$	6	1	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2/3^{\frac{1}{4}}$
$A_2^{(2)},\ (n = 1)$	$1/\sqrt{2}$	2	2	2	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ \sqrt{2}$
$D_4^{(3)},\ (n = 2)$	1	$2\sqrt{3}$	2	1	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2/3^{\frac{1}{4}}$
$F_4^{(1)},\ (n = 4)$	$1/\sqrt{6}$	$2^5 3$	$2^2 3$	1	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2^{\frac{3}{4}} 3^{\frac{1}{4}}$

Ψ (rank $= n$)	h^*	$\Pi_{r\in\Psi_n} \ \lvert\omega_r r\rvert$	ω	det $\mathbf{C}$	$h_{top}(\Phi_\lambda)$
$E_6^{(1)}$, $(n=6)$	$1/\sqrt{3}$	$2^3 3$	6	3	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 23^{\frac{1}{4}}$
$E_6^{(2)}$, $(n=4)$	$1/\sqrt{3}$	$2^3 3$	6	1	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2^{\frac{3}{4}}3^{\frac{1}{4}}$
$E_7^{(1)}$, $(n=7)$	$1/\sqrt{6}$	$2^5 3^2$	12	2	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2^{\frac{17}{14}}3^{\frac{3}{14}}$
$E_8^{(1)}$, $(n=8)$	$1/2\sqrt{15}$	$2^7 3^3 5$	$2^2 15$	1	$\mathfrak{r}(\Gamma)^{\frac{1}{n}}\ 2^{\frac{5}{8}}3^{\frac{1}{8}}5^{\frac{3}{8}}$

Say that $h \in \mathrm{Homeo}(T^*\Sigma)$ is *energy-preserving* if $h(\{H_1 = \frac{1}{2}\}) = \{H_2 = \frac{1}{2}\}$. *A priori*, an energy-preserving conjugacy need not map $\mathcal{V}_1$ onto $\mathcal{V}_2$, so the topological entropies of Theorem 3 are not obvious invariants of energy-preserving conjugacy. Invariance is proven below (Lemma 21), and thus:

Theorem 4. *Let φ_t^i be a hamiltonian flow contructed from the root basis Ψ_i, $i = 1, 2$. If there is an energy-preserving conjugacy of φ_t^1 with φ_t^2, then either $\Psi_1 = \Psi_2$, or $\{\Psi_1, \Psi_2\} = \{A_{2n}^{(2)}, D_{n+1}^{(2)}\}$ or n is even and $\{\Psi_1, \Psi_2\} = \{A_n^{(1)}, C_n^{(1)}\}$.*

Let λ_i be the normalization constant for φ_t^i. If there is an energy-preserving conjugacy of $\varphi_{\lambda_1 t}^1$ with $\varphi_{\lambda_2 t}^2$, then either $\Psi_1 = \Psi_2$ or $\Psi_2 = \check{\Psi}_1$ or $\Psi_1, \Psi_2 \in \{C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}\}$.

To improve Theorem 4, we show that the homology spectrum of φ^i is the graph of a quadratic form defined on $H_1(T^*\Sigma; \mathbb{Z})$, and an energy-preserving conjugacy of φ^1 with φ^2 implies that the two quadratic forms are congruent. Except when $n = 2$, resolution of the congruence question appears to be connected to a conjecture in transcendental number theory, namely,

Gelfond conjecture. (c.f. [25]) *Let $\mathcal{S}$ be a set of logarithms of algebraic numbers. If $\mathcal{S}$ is linearly independent over $\mathbb{Q}$, then $\mathcal{S}$ is algebraically independent over $\mathbb{Q}$.*

Theorem 5 and Corollary 3 rest, in fact, on a weaker version of Gelfond's conjecture, namely that rational independence of logarithms of algebraic numbers implies their *homogeneous* independence. On the other hand, it is worth mentioning a more general conjecture due to Schanuel. This conjecture says that if $\mathcal{S} \subset \mathbb{C}$ is rationally-independent, then the field $\mathbb{Q}(\mathcal{S}, \exp(\mathcal{S}))$ contains at least $\#\mathcal{S}$ algebraically-independent elements. The analogous conjectures for certain function fields have been proven, but the number-theory conjectures remain unproven [2, 13, 28, 25].

The Gelfond conjecture and this paper are connected through the identification of a maximal-abelian subgroup of $\mathrm{GL}(n+1; \mathbb{Z})$ with the unit-group of a number field of degree $n+1$. To explain, let $F/\mathbb{Q}$ be a totally real, normal extension of $\mathbb{Q}$ of degree $n+1$, let $\mathbf{G}$ be the Galois group of $F/\mathbb{Q}$, let $V = \mathbb{R}\mathbf{G}$ be the group ring, let $\bar{t} = \sum_{\sigma \in \mathbf{G}} \sigma$ and let $V_o = \ker \bar{t}$ be the augmentation ideal of V. Let $\mathbf{G}^*$ be the basis of V^* dual to $\mathbf{G}$ and $\hat{\mathbf{G}} = \{\hat{\sigma}|_{V_o} : \sigma \in \mathbf{G}\}$. For each $x \in \mathbb{Q}\mathbf{G}/\mathbb{Q}\bar{t}$, let $\bar{R}'_x : V_o^* \to V_o^*$ be the linear map induced by right-translation by x. In addition, let $\Omega := \{w_r r : r \in \Psi\}$ and let $\mathfrak{B}$ be the subset of linear isomorphisms $\phi : V_o^* \to \mathfrak{h}^*$ which satisfy $\phi(\hat{\mathbf{G}}) = \Omega$. The hamiltonians of Theorem 1 are parameterized by $\phi \in \mathfrak{B}$ – hence the $(n+1)!$ in the first sentence – so H_ϕ (resp. φ_t^ϕ) denotes the hamiltonian (resp. flow) constructed with the bijection ϕ. Finally, let $O(\mathfrak{h}^*)$ be the orthogonal group of $(\mathfrak{h}^*, \langle\langle , \rangle\rangle)$. If $\phi_1, \phi_2 \in \mathfrak{B}$, when is there an energy-preserving topological conjugacy of $\varphi_t^{\phi_1}$ with $\varphi_t^{\phi_2}$?

To answer the preceding question, recall that the unit-group of F is a natural $\mathbb{Z}\mathbf{G}$-module. Let $\mathcal{A} \subset \mathbb{Q}\mathbf{G}$ be the subring of elements that are integral with respect to this representation. It is known that $\mathbb{Z}\mathbf{G} + \mathbb{Q}\bar{t} \subseteq \mathcal{A}$ and $(n+1)\mathcal{A} \subseteq \mathbb{Z}\mathbf{G} + \mathbb{Q}\bar{t}$.

The following theorem, when $n = 2$, is a consequence of Gelfond's classical theorem that the ratio of logarithms of two algebraic numbers is either rational or transcendental [15].

Theorem 5. *If $n \geq 3$, assume the Gelfond conjecture. If the hamiltonian flows of H_{ϕ_1} and H_{ϕ_2} are topologically conjugate by an energy-preserving conjugacy then there is a unit r of the ring $S_o = \mathcal{A}/\mathbb{Q}\bar{t}$ such that*

$$\mu = \phi_1 \circ \bar{R}'_r \circ \phi_2^{-1} \in O(\mathfrak{h}^*).$$

Theorem 5 shows how to define an equivalence relation $\sim$ on $\mathfrak{B}$ that is possibly coarser than the equivalence relation induced by energy-preserving conjugacy. Section 4 proves

Corollary 2. *Let $F/\mathbb{Q}$ be a normal cubic extension. If*

1. $\Psi = A_2^{(1)}$ *or* $\Psi = C_2^{(1)}$, *then $\mathfrak{B}$ contains a single equivalence class;*
2. $\Psi = G_2^{(1)}$, *then $\mathfrak{B}$ contains 2 equivalence classes.*

Corollary 3. *Assume the Gelfond conjecture. Let $F/\mathbb{Q}$ be a normal totally-real quartic extension. If*

1.i $\mathbf{G} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and* $\Psi = A_3^{(1)}$ *or* $\Psi = B_3^{(1)}$, *then $\mathfrak{B}$ contains 3 equivalence classes;*

1.ii $\mathbf{G} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and* $\Psi = C_3^{(1)}$, *then $\mathfrak{B}$ contains 6 equivalence classes;*

2.i $\mathbf{G} = \mathbb{Z}_4$ *and* $\Psi = A_3^{(1)}$, *then $\mathfrak{B}$ contains 2 equivalence classes;*

2.ii $\mathbf{G} = \mathbb{Z}_4$ *and* $\Psi = B_3^{(1)}$ *or* $\Psi = C_3^{(1)}$, *then $\mathfrak{B}$ contains 3 equivalence classes.*

In both corollaries one also knows that if $\phi_i \in \mathfrak{B}_i$ – where Ψ_i are not necessarily isometric –, then there is an energy-preserving conjugacy of $\varphi_t^{\phi_1}$ with $\varphi_t^{\phi_2}$ iff $\Psi_1 = \Psi_2$ and $\phi_1 \sim \phi_2$. The conjugacies of Corollaries 2 and 3 arise from a natural action of the Galois group $\mathbf{G}$ on $T^*\Sigma$ and are analytic. The sharpness of these results is due to the triviality of the unit group of S_o – it is just $\pm\mathbf{G} + \mathbb{Q}\bar{t}$.

Remark (c.f. Example 4 of Sect. 4.3). To illustrate the construction behind Theorem 1 take the case where Γ is the group generated by

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Γ is conjugate by a $T \in \mathrm{SL}(3;\mathbb{R})$ to the group Γ' generated by

$$\mathcal{B}_1 = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_3 & 0 \\ 0 & 0 & \alpha_4 \end{bmatrix},$$

where $\alpha_j = \zeta^j + \zeta^{-j}$ for $j = 1, 2, 3$ and ζ is a primitive 7-th root of unity and $\alpha_4 = \alpha_1$. Let $\mathrm{N} = T(\mathbb{Z}^3)$ and $\Delta = \Gamma' \star \mathrm{N}$ so that $T^*\Sigma = T^*(\Delta \backslash \mathbb{R}^2 \times \mathbb{R}^3)$. If the matrix $\mathbf{M}$ with columns $\mathbf{M}_i$ is defined to be

$$\mathbf{M} = \begin{bmatrix} \ln|\alpha_1| & \ln|\alpha_2| & \ln|\alpha_3| \\ \ln|\alpha_2| & \ln|\alpha_3| & \ln|\alpha_4| \end{bmatrix},$$

while $(a, A, b + \mathrm{N}, B) \in T^*\mathbb{R}^2 \times T^*\mathbb{T}^3$ are the coordinates of $P \in T^*\hat{\Sigma}$ and $v \in \mathbb{R}^2$, then

$$A_v(P) = \langle v, A\rangle, \ \ \gamma_i(P) = B_i^2 \exp(2\langle a, \mathbf{M}_i\rangle),$$

define analytic functions on $T^*\Sigma$ whose Poisson brackets are

$$\{A_v, A_{v'}\} = \{\gamma_i, \gamma_j\} = 0, \quad \{\gamma_j, A_v\} = 2\langle v, \mathbf{M}_j\rangle\, \gamma_j.$$

Let v_i be vectors such that $\langle v_i, \mathbf{M}_j\rangle = \delta_{ij}$ for $i, j = 1, 2$. A calculation shows that

$$H = A_{v_1}^2 + A_{v_1}A_{v_2} + A_{v_2}^2 + \gamma_1 + \gamma_2 + \gamma_3$$

enjoys the first integrals

$$F = A_{v_1}^2 A_{v_2} + A_{v_1}A_{v_2}^2 + A_{v_1}(\gamma_2 - \gamma_1) - A_{v_2}(\gamma_1 + \gamma_3),$$

and

$$f = (\ln|B_1|, \ln|B_2|, \ln|B_3|) + \mathcal{L},$$

where $x \in \mathcal{L}$ iff $x_i = a_1 \ln|\alpha_i| + a_2 \ln|\alpha_{i+1}|$ and $a_j \in \mathbb{Z}$. If f “extends” smoothly to the set $B_1B_2B_3 = 0$ – and it does – then one sees H is completely integrable in the sense of Liouville. Indeed, what makes this construction work is that there is a Lie algebra isomorphism

$$A_{v_i} \to \xi_i, \quad \gamma_j \to \delta_j,$$

where $\delta_1 = \exp(2x_1)$, $\delta_2 = \exp(2x_2)$, $\delta_3 = \exp(-2x_1 - 2x_2)$ and (x_i, ξ_i) are canonical coordinates on $T^*\mathbb{R}^2$. H is the pullback of the $A_2^{(1)}$ Bogoyavlenskij-Toda hamiltonian

$$\mathfrak{T} = \xi_2^2 + \xi_1\xi_2 + \xi_2^2 + \delta_1 + \delta_2 + \delta_3,$$

and F is the pullback of a first-integral of $\mathfrak{T}$. H is, up to a scalar multiple, one of the hamiltonians constructed in Theorem 1.

One notices an arbitrariness in the construction of H. If one takes a permutation ρ of $\{1, 2, 3\}$ and uses the vectors v_i', where $\langle v_i', \mathbf{M}_{\rho(j)}\rangle = \delta_{ij}$ for $i, j = 1, 2$, in place of v_i to define H, then the new hamiltonian H' is integrable and looks very much like H. However, to determine if the hamiltonian flow of H' is the same as that of H requires the more precise, more intrinsic construction provided by algebraic number theory.

Similar comments apply to the $C_2^{(1)}$ and $G_2^{(1)}$ Bogoyavlenskij-Toda hamiltonians.

1.1. Background

Before the 20-th century, mathematicians working on dynamical problems were largely concerned with integrating differential equations. This point of view is expressed in Liouville's theorem concerning the integrability in quadratures of an n-degree of freedom system with n involutive and independent first integrals. The discovery of transverse homoclinic points and the resulting dynamical complexity showed that integrable systems are "rare." And much work has gone into making precise the meaning of the word "rare." (c.f. [30,24,27]). Research on integrable systems went into a lull until, in the 1960s, the partial differential equation that describes a 1-dimensional shallow-water wave was shown to be integrable. Lax [26] put the integrability of the KdV equation into perspective by showing that the equations are equivalent to an operator evolution equation $\dot{L} = [L, A]$. The eigenvalues of L are conserved quantities of motion.

Flaschka [14] showed that the equations of motion that describe n particles on the real line that interact via an exponential, repulsive potential also enjoys a so-called Lax-pair presentation. Hénon [17] showed that these first integrals are independent. This dynamical system, called the (periodic) Toda lattice, also has a "continuum limit" which is the KdV equation [17]. Bogoyavlenskij exposed an underlying Lie-algebraic structure by showing that a Toda-like lattice can be constructed for any set of *admissible* roots of a simple Lie algebra [3]. Adler, Kostant and Symes built on this by showing that Lie-algebraic structures also explain the integrability of the Bogoyavlenskij-Toda lattices [22,1,38]. The subsequent literature on this topic is enormous (c.f. [24,34]).

This paper originates from the following question: if Σ is a C^∞ manifold, does the existence of a smoothly integrable geodesic flow on $T^*\Sigma$

impose restrictions on the topology of Σ? In the category of compact analytic manifolds, the Kozlov-Taĭmanov theorem places strong restrictions on $\pi_1(\Sigma)$ – it must be almost abelian – and on $H^*(\Sigma)$ [23,39,40]. In the C^∞ category, the potential pathology of the singular set of a smooth map has frustrated a generalization of the Kozlov-Taĭmanov theorem [9]. Paternain's approach [31–33], based on the idea that integrable systems should have zero topological entropy, has been vitiated by the surprising example of Bolsinov and Taĭmanov which is smoothly integrable and has positive entropy (c.f. [5,12]). In a couple important respects, the examples in this paper are similar to those in [5,7–10].

1.2. Outline

Theorem 1 is proven by constructing a $2n+1$-dimensional solvable Poisson subalgebra $\mathfrak{s}$ of $C^\infty(T^*\Sigma)$. The Bogoyavlenskij-Toda hamiltonian and its first integrals are located inside the symmetric algebra of $\mathfrak{s}$. This construction provides only $n+1$ involutive and independent first integrals, so we need an additional n first integrals that are independent and commute with the first $n+1$. This is done by finding a second, n-dimensional abelian subalgebra $\mathfrak{a}$ of $C^\infty(T^*\Sigma)$ that centralizes $\mathfrak{s}$. The algebras $\mathfrak{s}$ and $\mathfrak{a}$ exist ultimately because the universal cover $\tilde{\Sigma}$ admits the structure of a solvable Lie group. This observation is pursued in the Remark after the proof of Theorem 1, where it is also shown how the constructions can be understood in terms of the traditional momentum map.

In Sect. 3, the topological entropies of Theorem 3 are computed. Section 4 uses asymptotic homology to prove the invariance of the entropies, and the homology spectrum of the flows to prove Theorem 5 and its corollaries.

2. The construction

2.1. Poisson geometry and the momentum map

Recall a few important concepts. Let Σ be a real-analytic ($= C^\omega$) manifold. The smooth functions on the cotangent bundle of Σ, $C^\infty(T^*\Sigma)$, has two canonical algebraic structures: it is an abelian algebra when equipped with the natural operations of point-wise addition and multiplication; and, coupled with the canonical *Poisson bracket*, $\{,\}$, $(C^\infty(T^*\Sigma), \{,\})$ is a Lie algebra of derivations of the algebra $C^\infty(T^*\Sigma)$. A hamiltonian $H \in C^\infty(T^*\Sigma)$ induces a vector field $Y_H := \{., H\}$. We are interested in the situation where Y_H has "many" independent first integrals. If $\mathcal{F} \subset C^\infty(T^*\Sigma)$, let $d\mathcal{F}_P = \operatorname{span}\{df_P \ :\ f \in \mathcal{F}\}$ and let $Z(\mathcal{F}) = \{f \in \mathcal{F}\ :\ \{\mathcal{F}, f\} \equiv 0\}$. Typically $\mathcal{F}$ is a Lie subalgebra of $C^\infty(T^*\Sigma)$, and $Z(\mathcal{F})$ is the centre of $\mathcal{F}$. Let $k = \sup\dim d\mathcal{F}_P$, $l = \sup\dim dZ(\mathcal{F})_P$. Let us say $P \in T^*\Sigma$ is $\mathcal{F}$-*regular* if there exist $f_1, \ldots, f_k \in \mathcal{F}$ such that P is a regular value for the map $F = (f_1, \dots, f_k)$ and $f_1, \dots, f_l \in Z(\mathcal{F})$; if P is not $\mathcal{F}$-regular then it is $\mathcal{F}$*-critical*. Let $L(\mathcal{F})$ be the set of $\mathcal{F}$-regular points.

H will be proper for the purposes of this paper.

Definition 1. [c.f. [4]] *Say that $H \in C^\infty(T^*\Sigma)$ is* **integrable** *if there is a Lie subalgebra $\mathcal{F} \subset C^\infty(T^*\Sigma)$ such that:*

(I1) $H \in Z(\mathcal{F})$;
(I2) $k + l = \dim T^*\Sigma$ and $L(\mathcal{F})$ is an open and dense subset of $T^*\Sigma$.

We will say that $\mathcal{F}$ is an **integrable subalgebra**. *If $\mathcal{F}^\omega = \mathcal{F} \cap C^\omega(T^*\Sigma)$ is also an integrable subalgebra and $H \in \mathcal{F}^\omega$ then we say H is real-analytically integrable.*

See [4] for an analogous definition and further explanation and references. The usual definition of complete integrability or non-commutative integrability are special cases of Definition 1 with $\mathcal{F} = \operatorname{span}\{f_1, \dots, f_k\}$ and $l = k$ (resp. $l \le k$) and the regular-point set of $F = (f_1, \dots, f_k)$ is dense. Definition 1 is both more intrinsic, and more suited to the examples of the present paper.

Definition 1 is equivalent to the integrability of the flow Φ_t of Y_H in the sense of the first sentence of the present paper. To see this, let G be the abelian group of C^∞ diffeomorphisms of $T^*\Sigma$ generated by the complete flows of Y_f, $f \in Z(\mathcal{F})$. The subalgebra $Z(\mathcal{F})$ defines a nonsingular distribution on $L(\mathcal{F})$, and by the Sussman-Stefan orbit theorem [20], the orbits of G are immersed C^∞ submanifolds. Condition (I2) implies that these orbits are actually embedded submanifolds of $L(\mathcal{F})$. Condition (I2) and the properness of H also imply that for each point in $L(\mathcal{F})$, there is a G-invariant open neighbourhood, U, and an action of $\mathbb{T}^l$ on U, such that the $\mathbb{T}^l$-orbits and G-orbits coincide. Since $H \in Z(\mathcal{F})$, the flow mapping Φ_t is a 1-parameter subgroup of G. Thus there is a C^∞ atlas $\mathfrak{A} = \{\varphi : U \to \mathbb{T}^l \times \mathbf{D}^k\}$ of $L(\mathcal{F})$, where $\mathbf{D}^k$ is an open disk in $\mathbb{R}^k$, and $\mathfrak{A}$ satisfies the universal property that for all 1-parameter subgroups g^t of G and $x \in \mathbb{T}^l$, $y \in \mathbf{D}^k$: $\varphi \circ g^t \circ \varphi^{-1}(x, y) = (x + t\xi(y), y)$ where ξ is smooth. From this discussion it follows that $L(\mathcal{F})$ has a natural structure of a $\mathbb{T}^l$ bundle over a smooth k-manifold B. It is possible that $L(\mathcal{F})$ is not a trivial bundle, as in the examples of this paper.

2.2. Algebraic preliminaries

If Γ is a subgroup of $\mathrm{GL}(n+1; \mathbb{Z})$, let G be the semi-direct product of Γ with $\mathbb{Z}^{n+1}$. If Γ is $\mathbb{R}$-split and acts irreducibly on $\mathbb{Z}^{n+1}$, then there is a totally real algebraic number field F of degree $n+1$, with integers $\mathcal{O}$ and unit group $\mathcal{U}$, and an embedding of G into the semi-direct product of $\mathcal{U}$ with $\mathcal{O}$ [36]. It is apparent that the most natural way to construct the manifolds Σ is to employ algebraic number theory. Thus, let us record:

Standing notation/hypotheses :

- F is a totally real algebraic number field of degree $n+1$ over $\mathbb{Q}$;
- E is the splitting field of F;
- $\mathbf{G}$ is the set of embeddings ($\mathbb{Q}$-isomorphisms) $\sigma : F \to \mathbb{R}$;
- $\mathcal{O}$ is the ring of integers of F and $\mathcal{U}$ is the group of units of $\mathcal{O}$;
- $V = \bigoplus_{\sigma \in \mathbf{G}} \mathbb{R}\sigma$ is the real vector space with basis $\mathbf{G}$;
- $\epsilon : V \to \mathbb{R}$ is the "augmentation map" that maps $\mathbf{G}$ to 1.
- $V_o = \sum_{\sigma,\tau \in \mathbf{G}} \mathbb{R}(\sigma - \tau)$ is the kernel of the "augmentation map";
- $\ell : \mathcal{U} \to V_o$ is the "logarithm map" defined by $\ell(u) = \sum_{\sigma \in \mathbf{G}} \ln|\sigma(u)|\,\sigma$;
- $\eta : \mathcal{O} \to V$ is the group isomorphism $\eta(\alpha) = \sum_{\sigma \in \mathbf{G}} \sigma(\alpha)\,\sigma$.

2.3. The configuration spaces Σ

The following are restatements of well-known facts from algebraic number theory [29]:

Lemma 1. *Let $\mathcal{L} = \ell(\mathcal{U})$. Then $\mathcal{L}$ is a discrete, cocompact subgroup of V_o that is isomorphic to $\mathbb{Z}^n$ and there is an index 2 subgroup $\mathcal{U}^+$ such that $\ell : \mathcal{U}^+ \to \mathcal{L}$ is an isomorphism.*

Lemma 2. *Let* $\mathrm{N} = \eta(\mathcal{O})$. *Then* N *is a discrete, cocompact subgroup of* V.

Define an action of $\mathcal{U}$ on $\hat{\Sigma} = V_o \oplus V/\mathrm{N}$ by:

$$u * (x, y + \mathrm{N}) = (x + \ell(u), \sum_{\sigma \in \mathbf{G}} \langle y, \hat{\sigma} \rangle \sigma(u)\sigma + \mathrm{N}), \tag{3}$$

for all $u \in \mathcal{U}$, $x \in V_o$ and $y + \mathrm{N} \in V/\mathrm{N}$. Since $\mathcal{U}$ is the unit group of $\mathcal{O}$, this action is well-defined and each $u \in \mathcal{U}$ acts as an analytic diffeomorphism of $\hat{\Sigma}$. The action of $\mathcal{U}^+$ is free, cocompact and uniformly discrete so $\Sigma = \mathcal{U}^+ \backslash \hat{\Sigma}$ is a compact real-analytic manifold. Let $\Delta = \mathcal{U}^+ \star \mathcal{O}$ be the semi-direct product of $\mathcal{U}^+$ with $\mathcal{O}$. It is clear that the fundamental group of Σ is isomorphic to Δ, and $H_1(\Sigma; \mathbb{Z}) \simeq \mathcal{L}$. It is also clear that the covering $\pi : \hat{\Sigma} \to \Sigma$ is the universal abelian covering of Σ, and the deck-transformation group of π is $\mathcal{L}$.

The covering map $T^*\hat{\Sigma} \to T^*\Sigma$ will be denoted by Π. Let $(x, y, \mathcal{X}, \mathcal{Y})$ denote canonical coordinates on $T^*\hat{\Sigma}$, where $x \in V_o$, $\mathcal{X} \in V_o^*$, $y \in V/\mathrm{N}$ and $\mathcal{Y} \in V^*$. The V_o-valued 1-form dx and the V_o^*-valued map $\mathcal{X}$ are $\mathcal{U}^+$-invariant (see next section), so the map $\hat{\mathbf{P}} : T^*\hat{\Sigma} \to T^*V_o$ given by $\hat{\mathbf{P}}((x, y, \mathcal{X}, \mathcal{Y})) = (x, \mathcal{X})$ descends to define a submersion $\mathbf{P} : T^*\Sigma \to T^*(V_o/\mathcal{L})$ given by $\mathbf{P}(\Delta(x, y, \mathcal{X}, \mathcal{Y})) = (x + \mathcal{L}, \mathcal{X})$. The following lemma is immediate:

Lemma 3. *The covering $\Pi : T^*\hat{\Sigma} \to T^*\Sigma$ is the universal abelian covering of $T^*\Sigma$. The deck transformation group* $\mathrm{Deck}(\Pi)$ *is $\mathcal{L}$. The projection $\mathbf{P} : T^*\Sigma \to T^*(V_o/\mathcal{L})$ induces a natural isomorphism of first integral homology groups.*

Let $u \in \mathcal{U}^+$ and $\sigma \in \mathbf{G}$. The diffeomorphism of $\hat{\Sigma}$ defined by equation (3) will also be denoted by u. Let $y_\sigma := \langle y, \hat{\sigma} \rangle$ denote the σ-component of y. It is immediate that

$$u^*(dy_\sigma) = \sigma(u)\, dy_\sigma. \tag{4}$$

Let T^*u be the canonical diffeomorphism of $T^*\hat{\Sigma}$ induced by u. Equation (4) implies that

$$(T^*u)^* \mathcal{Y}_\sigma = \sigma(u)^{-1}\, \mathcal{Y}_\sigma, \tag{5}$$

where $\mathcal{Y}_\sigma = \langle \sigma, \mathcal{Y} \rangle$ is the value of the covector $\mathcal{Y}$ evaluated at σ.

Fix a positive integer b_σ, and define $\gamma_\sigma : T^*\hat{\Sigma} \to \mathbb{R}$ by

$$\gamma_\sigma = \exp(2b_\sigma \langle x, \hat{\sigma} \rangle)\, \mathcal{Y}_\sigma^{2b_\sigma}. \tag{6}$$

Equations (3) and (5) imply that γ_σ is a $\mathcal{U}^+$-invariant analytic function. Hence, γ_σ will be regarded as an analytic function on $T^*\Sigma$.

Let $b = \operatorname{lcm}(b_\sigma : \sigma \in \mathbf{G})$ and let $c_\sigma = b/b_\sigma$. Because $\sum_{\sigma \in \mathbf{G}} \hat{\sigma}|_{V_o} = 0$, it follows that

$$\mathbf{k} = \Pi_{\sigma \in \mathbf{G}}\, \gamma_\sigma^{c_\sigma} = \Pi_{\sigma \in \mathbf{G}}\, \mathcal{Y}_\sigma^{2b}, \tag{7}$$

defines an analytic function on $T^*\Sigma$.

For each $v \in V_o$, let $\mathcal{X}_v$ be the induced linear function on V_o^*. $\mathcal{X}_v$ defines a function on $T^*\Sigma$, and

$$\{\gamma_\sigma, \mathcal{X}_v\} = 2b_\sigma \langle v, \hat{\sigma} \rangle \gamma_\sigma. \tag{8}$$

Equations (6) and (8) imply that

Lemma 4. *The subspace $\mathfrak{s}$ of $C^\omega(T^*\Sigma)$ spanned by $\{\gamma_\sigma : \sigma \in \mathbf{G}\}$ and $\{\mathcal{X}_v : v \in V_o\}$ is a solvable Lie subalgebra of dimension $2n+1$.*

Let $Sym(\mathfrak{s})$ denote the symmetric algebra on $\mathfrak{s}^*$; it is naturally viewed as the set of functions on $T^*\Sigma$ that are polynomial combinations of elements in $\mathfrak{s}$. Clearly, $Sym(\mathfrak{s})$ is a Lie subalgebra of $C^\omega(T^*\Sigma)$.

2.4. The Lie algebras $\mathfrak{s}$ and $\mathfrak{t}$

Let Ψ be a root basis of $(\mathfrak{h}^*, \langle\langle, \rangle\rangle)$. For each $r \in \Psi$, define the function

$$\delta_r = c_r^2 \exp(2\langle q, r \rangle), \tag{9}$$

for all $q \in \mathfrak{h}$ and $c_r \in \mathbb{R}$. For $s \in \mathfrak{h}$, let ξ_s be the linear function on $\mathfrak{h}^*$ induced by s. Let $[,]$ denote the canonical Poisson bracket on $\mathfrak{h} \times \mathfrak{h}^*$, so that:

$$[\delta_r, \xi_s] = 2\langle s, r \rangle\, \delta_r \tag{10}$$

for all $s \in \mathfrak{h}$ and $r \in \Psi$. Let $\Xi = T^*\mathfrak{h} \times \mathbb{R}^{n+1}$.

Lemma 5. *The subspace* $\mathfrak{t} \subset C^{\omega}(\Xi)$ *spanned by the functions* $\{\delta_r \ : \ r \in \Psi\}$ *and* $\{\xi_s \ : \ s \in \mathfrak{h}\}$ *is a* $2n+1$*-dimensional solvable Lie subalgebra of* $C^{\omega}(\Xi)$.

Let $Sym(\mathfrak{t})$ be the symmetric algebra of $\mathfrak{t}^*$. $Sym(\mathfrak{t})$ contains the distinguished Bogoyavlenskij-Toda hamiltonian:

$$\mathfrak{T}(q, \xi, c) := \frac{1}{2}\langle\langle \xi, \xi \rangle\rangle + \sum_{r \in \Psi} \delta_r. \tag{11}$$

The complete integrability of the Bogoyavlenskij-Toda lattices is by no means obvious. Here is a sketch of this fact, based on Bogoyavlenskij's generalization of Flaschka's transformation, which also shows that the integrals lie in $Sym(\mathfrak{t})$. View $\mathfrak{h}$ as the real part of a Cartan subalgebra of a simple Kac-Moody algebra $\mathfrak{g}$. For each root $r \in \Psi$, there are elements $e_{\pm r}$ such that $[h, e_{\pm r}] = \pm\langle h, r \rangle e_{\pm r}$ for all $h \in \mathfrak{h}$. Let $\kappa : \mathfrak{h}^* \to \mathfrak{h}$ denote the linear isomorphism induced by the Cartan-Killing form $\langle\langle , \rangle\rangle$. Hamilton's equations for $\mathfrak{T}$ are equivalent to the Lax equations $\dot{L} = [L, A]$ where $L = \kappa\xi + \sum_{r \in \Phi} \delta_r e_{-r} + e$, $A = -\kappa\xi - e$ and $e = \sum_{r \in \Phi} e_r$. From this it follows that Trace $\rho(L)^k$ is a first integal of $\mathfrak{T}$ for any representation ρ of $\mathfrak{g}$. These integrals suffice for the complete integrability of $\mathfrak{T}$. It is also apparent that these first integrals are polynomials in ξ and δ_r. See [3,34] for further information.

Let $\mathfrak{I}$ denote the algebra of integrals of $\mathfrak{T}$ contained in $Sym(\mathfrak{t})$. The following lemma is immediate:

Lemma 6. *Let* $L(\mathfrak{I}) \subset \Xi$ *be the set of regular points of* $\mathfrak{I}$. *Then* $L(\mathfrak{I})$ *is an open, real-analytic set.*

Let $w_r := \omega_r/\omega$ for each $r \in \Psi$, where $\omega = \mathrm{lcm}(\omega_r : r \in \Psi)$. Let ϕ be a linear map $\phi : V_o^* \to \mathfrak{h}^*$ that is also a bijection from $\hat{\mathbf{G}}$ to $\Omega = \{w_r r \ : \ r \ \in \ \Psi\}$. Write

$$\phi(\hat{\sigma}) = w_r r, \tag{12}$$

and let $\phi' : \mathfrak{h} \to V_o$ denote the transpose of ϕ. The map $\Phi : \mathfrak{t} \to \mathfrak{s}$ induced by ϕ is defined by

$$\Phi(\delta_r) := \gamma_\sigma, \tag{13}$$

$$\Phi(\xi_s) := \mathcal{X}_v, \tag{14}$$

where $\phi(\hat{\sigma}) = w_r r$ and $v = \phi'(s)$.

Lemma 7. Φ *is a Lie algebra isomorphism iff for all* $\sigma \in \mathbf{G}$, $b_\sigma = \omega/\omega_r$ *where* $r \in \Psi$ *is the unique root such that* $\phi(\hat{\sigma}) = w_r r$.

Proof. $\Phi([\delta_r, \xi_s]) = 2\langle s, r \rangle \gamma_\sigma$, while $\{\Phi(\delta_r), \Phi(\xi_s)\} = 2b_\sigma \langle v, \hat{\sigma} \rangle \gamma_\sigma$ where $\phi(\hat{\sigma}) = w_r r$ and $v = \phi'(s)$. On the one hand, $b_\sigma \langle v, \hat{\sigma} \rangle = b_\sigma \langle \phi'(s), \hat{\sigma} \rangle = b_\sigma w_r \langle s, r \rangle$, while on the other hand, we require that $b_\sigma \langle v, \hat{\sigma} \rangle = \langle s, r \rangle$. This proves the assertion.

Since ω is the lcm of the integers ω_r, b_σ is an integer and so the functions γ_σ are real-analytic for all $\sigma \in \mathbf{G}$. Thus Φ extends to an isomorphism – denoted also by Φ – of the respective symmetric algebras. There are a number of distinguished elements in $Sym(\mathfrak{s})$. To describe the most important one, let $\kappa : \mathfrak{h}^* \to \mathfrak{h}$ be the linear isomorphism induced by $\langle\langle , \rangle\rangle$, and define $\mathcal{Q} = \phi' \kappa \phi : V_o^* \to V_o$. Let $H = \Phi(\mathfrak{T})$; H is written as

$$H = \frac{1}{2}\langle \mathcal{Q}\mathcal{X}, \mathcal{X}\rangle + \sum_{\sigma \in \mathbf{G}} \gamma_\sigma. \tag{15}$$

$$= \frac{1}{2}\langle \mathcal{Q}\mathcal{X}, \mathcal{X}\rangle + \sum_{\sigma \in \mathbf{G}} \exp(2b_\sigma \langle x, \hat{\sigma}\rangle)\, \mathcal{Y}_\sigma^{2b_\sigma}.$$

Remark. An inspection of Figs. 3 and 4 shows that the only root bases for which $\omega_r = 1$ for all r are the root bases $A_n^{(1)}$ and $D_{n+1}^{(2)}$. Therefore, these are the only root bases for which the hamiltonian H is a fibre-wise positive-definite quadratic form.

It is also apparent that the definition of Φ and H is unique up to the choice of bijection ϕ. We return to this in Sect. 4.

2.5. *The algebra $\mathfrak{a}$ of first integrals*

Let $\hat{\mathfrak{s}} = \Pi^* \mathfrak{s}$. Since the functions in $\hat{\mathfrak{s}}$ are independent of y, the conjugate momenta $\mathcal{Y}$ are first integrals of $\hat{\mathfrak{s}}$. Thus:

Lemma 8. $\{\mathcal{Y}_w, \hat{\mathfrak{s}}\} \equiv 0$ *for all $w \in V$, where $\mathcal{Y}_w$ is the linear function on V^* induced by w.*

Let $\mathfrak{U} = \{P \in T^*\Sigma \ : \ \Pi_{\sigma \in \mathbf{G}}\, \gamma_\sigma \neq 0\}$. For $P \in \mathfrak{U}$, let

$$f(P) = \sum_{\sigma \in \mathbf{G}} \ln |\mathcal{Y}_\sigma| \sigma + \mathcal{L}. \tag{16}$$

Lemma 9. *$f : \mathfrak{U} \to V/\mathcal{L}$ is an analytic submersion.*

Proof. It suffices to show that f is well-defined. From the description of the action of $\mathcal{U}^+$ (Equation (5)) and the definition of $\mathfrak{U}$, it is clear that f is well-defined.

Let $C_0^\infty(V/\mathcal{L})$ be the set of smooth functions on $V/\mathcal{L}$ with compact support. Let $\mathbf{F} \subset C^\infty(\mathbb{R})$ be the set of functions which vanish on $(-\infty, c]$ for some $c > 0$. Let

$$\mathfrak{a} = \text{span}\left\{g \ : \ g(P) = \varphi(\mathbf{k}(P))\, h \circ f(P) \text{ where } h \in C_0^\infty(V/\mathcal{L}), \varphi \in \mathbf{F}\right\}.$$

Lemma 10. *$\mathfrak{a}$ is an abelian subalgebra of $C^\infty(T^*\Sigma)$ that commutes with $\mathfrak{s}$. Moreover, $\dim d\mathfrak{a}_P = n + 1$ for all $P \in \mathfrak{U}$.*

Proof. From the definition of $\mathbf{k}$ (Equation (7)) it is apparent that $\{\mathbf{k} \neq 0\} = \mathfrak{U}$. Therefore, if $g \in \mathfrak{a}$ and $g(P) = \varphi(\mathbf{k}(P))\, h \circ f(P)$, then $\varphi \circ \mathbf{k}$ vanishes to all orders on an open neighbourhood of $\mathfrak{U}^c$. Since h has compact support, it follows that g is smooth. Thus $\mathfrak{a} \subset C^\infty(T^*\Sigma)$.

Since any function in $\Pi^*\mathfrak{a}$ is a function of $\mathcal{Y}$ alone, $\mathfrak{a}$ is commutative under the Poisson bracket. Lemma 8 implies that $\{\mathfrak{a}, \mathfrak{s}\} \equiv 0$.

Let $P_o \in \mathfrak{U}$ and let $\mathbf{k}(P_o) = c$. Since $c > 0$, there is a $\varphi \in \mathbf{F}$ that is identically 1 on $[\frac{1}{2}c, \frac{3}{2}c]$. Let $D \subset V/\mathcal{L}$ be a small open disk around $f(P_o)$, and let $U \subset \mathfrak{U}$ be a small open disk around P_o such that $\mathbf{k}(U) \subset [\frac{1}{2}c, \frac{3}{2}c]$ and $f(U) \subset D$. The multi-valued function $\ln|\mathcal{Y}_\sigma|$ can be made single-valued on D by choosing a particular lift of D to V. Let h_σ be a smooth function in $C_0^\infty(V/\mathcal{L})$ that extends $\ln|\mathcal{Y}_\sigma|$ on D. Finally, let $g_\sigma(P) = \varphi(\mathbf{k}(P))\, h_\sigma \circ f(P)$; it is clear that $g_\sigma \in \mathfrak{a}$ for all $\sigma \in \mathbf{G}$ and it is also clear that span $\{dg_{\sigma\, P} : \sigma \in \mathbf{G}\} = \operatorname{im} df_P$ for all $P \in U$ near P_o. Since $P_o \in \mathfrak{U}$ was arbitrary, the previous lemma implies that $\dim d\mathfrak{a}_P = n + 1$ for all $P \in \mathfrak{U}$.

Lemma 11. *Let $\mathfrak{f} = \mathfrak{a} + \mathfrak{s}$. Then $\dim d\mathfrak{f}_P = 3n + 1$ for all $P \in \mathfrak{U}$.*

Proof. Let dx_σ be the 1-form on $V_o/\mathcal{L}$ induced by $\hat{\sigma}$ on V_o. The sole linear dependence relation amongst these 1-forms is $\sum_{\sigma\in\mathbf{G}} dx_\sigma = 0$. From the Equation (6) and the definition of $\mathfrak{s}$,

$$d\mathfrak{s}_P = \operatorname{span}\left\{d\mathcal{X}_v, dx_\sigma + d\ln|\mathcal{Y}_\sigma| \;:\; v \in V_o^*, \sigma \in \mathbf{G}\right\},$$

for all $P \in \mathfrak{U}$. This shows that $\dim d\mathfrak{s}_P = 2n + 1$.

On the other hand, the previous lemma showed that

$$d\mathfrak{a}_P = \operatorname{span}\{d\ln|\mathcal{Y}_\sigma| \;:\; \sigma \in \mathbf{G}\}.$$

It is clear that $d\mathfrak{a}_P \cap d\mathfrak{s}_P = \operatorname{span}\{\sum_{\sigma\in\mathbf{G}} d\ln|\mathcal{Y}_\sigma|\} = \operatorname{span}\{d\mathbf{k}\}$ and is 1-dimensional. Since $d\mathfrak{a}_P$ has dimension $n + 1$, the lemma now follows.

Proof of Theorem 1, parts (i) and (ii). Fix the root system Ψ, the linear isomorphism $\phi : V_o^* \to \mathfrak{h}^*$ (Equation (12)) and hence the isomorphism $\Phi : Sym(\mathfrak{t}) \to Sym(\mathfrak{s})$ (Equation (13)). Let $\mathfrak{b} = \Phi(\mathfrak{I}) \subset \mathfrak{s}$ be the algebra of first integrals of H obtained from the Toda first integrals. Let $\mathcal{F} = \mathfrak{a} + \mathfrak{b}$. The $\mathcal{F}$-regular-point set $L(\mathfrak{I})$ is an open real-analytic set (Lemma 6) and on this set $\dim d\mathfrak{I} = n$. In addition, $d\mathfrak{b}_P \cap d\mathfrak{a}_P = \{0\}$ since the Toda hamiltonian possesses no non-trivial first integrals that are functions of the potentials δ_r alone. It follows from the previous lemma that $L(\mathcal{F})$ is an open real-analytic set and that $\dim d\mathcal{F}_P = 2n + 1 = \dim \Sigma$. Since H is a proper function and $\mathcal{F}$ is abelian, conditions (I1) and (I2) of Definition 1 are satisfied. This proves part (i).

Part (ii) is proven in the remark immediately following Lemma 7.

Part (iii) is proven in Sect. 3.

Remark. There is an alternative proof of Theorem 1 that proceeds by noting that $V_o \times V$ admits the structure of a solvable Lie group $\mathfrak{S}$ where multiplication is defined by:

$$(x, y) * (x', y') := (x + x', y + \sum_{\sigma \in \mathbf{G}} \exp(\langle x, \hat{\sigma} \rangle)\, y'_\sigma\, \sigma), \tag{17}$$

for all $x, x' \in V_o$ and $y, y' \in V$. A finite-index subgroup, Δ_1, of $\Delta = \mathfrak{U}^+ \star \mathcal{O}$ embeds as a discrete, cocompact subgroup of $\mathfrak{S}$ so Σ is finitely-covered by $\Delta_1 \backslash \mathfrak{S}$. The linear space $\mathrm{Lie}(\mathfrak{S})^*$ can be identified with $V_o^* \oplus V^*$, so the momentum map of $\mathfrak{S}$'s left (resp. right) action on $T^*\mathfrak{S}$, in the coordinates $P = (x, y, \mathcal{X}, \mathcal{Y})$, is:

$$\psi_L(P) = \mathcal{X} \oplus \sum_{\sigma \in \mathbf{G}} \exp(\langle x, \hat{\sigma} \rangle)\, \mathcal{Y}_\sigma\, \hat{\sigma}, \tag{18}$$

$$\psi_R(P) = \left[\mathcal{X} + \sum_{\sigma \in \mathbf{G}} y_\sigma \mathcal{Y}_\sigma \hat{\sigma} \right] \oplus \mathcal{Y}. \tag{19}$$

Let $\tilde{\mathfrak{s}}$ (resp. $\tilde{\mathfrak{a}}$) denote the algebra $\mathfrak{s}$ (resp. $\mathfrak{a}$) lifted to $T^*\mathfrak{S}$. Let σ^{2b_σ} be the polynomial function on $\mathrm{Lie}(\mathfrak{S})^*$ defined by $\sigma^{2b_\sigma}(\mathcal{X} \oplus \mathcal{Y}) = \langle \sigma, \mathcal{Y} \rangle^{2b_\sigma}$. Let $\mathfrak{s}_o \subset C^\omega(\mathrm{Lie}(\mathfrak{S})^*)$ be the subalgebra spanned by V_o and $\{\sigma^{2b_\sigma} \ :\ \sigma \in \mathbf{G}\}$. Then it is clear that $\tilde{\mathfrak{s}} = \psi_L^* \mathfrak{s}_o$.

On the other hand, if we let identify $C^\infty(V^*)$ with the abelian subalgebra of $C^\infty(\mathrm{Lie}(\mathfrak{S})^*)$ generated by the linear functions V, then $\tilde{\mathfrak{a}}$ is a subalgebra of $\psi_R^* C^\infty(V^*)$.

Finally, the coadjoint orbits of $\mathfrak{S}$ in $\mathrm{Lie}(\mathfrak{S})^*$ are non-naturally identified with the so-called "Toda" orbits in the loop algebra $\mathfrak{L}(\mathfrak{g})$ via the map ϕ (see [34]). It follows then that $\tilde{\mathfrak{b}} = \psi_L^*(\mathfrak{t})$, where $\mathfrak{t} \subset C^\infty(\mathfrak{L}(\mathfrak{g})^*)$ is the algebra of integrals of the Toda lattice.

3. Entropy and the dynamics on the singular set

Let $\hat{H} = \Pi^* H \in C^\omega(T^*\hat{\Sigma})$ denote the hamiltonian induced by H (Equation (15)) and let $\hat{\mathbf{V}}^\perp = \Pi^{-1}(\mathbf{V}^\perp)$. The hamiltonian flow of $\hat{H}$ is

$$\hat{\varphi}_t(x, y + \mathrm{N}, \mathcal{X}, 0) = (x + t\mathcal{Q}\mathcal{X}, y + \mathrm{N}, \mathcal{X}, 0), \tag{20}$$

for all $P = (x, y + \mathrm{N}, \mathcal{X}, 0) \in \hat{\mathbf{V}}^\perp$. Thus $\varphi_t | \{\mathcal{X} = \mathit{const.}, \mathcal{Y} = 0\}$ is topologically conjugate to the flow $\hat{\varphi}_t^v : \hat{\Sigma} \to \hat{\Sigma}$ defined for all $(x, y + \mathrm{N}) \in \hat{\Sigma}$ by

$$\hat{\varphi}_t^v(x, y + \mathrm{N}) = (x + tv, y + \mathrm{N}), \tag{21}$$

where $v = \mathcal{Q}\mathcal{X}$. Both flows are $\mathcal{U}^+$-equivariant and the conjugacy factors through $\mathcal{U}^+$. Let $\varphi_t^v : \Sigma \to \Sigma$ be the flow induced by $\hat{\varphi}_t^v$. Part (iii) of

Theorem 1 now follows by taking $v \in \mathcal{L}$ and φ_1^v restricted to the invariant fibre $V/\mathrm{N} = \mathbf{p}^{-1}(0+\mathcal{L})$. In addition, the topological entropy of $\Phi_1 = \varphi_1|\mathcal{V}$ is

$$h_{top}(\Phi_1) = \max_{v\,:\,v=\mathcal{Q}\mathcal{X} \text{ and } \langle\mathcal{Q}\mathcal{X},\mathcal{X}\rangle=1} h_{top}(\varphi_1^v). \tag{22}$$

Lemma 12.

$$h_{top}(\varphi_1^v) = \sum_{\sigma\in\mathbf{G}} \langle v, \hat{\sigma}\rangle^+, \tag{23}$$

where $u^+ = \frac{1}{2}(u + |u|)$.

Proof. Let $\tilde{\varphi}_t^v : \tilde{\Sigma} \to \tilde{\Sigma}$ be the flow on the universal cover of Σ induced by $\hat{\varphi}_t^v$. According to the remark after the proof of Theorem 1, $\tilde{\Sigma} = V_o \times V$ admits the structure of a solvable Lie group $\mathfrak{S}$. Let $R_v(x, y) = (x + v, y)$, $L_v(x, y) = (v, 0) * (x, y)$ and let $A_v = R_v \circ L_{-v}$. The transformation $R_v : \mathfrak{S} \to \mathfrak{S}$ is right-translation by the element $(v, 0)$, L_v is left-translation by the same element and A_v is an inner autmorphism of $\mathfrak{S}$. Then $\tilde{\varphi}_1^v = L_v A_v$ can be written as an affine transformation of $\mathfrak{S}$. Assume first that Δ embeds as a subgroup of $\mathfrak{S}$ so that $\Sigma = \Delta\backslash\mathfrak{S}$. By Theorem 20 of [6] and the following remark, $h_{top}(\varphi_1^v) = h_{top}(A_v)$. By the Bowen formula, Corollary 16 of [6], $h_{top}(A_v)$ is the sum of the positive logarithms of the eigenvalues of $dA_v|T_e\mathfrak{S}$. From the explicit form of multiplication in $\mathfrak{S}$ (Equation (17)), this is $h_{top}(A_v) = \sum_{\sigma\in\mathbf{G}} \langle v, \hat{\sigma}\rangle^+$.

In the general case, there is a finite index normal subgroup Δ_1 of Δ that embeds in $\mathfrak{S}$. $\tilde{\varphi}_1^v$ covers a diffeomorphism A of $\Sigma_1 := \Delta_1\backslash\mathfrak{S}$. The previous paragraph shows that $h_{top}(A)$ is given by the right-hand side of Equation (23). Since A covers φ_1^v and Δ/Δ_1 is a compact group, their entropies are equal by Theorem 19 of [6].

Here is a more amenable form for Equation (23). Let $s = \kappa\phi(\mathcal{X})$ and $v = \mathcal{Q}\mathcal{X}$, then $\langle\langle s, s\rangle\rangle = \langle\mathcal{Q}\mathcal{X}, \mathcal{X}\rangle$ and $\langle v, \hat{\sigma}\rangle = \langle s, w_r r\rangle$ where $\phi(\hat{\sigma}) = w_r r$. Equation (23) implies that

$$h_{top}(\Phi_1) = \max_{\langle\langle s,s\rangle\rangle=1} \sum_{r\in\Psi} \langle s, w_r r\rangle^+.$$

For each subset $I \subseteq \Psi$, let $r_I := \sum_{r\in I} w_r r$, where a sum over the empty set is zero. Let $h : \mathfrak{h} \to \mathbb{R}$ be defined by $h(s) = \sum_{r\in\Psi} \langle s, w_r r\rangle^+$. It is clear that $h(s) \geq \langle s, r_J\rangle$ for all $J \subseteq \Psi$, and that there is an $I = I(s)$ such that $h(s) = \langle s, r_I\rangle$. The maximum value of h on the unit sphere, call it h^*, is therefore the maximum over all subsets I of Ψ of the maximum value attained by r_I on the unit sphere. Since the latter is $|r_I|$

$$h^* = \max_{I\subseteq\Psi} \left|\sum_{r\in I} w_r r\right|.$$

Let $\mathrm{H}(I) := |\sum_{r\in I} \omega_r r|$ for each $I \subseteq \Psi$. Then

Lemma 13.
$$h^* = h_{top}(\Phi_1) = \frac{1}{\omega} \max_{I \subseteq \Psi} \mathrm{H}(I).$$

3.1. Calculation of h^*

We will now compute $H^* = \max\{\, \mathrm{H}(I) \;:\; I \subseteq \Psi \,\}$, hence h^*, for each root basis. To derive H^* a few lemmas are needed that give necessary conditions for a set I to be a maximum point for H. Before stating these lemmas, recall that for each root basis Ψ there is a labeled graph $\Gamma(\Psi)$, called the Dynkin diagram, whose vertices are the points of Ψ. A pair of distinct vertices r, s have $4\langle\langle r, s\rangle\rangle^2/|r|^2|s|^2$ edges connecting them, and if $|r| > |s|$ then there is an arrow pointing from r to s. The vertex r has the label ω_r.

Let $I \subset \Psi$ be non-empty. The *restricted* Dynkin diagram of I, $\Gamma(I)$, is the labeled subgraph of $\Gamma(\Psi)$ that contains all vertices in I and edges (r, s) if $r, s \in I$. For each $v \in \Psi$, let $\mathrm{ST}(v)$, called the *star* of v, be the labeled subgraph of $\Gamma(\Psi)$ that contains v, all vertices $r \in \Psi$ such that $\langle\langle r, v\rangle\rangle \neq 0$ and all edges incident to v. $\mathrm{ST}(v, I) = \mathrm{ST}(v) \cap \Gamma(I)$ is the star of v in I, and $\mathrm{VST}(v, I)$ denotes the vertex set of $\mathrm{ST}(v, I)$ less $\{v\}$.

Let $\mathbf{H}(I) = \mathrm{H}(I)^2$ and let $\mathbf{H}^* = \max\{\mathbf{H}(I) \;:\; I \subseteq \Psi\}$. Since
$$\mathbf{H}(I) = \sum_{r,s \in I} \omega_r \omega_s \langle\langle r, s\rangle\rangle,$$
if $v \in I$, then one has
$$\mathbf{H}(I) = \mathbf{H}(I - \{v\}) + \omega_v \left(\omega_v |v|^2 + \sum_{r \in \mathrm{VST}(v,I)} 2\omega_r \langle\langle v, r\rangle\rangle \right). \tag{24}$$
The following is a trivial consequence of the linear dependence relation $\sum_{r\in\Psi} \omega_r r = 0$:

Lemma 14 (Complementarity principle). $\mathbf{H}(I^c) = \mathbf{H}(I)$.

Lemma 15. *(see Fig. 1) Assume that $a, b, c, d \geq 0$. Let*

I.i	$b \leq a$ *and* $\mathrm{ST}(v, I) = \mathrm{I.i}$;
I.ii	$b \leq 2a$ *and* $\mathrm{ST}(v, I) = \mathrm{I.ii}$;
I.iii	$b \leq 3a$ *and* $\mathrm{ST}(v, I) = \mathrm{I.iii}$;
I.iv	$b \leq a$ *and* $\mathrm{ST}(v, I) = \mathrm{I.iv}$;
I.v	$b \leq a$ *and* $\mathrm{ST}(v, I) = \mathrm{I.v}$;
II	$b \leq a + c$ *and* $\mathrm{ST}(v, I) = \mathrm{II}$;
III	$b \leq a + c + d$ *and* $\mathrm{ST}(v, I) = \mathrm{III}$;
IV	$b \leq 2a + c$ *and* $\mathrm{ST}(v, I) = \mathrm{IV}$;
V	$b \leq a + c$ *and* $\mathrm{ST}(v, I) = \mathrm{V}$;
VI	$2b \leq 2a + c + d$ *and* $\mathrm{ST}(v, I) = \mathrm{VI}$;
VII	$b \leq 2a + 2c$ *and* $\mathrm{ST}(v, I) = \mathrm{VII}$;
VIII	$3b \leq 3a + c$ *and* $\mathrm{ST}(v, I) = \mathrm{VIII}$;
IX	$b \leq 3a + c$ *and* $\mathrm{ST}(v, I) = \mathrm{IX}$;

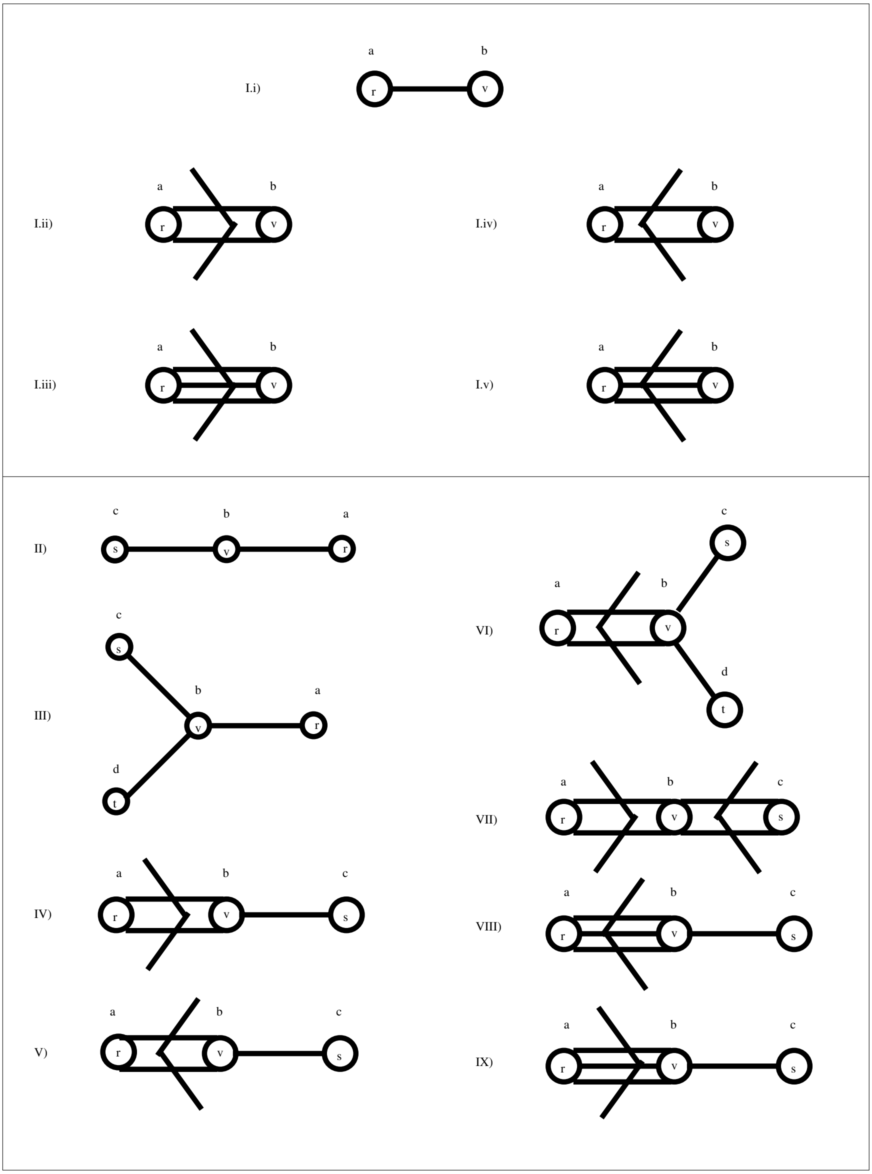

Fig. 1. Vertex stars

Then $\mathbf{H}(I - \{v\}) \geq \mathbf{H}(I)$ *with equality (resp. strict inequality) implying equality (resp. strict inequality).*

Proof. Case (I.i). From the Dynkin diagram (Fig. 1) and Equation (24), $\mathbf{H}(I) = \mathbf{H}(I - \{v\}) + b\left(b|v|^2 + 2a\,\langle\langle v, r\rangle\rangle\right)$. Since only one edge joins r to v, r and v have the same length. This forces $2\langle\langle v, r\rangle\rangle = -|v|^2$. Thus: $\mathbf{H}(I) = \mathbf{H}(I - \{v\}) + b(b - a)|v|^2 \leq \mathbf{H}(I - \{v\})$.

Cases (I.ii–IX) are proved similarly.

Say that $I \in \mathbf{H}^{-1}(\mathbf{H}^*)$ is a *maximizer*.

Lemma 16. *If $I \subset \Psi$ is a maximizer, then for all $v \in I^c$, $\langle\langle v, I\rangle\rangle \neq 0$.*

Proof. If there is a $v \notin I$ such that $\langle\langle v, I\rangle\rangle = 0$, then Equation (24) implies $\mathbf{H}(I \cup \{v\}) = \mathbf{H}(I) + b^2|v|^2 > \mathbf{H}(I)$.

Lemma 17. *Let* $\mathrm{ST}(v, \Psi) \in \{\mathrm{II}, \ldots, \mathrm{IX}\}$. *If I is a maximizer, then either $v \in I$ or $r \in I$.*

Proof. If $r, v \notin I$, then $\langle\langle v, I\rangle\rangle = 0$, which contradicts the previous lemma.

Lemma 18. *If I is a maximizer, then for each $v \in I$,* $\mathrm{ST}(v, I)$ *has one of the following restricted Dynkin diagrams (see Fig. 2).*

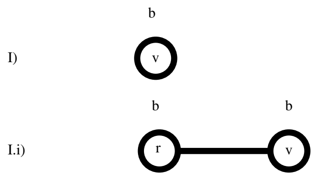

Fig. 2. Vertex stars in a maximizer

Proof. Let I be a maximizer. If $\mathrm{ST}(v, I) = \mathrm{I}$ for all $v \in I$, then there is nothing to prove. Therefore, assume that $\Gamma(I)$ contains a subgraph with one or more edges. First, we consider the case when $\Gamma(I)$ contains a subgraph with two or more edges. By inspection of the Dynkin diagrams in Figs. 3 and 4 there is a $v \in I$ such that $\mathrm{ST}(v, I) \in \{\mathrm{II}, \ldots, \mathrm{IX}\}$ for some choice of coefficients a, b, c and possibly d. Henceforth, we will agree that $d = 0$ if $\mathrm{ST}(v, I)$ contains only three vertices. Inspection of Figs. 3 and 4 shows that if $\mathrm{ST}(v, I) \in \{\mathrm{II}, \ldots, \mathrm{IX}\}$, then $b \leq a + c + d$; and strict inequality occurs in all cases except when $b = 2$ and $a = c = 1$ and $\mathrm{ST}(v, I) \in \{\mathrm{II}, \mathrm{IX}\}$. On the other hand, if $\mathrm{ST}(v, I) \notin \{\mathrm{VI}, \mathrm{VIII}\}$, then Lemma 15 implies that $\mathbf{H}(I - \{v\}) \geq \mathbf{H}(I)$ with strict inequality in all but the case of II with $b = 2$ and $a = c = 1$. Since I is a maximizer, it follows that either $\mathrm{ST}(v, I) \in \{\mathrm{VI}, \mathrm{VIII}\}$ or $\mathrm{ST}(v, I) = \mathrm{II}$, $b = 2$, $a = c = 1$ and $\mathbf{H}(I - \{v\}) = \mathbf{H}(I)$.

If $\mathrm{ST}(v, I) = \mathrm{VI}$, then $\Psi = \mathrm{VI} = B_3^{(1)}$. In this case $a = b = 2$ and $c = d = 1$ and $2b < 2a+c+d$. Lemma 15 implies that I is not a maximizer.

If $\mathrm{ST}(v, I) = \mathrm{VIII}$, then $\Psi = \mathrm{VIII} = G_2^{(1)}$. In this case $a = 3$, $b = 2$, $c = 1$, $d = 0$ and $3b < 3a + c$. Lemma 15 implies that I is not a maximizer.

If $\mathrm{ST}(v, I) = \mathrm{II}$ and $b = 2$, $a = c = 1$, then by inspection, Ψ is one of $B_n^{(1)}$, $D_n^{(1)}$ or $A_{2n-1}^{(2)}$ and we can assume that the vertices enjoy the numbering $(r, v, s) = (1, 2, n + 1)$. Then r, s are orthogonal to I^c so by Lemma 16 I^c, hence I, is not a maximizer.

Therefore, if I is a maximizer then $\Gamma(I)$ contains no subgraphs with two or more edges.

Assume now that I is a maximizer with a vertex v such that $\mathrm{ST}(v, I)$ contains only two distinct vertices. It follows that $\mathrm{ST}(v, I) \in \{\mathrm{I.i}, \ldots, \mathrm{I.v}, A_1^{(1)}, A_2^{(2)}\}$. In the latter two cases $ar + bv = 0$ so I cannot be a maximizer. Assume that $\mathrm{ST}(v, I) \in \{\mathrm{I.ii}, \mathrm{I.iii}\}$; the above observations imply that $\mathrm{ST}(r, I) \in \{\mathrm{I.iv}, \mathrm{I.v}\}$. Since $\mathbf{H}(I) \geq \mathbf{H}(I - \{v\})$, Lemma 15 implies that $b \geq ka$ where $k = 2$ or 3. On the other hand, since $\mathbf{H}(I) \geq \mathbf{H}(I - \{r\})$, Lemma 15 implies that $a \geq b$. Thus $a = 0$. Absurd. By symmetry, we conclude that $\mathrm{ST}(v, I) \notin \{\mathrm{I.ii}, \ldots, \mathrm{I.v}\}$. Thus, the only possibility is that $\mathrm{ST}(v, I) = \mathrm{I.i}$, $b = a$ and $|v| = |r|$.

Recall that a graph has a natural topological structure of a simplicial complex:

Lemma 19. *Let $v \in \Psi$. There is a maximizer J such that $\Gamma(J)$ is totally disconnected and $v \in J$.*

Proof. Let $I =: J_0$ be a maximizer; since I^c is also a maximizer we can assume that $v \in J_0$. Let $J_0 = A_0 \cup B_0$ where $r \in A_0$ (resp. $r \in B_0$) iff $\mathrm{ST}(r, J_0) = \mathrm{I}$ (resp. $\mathrm{ST}(r, J_0) = \mathrm{I.i}$). For each $r \in B_0$ there is a unique $s = s_r \in B_0$, $s \neq r$, such that $\mathrm{ST}(r, J_0) = \mathrm{ST}(s, J_0)$. If $B_0 = \emptyset$, then we let $J = J_0$; otherwise, let $r \in B_0$ be such that $r \neq v$ and let $J_1 := J_0 - \{r\}$. By Lemmas 15 and 18 J_1 is also a maximizer and $v \in J_1$. If J_1 is decomposed into the sets A_1, B_1 as above, then B_1 obviously has 2 fewer elements than B_0. Thus, this process inductively yields a maximizer J such that $\Gamma(J)$ is totally disconnected and $v \in J$.

Recall the notion of duality between root bases: let $\check{r} = r/|r|^2$ for each $r \in \Psi$. The set $\check{\Psi} = \{\check{r} : r \in \Psi\}$ is a root basis, too; it is the *dual* of Ψ [16]. Note that $\check{\check{r}} = r$ for all r. The labels on $\Gamma(\check{\Psi})$ satisfy $\omega_{\check{r}} = \eta |r|^2 \omega_r$ for each $r \in \Psi$, where $\eta^{-1} = \gcd(|r|^2\omega_r)_{r\in\Psi}$. We will also let $\check{\omega}_r = |r|^2\omega_r$ so that $\omega_{\check{r}} = \eta\check{\omega}_r$ and $\check{\omega}_r\check{r} = \omega_r r$ for all r. It follows that $\check{\mathbf{H}}(\check{I}) = |\sum_{r\in I}\omega_{\check{r}}\check{r}|^2 = \eta^2\mathbf{H}(I)$, $\check{h}^* = \frac{\omega}{\check{\omega}}\eta h^*$ and $\Pi_{r\in I}|\omega_{\check{r}}\check{r}| = \eta^{|I|}\,\Pi_{r\in I}|\omega_r r|$. Note that the Dynkin diagram of $\check{\Psi}$ is obtained from $\Gamma(\Psi)$ by reversing all arrows and relabeling the vertex r with $\omega_{\check{r}}$. This implies that the Cartan matrix of $\check{\Psi}$ is the transpose of that of Ψ.

Finally, note that if $\Psi = B_n^{(1)}, C_n^{(1)}, F_4^{(1)}, G_2^{(1)}$ then $\check{\Psi} = A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$; all other root systems are self-dual.

Proof of Theorem 3. We will show the proof for $B_n^{(1)}$ only. The remaining cases are similar and/or handled by the foregoing lemmata.

By Lemma 19, we can assume that I is a maximizer containing r_1 such that $\Gamma(I)$ is totally disconnected. Then $r_{n+1} \in I$ and $r_2 \notin I$. There are two mutually exclusive possibilities: (i) either $r_{n-1} \in I$; or (ii) $r_n \in I$. In case (i) we conclude that $r_n, r_{n-2} \notin I$ so the Dynkin diagram of $I' =$

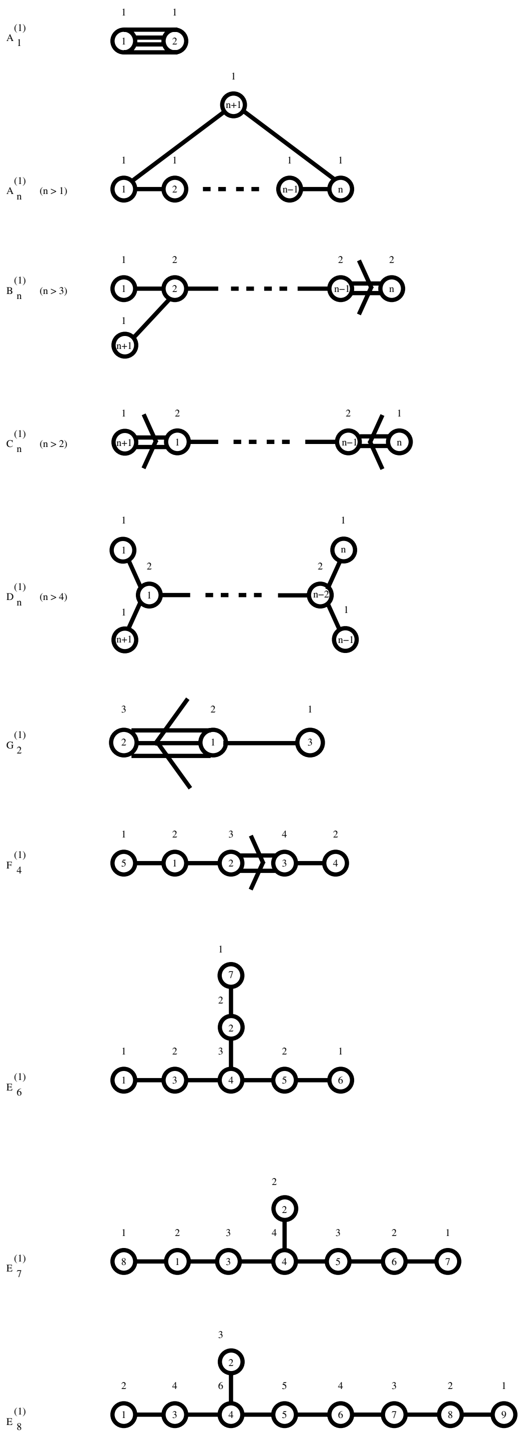

Fig. 3. Dynkin diagrams of rank n, type $\mathfrak{g}^{(1)}$

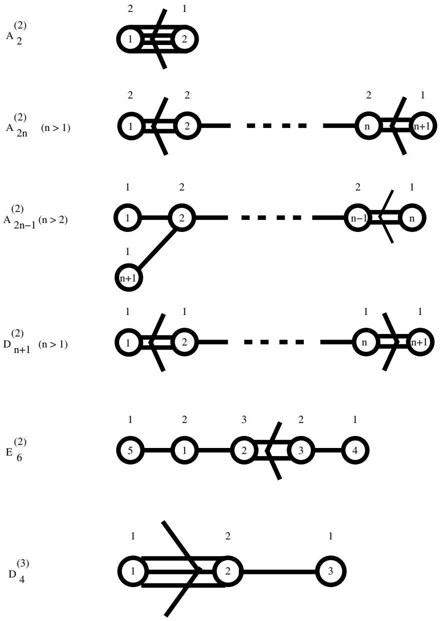

Fig. 4. Dynkin diagrams of rank n, types $\mathfrak{g}^{(2)}$ and $\mathfrak{g}^{(3)}$

$I - \{r_1, r_{n+1}, r_{n-1}\}$ is $\overset{2}{\bullet} \cdots \overset{2}{\bullet}$. Since I is totally disconnected, $\mathbf{H}(I) = \mathbf{H}(I') + \mathbf{H}(\{r_1, r_{n+1}, r_{n-1}\})$; since I is a maximizer, I' is a maximizer in the subgraph with vertices $r_3, \ldots, r_{n-3}$. If $n = 2m + 1$ (resp. $n = 2m$) then $I' = \{r_3, \ldots, r_{2m-3}\}$. (resp. $I' = \{r_3, \ldots, r_{2m-3}\}$) is a maximizer. Then $\mathbf{H}(I) = 8(m - 1) + 4 = 4(2m - 1)$. In case (ii), we conclude that for $n = 2m + 1$ (resp. $n = 2m$) $I' = \{r_3, \ldots, r_{2m-1}\}$ (resp. $I' = \{r_3, \ldots, r_{2m-3}\}$) and $\mathbf{H}(I) = 8m$ (resp. $\mathbf{H}(I) = 8(m - 1)$). Comparison of $\mathbf{H}(I)$ for the candidate maximizers shows that $\mathbf{H}^* = 4(n - 1)$ for all $n \geq 3$. Since $\omega = 2$, this shows

$$h^* = \frac{1}{\omega}\sqrt{\mathbf{H}^*} = \sqrt{n - 1}.$$

4. Uniqueness of flows up to topological conjugacy

Recall that flows $\phi : M \to M$ and $\varphi : N \to N$ are topologically conjugate if there is a homeomorphism $h : M \to N$ such that $h\phi_t = \varphi_t h$ for all t. Let P_ϕ be the set of periodic orbits of the flow ϕ. For each periodic orbit γ of ϕ, let the homology class of γ be denoted by $\bar{\gamma}$ and its period by $\mathrm{Period}(\gamma)$. The following two definitions originate in Schwartzman's work [35].

Definition 2. *Let* $\mathcal{M}_\phi = \{(\bar{\gamma}, \mathrm{Period}(\gamma)) : \gamma \in \mathrm{P}_\phi\}$. *We call* $\mathcal{M}_\phi$ *the* homology spectrum *of* ϕ.

The homology spectrum is a subset of $H_1(M;\mathbb{Z})\times\mathbb{R}$ that is an invariant of topological conjugacy in the following sense: if ϕ and φ are topologically conjugate then

$$(h_*\times id_{\mathbb{R}})(\mathcal{M}_\phi)=\mathcal{M}_\varphi,$$

where $h_*:H_1(M;\mathbb{Z})\to H_1(N;\mathbb{Z})$ is the obvious isomorphism.

Let $\pi:\hat{M}\to M$ be the universal abelian covering of M. The flow ϕ is covered by a flow $\hat{\phi}:\hat{M}\to\hat{M}$. Let $F\subset\hat{M}$ be a fundamental domain for $\mathrm{Deck}(\pi)$. For each $p\in M$ choose $\hat{p}\in F\cap\pi^{-1}(p)$. For each t there is a $g\in\mathrm{Deck}(\pi)$ such that $\hat{\phi}_t(p)\in g.F$; let $g_t(p)$ be one such element and let $\frac{1}{t}g_t(p)\in\mathrm{Deck}(\pi)\otimes_{\mathbb{Z}}\mathbb{R}$. Recall that $\mathrm{Deck}(\pi)\otimes_{\mathbb{Z}}\mathbb{R}\simeq H_1(M;\mathbb{R})$. Let

Definition 3.

$$\eta_\phi(p):=\cap_{T\geq 0}\overline{\left\{\frac{1}{t}g_t(p)\ :\ t\geq T\right\}}$$

be the asymptotic homology *of $p\in M$. Let $\eta_\phi^\pm=\eta_{\phi^\pm}$ where $\phi_t^\pm=\phi_{\pm t}$.*

One can show that $\eta_\phi(p)$ is independent of the choice of representatives and if M is compact then $\eta_\phi(p)$ is non-empty for all p. It is also clear that if φ is conjugate to ϕ then $h_*\eta_\phi^\pm(p)=\eta_\varphi^\pm(h(p))$.

Example 1. Let $\mathcal{Q}:V_o^*\to V_o$ be a linear isomorphism and $M=V_o/\mathcal{L}\times V_o^*$. Let $\varphi_t(x,\mathcal{X})=(x+t\mathcal{Q}\mathcal{X}\ \mathrm{mod}\ \mathcal{L},\mathcal{X})$. Clearly, $\eta^\pm(x,\mathcal{X})=\{\pm\mathcal{Q}\mathcal{X}\}$ for all $(x,\mathcal{X})\in M$.

Let $\mathcal{V}_o=\{(x,\mathcal{X})\in M\ :\ \langle\mathcal{Q}\mathcal{X},\mathcal{X}\rangle=1\}$, $\Phi^o=\varphi|\mathcal{V}_o$ and $|m|_{\mathcal{Q}}=\sqrt{|\langle\mathcal{Q}^{-1}m,m\rangle|}$ for all $m\in V_o$. The homology spectrum of Φ^o is seen to be $\mathcal{M}_{\Phi^o}=\{(m,|m|_Q)\ :\ m\in\mathcal{L}-\{0\}\}$.

Example 2. Let $\mathcal{Q}$ be as above, and let $f_t(x,y,\mathcal{X})=\Delta(x+t\mathcal{Q}\mathcal{X},y,\mathcal{X})$ be a flow on $\mathbf{V}^\perp\subset T^*\Sigma$. Let $\Phi_t=f_t|\mathcal{V}$. Let $\mathbf{P}$ denote the projection map $\mathcal{V}\to\mathcal{V}_o$. We have that $\mathbf{P}\Phi_t=\Phi_t^o\mathbf{P}$. Thus $(\mathbf{P}_*\times id)\mathcal{M}_\Phi=\mathcal{M}_{\Phi^o}$. Since $\mathcal{V}\simeq\Sigma\times S^{n-1}$, $H_1(\mathcal{V};\mathbb{Z})=H_1(\Sigma;\mathbb{Z})\oplus H_1(S^{n-1};\mathbb{Z})$. By the structure of the flow f_t it is clear that the projection of a periodic orbit's homology class to $H_1(S^{n-1};\mathbb{Z})$ is trivial. Thus $(\mathbf{P}_*\times id)$ is a bijection between $\mathcal{M}_\Phi$ and $\mathcal{M}_{\Phi^o}$.

Lemma 20. *Let H be a hamiltonian defined by Equation (15), and let $\varphi:T^*\Sigma\to T^*\Sigma$ denote its hamiltonian flow. Let $\mathfrak{U}_\sigma=\{\gamma_\sigma\neq 0\}$ for each $\sigma\in\mathbf{G}$. If $P\in\mathfrak{U}_\sigma$ then*

$$\left\langle\eta_\varphi^\pm(P),\hat{\sigma}\right\rangle\leq 0.$$

Proof. Let $\hat{P}=(x,y+\mathrm{N},\mathcal{X},\mathcal{Y})\in\hat{\mathfrak{U}}_\sigma$ and let $P=\Pi(\hat{P})$. Since $\gamma_\sigma\neq 0$, $\mathcal{Y}_\sigma\neq 0$. If $v\in\eta_\varphi^\pm(P)$, then there is a sequence $T_k\to\pm\infty$ such that

$$v=\lim_{k\to\infty}\frac{1}{|T_k|}(x(T_k)-x(0)),$$

where $\hat{\varphi}_t(x, y + \mathrm{N}, \mathcal{X}, \mathcal{Y}) = (x(t), y(t) + \mathrm{N}, \mathcal{X}(t), \mathcal{Y}(t))$ and $\hat{\varphi}_t$ is the lift of φ_t. Thus:

$$\langle v, \hat{\sigma} \rangle = \lim_{k \to \infty} \frac{1}{|T_k|} \langle x(T_k), \hat{\sigma} \rangle.$$

On the other hand $\hat{H}$ and $\mathcal{Y}_\sigma$ are first integrals of $\hat{\varphi}_t$. Inspection of Equation (15) shows that $\hat{H}(\hat{P}) \geq \mathcal{Y}_\sigma^{2b_\sigma} \exp(2b_\sigma \langle x(T), \hat{\sigma} \rangle)$ for all T. Since $b_\sigma > 0$ and $\mathcal{Y}_\sigma \neq 0$, this inequality implies that

$$\frac{1}{|T_k|} \langle x(T_k), \hat{\sigma} \rangle \leq \frac{1}{|T_k|} \left(\frac{1}{2b_\sigma} \ln \hat{H} - \ln \mathcal{Y}_\sigma^2 \right) \xrightarrow{k \to \infty} 0.$$

Since $v \in \eta_\varphi^\pm(P)$ was arbitrary, this proves the lemma.

Lemma 21. *Let H_1, H_2 be defined by Equation (15) corresponding to root bases Ψ_1, Ψ_2 . If $h : T^*\Sigma \to T^*\Sigma$ conjugates the hamiltonian flows of H_1 and H_2, then*

$$h(\mathbf{V}^\perp) = \mathbf{V}^\perp.$$

Proof. Let U be the set of points in $\mathbf{V}^\perp$ that are mapped out of $\mathbf{V}^\perp$ under h: $U = h^{-1}(\cup_{\sigma \in \mathbf{G}} \mathfrak{U}_\sigma) \cap \mathbf{V}^\perp$. It suffices to prove that U is empty, since a symmetric argument applies to h^{-1}. Clearly, it suffices to prove that $U_\sigma = h^{-1}(\mathfrak{U}_\sigma) \cap \mathbf{V}^\perp$ is empty for all σ. Since $\mathfrak{U}_\sigma$ is open, U_σ is an open subset of $\mathbf{V}^\perp$, so to prove that it is empty, it suffices to show that U_σ is nowhere dense. Remark that $\mathbf{V}^\perp$ is naturally diffeomorphic to $\Sigma \times V_o^*$. Let $\pi_o : \mathbf{V}^\perp \to V_o^*$ denote the projection onto the second factor. Clearly, π_o is an open map and $\pi_o(P) = \mathcal{X}$ where $P = \Pi(x, y, \mathcal{X}, 0) \in \mathbf{V}^\perp$. It suffices to show that $\pi_o(U_\sigma)$ lies in a hyperplane to prove the lemma.

Let φ^i be the hamiltonian flow of H_i, and $\mathcal{Q}_i$ the quadratic form used to define H_i (Equation (15)). If $P \in U_\sigma$, then $P \in \mathbf{V}^\perp$ so

$$\eta_{\varphi^1}^\pm(P) = \{\pm \mathcal{Q}_1 \, \mathcal{X}\},$$

while $h(P) \in \mathfrak{U}_\sigma$ so from the previous lemma

$$\left\langle \eta_{\phi^2}^\pm(h(P)), \hat{\sigma} \right\rangle \leq 0.$$

If $h\varphi_t^1 = \varphi_t^2 h$, then

$$h_* \eta_{\varphi^1}^\pm(P) = \eta_{\varphi^2}^\pm \circ h(P),$$

which implies that

$$\pm \langle h_* \mathcal{Q}_1 \mathcal{X}, \hat{\sigma} \rangle \leq 0.$$

Since $h_* \mathcal{Q}_1$ is non-degenerate, $\mathcal{X} = \pi_o(P)$ must lie in a fixed hyperplane. Thus, $\pi_o(U_\sigma)$ lies in a hyperplane. Since π_o is an open map, U_σ is empty.

Recall that $h : T^*\Sigma \to T^*\Sigma$ is *energy-preserving* if $h(\{H_1 = \frac{1}{2}\}) = \{H_2 = \frac{1}{2}\}$. We use the notation of Lemma 21 and its proof:

Theorem 6. *Let H_1, H_2 be defined by Equation (15) corresponding to root bases Ψ_1, Ψ_2. If $h \in \mathrm{Homeo}(T^*\Sigma)$ is an energy-preserving conjugacy of φ^1 with φ^2, then the norms $|\cdot|_{\mathcal{Q}_1}$ and $|\cdot|_{\mathcal{Q}_2}$ are equivalent over* $\mathrm{Aut}(\mathcal{L})$.

Proof. Let $\mathcal{V}_i = \mathbf{V}^\perp \cap H_i^{-1}(\frac{1}{2})$. Since h is energy preserving, Lemma 20 implies that $h(\mathcal{V}_1) = \mathcal{V}_2$. Let $\varphi^i|\mathcal{V}_i$ be denoted by Φ^i and let $h|\mathcal{V}_1$ continue to be denoted by h. From Examples 1 and 2

$$\mathcal{M}_{\Phi^i} = \{(m, |m|_{\mathcal{Q}_i}) \ : \ m \in \mathcal{L} - \{0\}\}$$

for $i = 1, 2$. By hypothesis $h\Phi^1$ equals $\Phi^2 h$, so $(h_* \times id)(\mathcal{M}_{\Phi^1}) = \mathcal{M}_{\Phi^2}$. This means that h_* induces an automorphism f of $\mathcal{L}$ such that $|fm|_{\mathcal{Q}_2} = |m|_{\mathcal{Q}_1}$ for all $m \in \mathcal{L}$.

Let us dualize Theorem 6. Let $\phi_i \in \mathfrak{B}_i$, where $\mathfrak{B}_i$ is the set of linear isomorphisms $V_o^* \to \mathfrak{h}_i^*$ induced by bijections $\hat{\mathbf{G}} \to \Omega_i = \{w_r r \ : \ r \in \Psi_i\}$. The norms $|\cdot|_{\mathcal{Q}_i}$ on $\mathcal{L}$ are equivalent modulo $\mathrm{Aut}(\mathcal{L})$ iff the dual norms $|\cdot|^*_{\mathcal{Q}_i}$ on $\mathcal{L}^*$ are equivalent modulo $\mathrm{Aut}(\mathcal{L}^*)$. Since $|\mathcal{X}|^*_{\mathcal{Q}_i} = \sqrt{\langle\langle \phi_i(\mathcal{X}), \phi_i(\mathcal{X})\rangle\rangle_i}$, Theorem 6 implies

Corollary 4. *If φ^1 and φ^2 are topologically conjugate by an energy-preserving homeomorphism, then there exists $\mu \in \mathrm{Isom}(\mathfrak{h}_2^*; \mathfrak{h}_1^*)$ and $g \in \mathrm{Aut}(\mathcal{L}^*)$ such that*

$$\mu = \phi_1 g \phi_2^{-1}. \tag{25}$$

Let $\mathfrak{C}$ be the union of the sets $\mathfrak{B}$ for all root bases. Equation (25) defines an equivalence relation $\sim$ on $\mathfrak{C}$ that is coarser than that defined by energy-preserving topological conjugacy. In the next section, we show that knowledge of the equivalence relation $\sim$ is highly non-trivial.

4.1. Conjugacies, $\mathbb{Q}$-structures and Gelfond's conjecture

Let L be a subfield of $\mathbb{C}$ and let W be a finite-dimensional vector space over L. A subset U of W is a *rational structure* on W if U is a $\mathbb{Q}$-vector space such that $\dim_{\mathbb{Q}} U = \dim_L W$. If L is a subring of $\mathbb{C}$, let $V_L = \mathrm{span}_L\, \mathbf{G}$, $V_{o,L} = \ker \epsilon \cap V_L$ and $\mathfrak{h}_L^* = \mathrm{span}_L\, \Psi$. It follows that $V_{\mathbb{Q}}$ (resp. $V_{o,\mathbb{Q}}$, $\mathfrak{h}^*_{\mathbb{Q}}$) is a $\mathbb{Q}$-structure on V (resp. V_o, $\mathfrak{h}^*$), and also the complex form of these vector spaces. Because $\hat{\mathbf{G}}$ (resp. Ω) is a spanning set of $V_{o,\mathbb{Q}}$ (resp. $\mathfrak{h}^*_{\mathbb{Q}}$), the following is obvious:

Lemma 22. *If $\phi \in \mathfrak{B}$, then $\phi(V^*_{o,\mathbb{Q}}) = \mathfrak{h}^*_{\mathbb{Q}}$.*

V_o^* has a second, non-standard $\mathbb{Q}$-structure induced by $\mathcal{L}$: $\mathrm{span}_{\mathbb{Q}}\, \mathcal{L}^*$. If $\phi_i \in \mathfrak{B}_i$ and there is a solution to the congruence equation (Equation (25)), then, on the one hand g must be an automorphism of this non-standard $\mathbb{Q}$-structure, while on the other hand g is rationally equivalent to an element in a $\mathbb{Q}$-algebraic group. This tension should force g to have a rather restricted form.

Let $E/\mathbb{Q}$ be the splitting field of F. Let $\mathcal{U}_E$ be the group of units of the ring of integers of E, let $v_1, \ldots, v_m, -1$ be a basis of $\mathcal{U}_E$ and let $\xi_i = \ln|v_i|$ for $i = 1, \ldots, m$.

Let $f \in \mathrm{Aut}(\mathcal{L})$. Then f can be decomposed as $f = BCB^{-1}$ where $B : V_o \to V_o$ is a linear automorphism that maps $V_{o,\mathbb{Z}}$ onto $\mathcal{L}$ and $C \in \mathrm{Aut}(V_{o,\mathbb{Z}})$. Then for all $\tau \neq 1$, there is a $u_\tau \in \mathcal{U}^+$ such that $B(\tau - 1) = \sum_{\sigma\in\mathbf{G},\sigma\neq 1} \ln|\sigma(u_\tau)|\,(\sigma - 1)$. Since $|\sigma(u_\tau)| \in \mathcal{U}_E$, it follows that the entries B_{ij} of B relative to a $\mathbb{Q}$-basis of V_o are rational linear combinations of $\xi_1, \ldots, \xi_m$. Let $B_{ij}(\mathbf{x}) = \sum_{a=1}^m b_{ij}^a x_a$ be rational linear functions such that $B_{ij}(\xi) = B_{ij}$ where $\xi = (\xi_1, \ldots, \xi_m)$. Define a map $\mathbb{C}^m \to \mathrm{Hom}(V_{o,\mathbb{C}})$ by

$$f(\mathbf{x}) = \frac{1}{s(\mathbf{x})}\, B(\mathbf{x})\, C \,\mathrm{adj}\, B(\mathbf{x}) \tag{26}$$

where adj is the classical adjoint matrix and $s(\mathbf{x}) = \det B(\mathbf{x})$. Clearly $f(\xi) = f$ and $B(\mathbf{x})$ (resp. $s(\mathbf{x})$) is a homogeneous polynomial in $\mathbf{x}$ of degree 1 (resp. n) with rational coefficients. Dualizing this observation gives

Lemma 23. *Let $g \in \mathrm{Aut}(\mathcal{L}^*)$. Then there is a $c \in \mathrm{Aut}(V_{o,\mathbb{Z}}^*)$ and a linear map $\mathbf{x} \mapsto b(\mathbf{x}) : \mathbb{C}^m \to \mathrm{Hom}(V_{o,\mathbb{C}}^*)$, with rational coefficients, such that*

$$g(\mathbf{x}) = \frac{1}{s(\mathbf{x})}\, \mathrm{adj}\, b(\mathbf{x})\; c\; b(\mathbf{x}) \tag{27}$$

where $s(\mathbf{x}) = \det b(\mathbf{x})$ satisfies $g(\xi) = g$.

Note that if one fixes a basis of V_o and its dual basis of V_o^*, then the matrix $g(\mathbf{x})$ is the inverse transpose of $f(\mathbf{x})$ and $b(\mathbf{x})$ is the transpose of $B(\mathbf{x})$.

Let $\phi_i \in \mathfrak{B}_i$ and assume that $\mu \in \mathrm{Isom}(\mathfrak{h}_2^*; \mathfrak{h}_1^*)$, $g \in \mathrm{Aut}(\mathcal{L}^*)$ solves Equation (25). Define a polynomial map $\mathbb{C}^m \to \mathrm{Hom}(\mathfrak{h}_{\mathbb{C}}^*)$, $\mathbf{x} \mapsto T(\mathbf{x})$ by $T(\mathbf{x}) := \phi_1 \,\mathrm{adj}\, b(\mathbf{x})\, c\, b(\mathbf{x})\, \phi_2^{-1}$. From Lemmas 23 and 22, both $s(\mathbf{x})$ and $T(\mathbf{x})$ are homogeneous polynomials of degree n in $\mathbf{x}$ with rational coefficients. Therefore, there are polynomials $T_{ij}(\mathbf{x}), s(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$, all homogeneous and of degree n, such that

$$\mu_{ij} = \frac{T_{ij}(\xi)}{s(\xi)},$$

where $[\mu_{ij}]$ are the entries of μ relative to $\mathbb{Q}$-bases of $\mathfrak{h}_1^*$ and $\mathfrak{h}_2^*$. Let $R_{k,ij} \in \mathbb{Q}$ denote the entries of $\langle\langle, \rangle\rangle_k$ relative to these $\mathbb{Q}$-bases. Since $\mu \in \mathrm{Isom}(\mathfrak{h}_2^*; \mathfrak{h}_1^*)$, the equation for μ_{ij} implies that for all i, j the polynomial

$$Q_{ij}(\mathbf{x}) := \sum_{a,b=1}^{n} T_{ia}(\mathbf{x})\, T_{jb}(\mathbf{x})\, R_{2,ab} - s(\mathbf{x})^2 R_{1,ij} \tag{28}$$

has a zero at $\mathbf{x} = \xi$. In addition, $Q_{ij}(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ for all i, j.

Theorem 7. *If $m \geq 3$, assume the Gelfond conjecture. If Equation (25) is satisfied, then* $\mathbf{x} \mapsto g(\mathbf{x})$ *is a constant map. Thus,* $g \in \mathrm{Aut}(\mathcal{L}^*) \cap \mathrm{Aut}(V^*_{o,\mathbb{Q}})$ *and* $\mu \in \mathrm{Isom}(\mathfrak{h}^*_{2,\mathbb{Q}}; \mathfrak{h}^*_{1,\mathbb{Q}})$.

Remark 1. One requires only that the rational independence of logarithms of algebraic numbers imply their homogeneous independence for the following proof. Gelfond himself showed this to be true for pairs of algebraic numbers with rationally independent logarithms. Thus Theorem 7 is true independent of the Gelfond conjecture when $F/\mathbb{Q}$ is a normal, cubic extension.

Proof. By hypothesis, $\xi_1, \ldots, \xi_m$ are rationally independent logarithms of algebraic numbers. The Gelfond conjecture therefore implies that $Q_{ij}(\mathbf{x}) \equiv 0$ (Equation (28)) for all i, j. In terms of matrices, this means that for all $\mathbf{x}$, ${}^tT(\mathbf{x})\ R_2\ T(\mathbf{x}) \equiv s(\mathbf{x})^2\ R_1$ where ${}^tT_{ij} = T_{ji}$. From the non-degeneracy of R_k, it follows that $s(\mathbf{x}) = 0$ iff $T(\mathbf{x}) = 0$. Therefore, for all i and j, $T_{ij}(\mathbf{x})$ and $s(\mathbf{x})$ are homogeneous polynomials of degree n in $\mathbf{x}$ that vanish on the same set of $\mathbf{x} \in \mathbb{C}^m$. A standard algebro-geometric argument shows that $s(\mathbf{x})$ divides $T_{ij}(\mathbf{x})$, and since they have the same degree, $T_{ij}(\mathbf{x})/s(\mathbf{x})$ is a rational number. Let $\mu(\mathbf{x}) = \frac{1}{s(\mathbf{x})} T(\mathbf{x})$; evaluation of $\mu(\mathbf{x})$ at $\mathbf{x} \in \mathbb{Q}^m$ gives a matrix with rational entries, but since $\mu(\mathbf{x})$ is independent of $\mathbf{x}$, $\mu = \mu(\xi)$ must have rational entries. This proves that μ is an isometry between $\mathbb{Q}$-vector spaces. On the other hand, since $g(\mathbf{x}) = \phi_1^{-1}\, \mu(\mathbf{x})\, \phi_2$, it is also a constant, which must be g. Lemma 22 and the fact that μ is an isomorphism of $\mathbb{Q}$-vector spaces shows that g is, too.

Example 3. Theorem 7 can already be applied. Let F be a degree 3 totally real extension of $\mathbb{Q}$ that is not normal. For example, if we take a root α of the polynomial $x^3 - 4x + 2$ which has discriminant $d = 148$, then $F = \mathbb{Q}(\alpha)$ is totally real since $d > 0$ and non-normal since $\sqrt{d} \notin \mathbb{Q}$ [29]. The splitting field of F is of degree 6 and the conjugate subfields of F intersect pairwise in $\mathbb{Q}$. Let $\mathbf{G} = \{\sigma_1, \sigma_2, \sigma_3\}$ be the embeddings of F into $\mathbb{R}$, let $u, v, -1$ generate $\mathcal{U}_F$ and let $\alpha_i = \ln|\sigma_i(u)|$, $\beta_i = \ln|\sigma_i(v)|$. The non-normality of F implies that $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ is $\mathbb{Q}$-linearly independent.

Let $B(\sigma_1 - \sigma_3) = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$ and $B(\sigma_2 - \sigma_3) = \beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3$, so $B : V_{o,\mathbb{Z}} \to \mathcal{L}$ is an isomorphism. Relative to the $\mathbb{Q}$-basis $\{\sigma_1 - \sigma_3, \sigma_2 - \sigma_3\}$ of V_o:

$$B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}, \tag{29}$$

since the α_i's and β_i's separately sum to zero. By the Gelfond conjecture, the rational independence of the entries of B implies that the map $\mathbf{x} \to B(\mathbf{x})$ (Equation (26)) surjects onto $\mathrm{Hom}(V_{o,\mathbb{C}})$. Thus, the only constant map $\mathbf{x} \to g(\mathbf{x})$ (Equation (27)) is the identity and its multiples; since g is also a lattice automorphism, $g = \pm 1$. Therefore, up to an inessential factor of ± 1, $\mu = \phi_1\phi_2^{-1}$ is a linear isometry from $\mathfrak{h}_2^*$ to $\mathfrak{h}_1^*$ that maps Ω_2 to Ω_1. To proceed, it is necessary to investigate such maps. To do this, it will be useful to define the notion of an automorphism of a Dynkin diagram.

A permutation $\rho \in S(\Psi)$ is an *automorphism* of the Dynkin diagram $\Gamma(\Psi)$ iff the permutation leaves the Dynkin diagram unchanged with the exception of the numbering of the roots. $\mathrm{Aut}(\Gamma(\Psi))$ is the automorphism group of $\Gamma(\Psi)$. Note that $\rho \in \mathrm{Aut}(\Gamma(\Psi))$ iff $\omega_r = \omega_{\rho(r)}$ and $\langle\langle r, s\rangle\rangle = \langle\langle \rho(r), \rho(s)\rangle\rangle$ for all $r, s \in \Psi$.

Lemma 24. *Let $\mu \in \mathrm{Isom}(\mathfrak{h}_2^*; \mathfrak{h}_1^*)$ be an isometry that maps Ω_2 to Ω_1. If $\Psi_1, \Psi_2 \neq A_{2n}^{(2)}$, then $\Psi_1 = \Psi_2 = \Psi$, and μ induces an automorphism of the Dynkin diagram $\Gamma(\Psi)$.*

Proof. For each $r \in \Psi_2$ there is a unique $v \in \Psi_1$ such that $\mu(w_r r) = w_v v$. Define ρ to be the map that sends r to v. Since μ is an isometry, it follows that for all $r, s \in \Psi_2$

$$\langle\langle r, s\rangle\rangle_2 = \frac{\omega_{\rho(r)}\omega_{\rho(s)}}{\omega_r\omega_s} \langle\langle \rho(r), \rho(s)\rangle\rangle_1. \tag{30}$$

For $r = s$, Equation (30) implies that $|r|_2^2/|\rho(r)|_1^2 = \omega_{\rho(r)}^2/\omega_r^2$. If $|r|_2/|\rho(r)|_1$ is irrational, then $\omega_r/\omega_{\rho(r)}$ is irrational, which is absurd. If neither Ψ_1 nor Ψ_2 is $A_{2n}^{(2)}$, then the ratios of root lengths $|r|_2/|\rho(r)|_1$ is one of 1, $\sqrt{2}$ or $\sqrt{3}$ or their reciprocals. Thus, in all cases but where one of Ψ_i equals $A_{2n}^{(2)}$, $\rho(r)$ is the same length as r. Thus, $\rho : \Psi_2 \to \Psi_1$ is an isometry and

$$\omega_r = \omega_{\rho(r)} \tag{31}$$

for all r. Hence $\Psi_2 = \Psi_1$ are the same root bases and ρ induces an automorphism of the Dynkin diagram $\Gamma(\Psi)$.

Returning to the example, Lemma 24 implies that Equation (25) has a solution iff

$$\phi_1 \in \mathrm{Aut}(\Gamma(\Psi))\phi_2.$$

If $\Psi = A_2^{(1)}$, then $\mathrm{Aut}(\Gamma(\Psi)) = S_3$, so this shows that there is only one equivalence class of bijections; for $\Psi = C_2^{(1)}$ the automorphism group of $\Gamma(\Psi)$ is $\langle\,(2\ 3)\,\rangle$ so $\mathfrak{B}$ is partitioned into 3 equivalence classes with 2 elements each; for $\Psi = G_2^{(1)}$ the automorphism group is trivial, so $\mathfrak{B}$ is partitioned into 6 equivalence classes. Thus

Theorem 8. *Assume the Gelfond conjecture. Let $F/\mathbb{Q}$ be a non-normal, totally real, cubic extension. Then there are at least $10 = 1 + 3 + 6$ hamiltonian flows constructed in Sect. 2 that are not topologically conjugate by an energy-preserving conjugacy.*

4.2. Normal extensions of $\mathbb{Q}$

Recall that $F/\mathbb{Q}$ is a normal field extension if the set of embeddings of F is the Galois group of field automorphisms of F. Henceforth, we will assume that $\mathbf{G} = \mathrm{Gal}(F/\mathbb{Q})$. The vector space $V = \mathbb{R}\mathbf{G}$ is therefore also the group ring of $\mathbf{G}$, and V_o is a 2-sided ideal in $\mathbb{R}\mathbf{G}$. $\mathcal{U}^+$ – hence $\mathcal{L}$ – is a natural $\mathbb{Z}\mathbf{G}$-module, so let $\mathcal{A}$ be the subring of $\mathbb{Q}\mathbf{G}$ which is integral with respect to this representation. Let $S_o = \mathcal{A}/\mathbb{Q}\bar{t}$ where $\bar{t} = \sum_{\sigma\in\mathbf{G}} \sigma$.

Theorem 9. *Let* $f \in \mathrm{Aut}(V_{o,\mathbb{Q}}) \cap \mathrm{Aut}(\mathcal{L})$. *Then there is an* $r \in \mathcal{A}$, *with* $r + \mathbb{Q}\bar{t}$ *a unit of* S_o, *such that*

$$f = R_r | V_o,$$

where $R_r : V \to V$ *is the right-translation map* $y \mapsto yr$.

Proof. Let $t = \frac{1}{n+1}\bar{t}$, $s = 1 - t$ and write $V_{\mathbb{Q}} = sV_{\mathbb{Q}} \oplus tV_{\mathbb{Q}} = V_{o,\mathbb{Q}} \oplus \mathbb{Q}t$ as a direct sum of $\mathbb{Q}\mathbf{G}$ modules. It is convenient to extend f to an automorphism of $V_{\mathbb{Q}}$ by letting f fix t. This extension will also be denoted by f. For each $\sigma \in \mathbf{G}$, $f(\sigma) = \sum_{\tau\in\mathbf{G}} f_{\sigma,\tau}\, \tau$; the coefficients $f_{\sigma,\tau}$ are rational.

From the definition of $\mathcal{L}$ and the hypothesis that $f \in \mathrm{Aut}(\mathcal{L})$, there is the following commutative diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{f} & \mathcal{L} \\ \ell \uparrow & & \uparrow \ell \\ \mathcal{U}^+ & \xrightarrow{\alpha} & \mathcal{U}^+ \end{array},$$

where α is an automorphism of $\mathcal{U}^+$. Thus, if $\mathfrak{u} = \ell(u)$, then

$$\begin{aligned} f(\mathfrak{u}) &= \sum_{\tau\in\mathbf{G}} \left(\sum_{\sigma\in\mathbf{G}} f_{\sigma,\tau} \ln|\sigma(u)| \right) \tau, \quad \text{and} \\ &= \sum_{\tau\in\mathbf{G}} \ln|\tau(\alpha(u))|\tau. \end{aligned}$$

Until now, it has been assumed that $\mathcal{U}^+$ is an index 2, torsion-free subgroup of $\mathcal{U}$. Let us be precise and let $\mathcal{U}^+$ be the set of all positive units. In addition, let $\mathcal{U}^>$ be the subgroup of units all of whose conjugates are positive; this is a finite index subgroup of $\mathcal{U}^+$ since it contains the subgroup of squared units. The above equation for $f(\mathfrak{u})$ implies that for all $u \in \mathcal{U}^>$ and $\tau \in \mathbf{G}$

$$\tau(\alpha(u)) = \Pi_{\sigma\in\mathbf{G}}\, \sigma(u)^{f_{\sigma,\tau}}. \tag{32}$$

Since τ is an automorphism, Equation (32) implies that

$$\alpha(u) = \Pi_{\sigma\in\mathbf{G}}\, \sigma(u)^{f_{\tau\sigma,\tau}}.$$

Since $\mathcal{U}^{>}$ is a finite-index subgroup of $\mathcal{U}^{+}$, this implies that $f|V_o$ is determined by the coefficients $f_{\tau,1} : \tau \in \mathbf{G}$. Thus, without altering $f|V_o$, the extension of f can be altered so that $f_{\sigma,\tau} = f_{1,\sigma^{-1}\tau}$ for all $\sigma, \tau \in \mathbf{G}$.

Let $x_\tau = f_{1,\tau}$ and let $x = \sum_{\tau\in\mathbf{G}} x_\tau\, \tau$. Then

$$R_x(\sigma) = \sum_{\tau\in\mathbf{G}} x_{\sigma^{-1}\tau}\, \tau, \tag{33}$$

which is equal to $f(\sigma)$ by the hypothesis on the coefficients $f_{\sigma,\tau}$. Note that if we let $r = xs + t$, then $R_r|V_o = f|V_o$ and $R_r(t) = t$. Thus, R_r equals the extension of f that fixes t. Since $\alpha = \ell^{-1} \circ R_r \circ \ell$, and ℓ is a $\mathbb{Z}\mathbf{G}$-isomorphism between $\mathcal{L}$ and $\mathcal{U}^{+}$, it is clear that $r \in \mathcal{A}$ and $r + \mathbb{Q}\bar{t}$ is a unit of S_o. The theorem is proved.

How large is $\mathcal{A}$? If $y \in \mathcal{A}$, then R_y is an endomorphism of $\mathcal{L}$, so $x = ys + t$ defines an endomorphism of the lattice subgroup $\mathcal{L} \oplus \mathbb{Z}t$ of V. Hence, Trace(R_x) is a rational integer. Write $x = \frac{1}{n+1}\sum_{\sigma\in\mathbf{G}} x_\sigma\, \sigma$. Equation (33) shows that Trace$(R_x) = x_1$. Thus $x_1 \in \mathbb{Z}$. Similar reasoning shows that $x_\sigma \in \mathbb{Z}$ for all σ. Since $x \equiv y \bmod \mathbb{Q}t$, this shows that

$$\mathbb{Z}\mathbf{G} + \mathbb{Q}t \subseteq \mathcal{A} \subseteq (n+1)^{-1}\mathbb{Z}\mathbf{G} + \mathbb{Q}t.$$

This means that it is practical to compute the $\sim$ equivalence classes, at least for small n. However, Theorem 9 is probably too weak: my belief is that Equation (25) has a solution, regardless of the normality of F, iff $g = R'_\sigma|V_o$ for some automorphism σ of the field F.

Proof of Theorem 5. Assuming the hypotheses of the theorem, Corollary 4 implies that there is a $g \in \mathrm{Aut}(\mathcal{L}^*)$ and a $\mu \in O(\mathfrak{h}^*)$ such that $\mu = \phi_1 g \phi_2^{-1}$. By Theorem 7, the map $g \in \mathrm{Aut}(V^*_{o,\mathbb{Q}}) \cap \mathrm{Aut}(\mathcal{L}^*)$. Let $f = g'$; Theorem 9 now implies Theorem 5.

4.3. Applications of Theorem 5

Let us establish a convention that will be helpful to do the computations of the following examples. By enumerating the elements of $\mathbf{G}$ (and Ψ, Ω), $\mathbf{G}$ (and Ψ, Ω) will be identified with the set $\{1, \ldots, n+1\}$. The enumeration of Ψ and Ω will be the enumeration from its Dynkin diagram, while $\mathbf{G}$ will usually suggest a convenient enumeration. Permutations of, and bijections between, these sets are then naturally identified with permutations of $n+1$ symbols. The group of permutations of the set $\bullet$ is denoted by $S(\bullet)$.

Example 4. Let's apply Theorem 5 to a totally real, normal, cubic extension. For example, let $F = \mathbb{Q}(\zeta + \zeta^{-1})$ where ζ is a primitive 7-th root of unity. In this case $\mathbf{G} = \{1, \sigma, \sigma^2\}$ is a cyclic group of order 3.

Units of S_o [c.f. [18,37]]. The ring $S_o = \mathcal{A}/\mathbb{Q}\bar{t}$ is isomorphic to $\mathbb{Z}[\omega]$, where ω is a primitive 3-rd root of unity. The unit group of $\mathbb{Z}[\omega]$ is $\{\pm 1, \pm\omega, \pm\omega^2\}$, so the unit group of S_o, $\mathcal{U}(S_o)$, is $\pm\mathbf{G}+\mathbb{Q}\bar{t}$. Let $\mathcal{U}(S_o)^+ = \mathbf{G}+\mathbb{Q}\bar{t}$; $\mathcal{U}(S_o)^+$ is naturally isomorphic to $\mathbf{G}$.

Solutions to Equation (25). Let $\phi_1, \phi_2 \in \mathfrak{B}$ and assume that $\mu = \phi_1 g \phi_2^{-1}$ solves Equation (25) with $g = \bar{R}'_r$ for some $r \in \mathcal{U}(S_o)$. Then Equation (25) also has the solution $-\mu$, $-g = \bar{R}'_{-r}$ and $-r \in \mathcal{U}(S_o)$. This shows that it can be assumed that $r \in \mathcal{U}(S_o)^+$.

Since $\mathcal{U}(S_o)^+$ acts naturally on $\hat{\mathbf{G}}$ by permutations, $\mu = \phi_1 \bar{R}_r \phi_2^{-1}$ is a permutation of Ω and an element in $O(\mathfrak{h}^*)$. Thus, μ naturally induces an automorphism of the Dynkin diagram $\Gamma(\Psi)$ by Lemma 24.

To conclude the calculations, identify $\mathbf{G}$, Ψ and Ω, etc. with $\{1, 2, 3\}$. The permutation group S_3 on 3 symbols is generated by the transposition $a = (1\ 2)$ and the 3 cycle $b = (1\ 2\ 3)$ and $C_3 = \langle b \rangle$ is a normal subgroup of S_3. The permutation representation of $\mathcal{U}(\mathbf{G})^+$ on $\hat{\mathbf{G}}$ induces the map $\sigma + \mathbb{Q}\bar{t} \to (1\ 2\ 3)$. Since C_3 is normal in S_3, Equation (25) has a solution iff

$$\phi_1\phi_2^{-1}C_3 \cap \mathrm{Aut}(\Gamma(\Psi)) \neq \emptyset.$$

If $\Psi = A_2^{(1)}$, then $\mathrm{Aut}(\Gamma(\Psi)) \equiv S_3$, so $\mathfrak{B}$ has only one equivalence class of bijections. If $\Psi = C_2^{(1)}$, then $\mathrm{Aut}(\Gamma(\Psi)) \equiv \langle (2\ 3) \rangle$. Since $C_3 \cup (2\ 3)C_3 = S_3$, the equivalence relation is again trivial. If $\Psi = G_2^{(1)}$, then $\mathrm{Aut}(\Gamma(\Psi)) = 1$. It follows that there are 2 equivalence classes associated with the cosets $1.C_3$ and $(1\ 2)C_3$.

The contrast with Example 3, where there were 1, 3 and 6 equivalence classes, is striking. To conclude

Theorem 10. *Let $F/\mathbb{Q}$ be a normal, cubic extension. Then there are exactly $1 + 1 + 2 = 4$ flows defined by Theorem 1 that are not topologically conjugate by an energy-preserving homeomorphism.*

This is a corollary of

Theorem 11. *Assume that $\Psi_i \neq A_{2n}^{(2)}$. If $\mathcal{U}(S_o) = \pm\mathbf{G} + \mathbb{Q}t$ and $\phi_i \in \mathfrak{B}_i$, then $\phi_1 \sim \phi_2$ iff $\Psi_1 = \Psi_2$ and $\varphi_t^{\phi_1}$ is conjugate to $\varphi_t^{\phi_2}$ by an energy-preserving homeomorphism.*

Proof. It suffices to prove that if $\phi_1 \sim \phi_2$, then there is a symplectic diffeomorphism $h : T^*\Sigma \to T^*\Sigma$ such that $H_{\phi_1} \circ h = H_{\phi_2}$.

If $\phi_1 \sim \phi_2$ and $\mathcal{U}(S_o) = \pm\mathbf{G} + \mathbb{Q}t$, then there is a $\tau \in \mathbf{G}$ such that $\mu = \phi_1 \circ \bar{R}'_\tau \circ \phi_2^{-1}$ solves Equation (25). Since $\bar{R}'_\tau$ acts as a permutation of $\hat{\mathbf{G}}$, μ is an isometry of Ω_2 with Ω_1. By Lemma 24, Ψ_1 and Ψ_2 equal a common root basis Ψ and μ acts as a permutation of Ω induced by an automorphism of $\Gamma(\Psi)$.

Define:

$$\hat{h}(x, y, \mathcal{X}, \mathcal{Y}) = \left(\bar{R}_{\tau^{-1}}x,\, R_{\tau^{-1}}y,\, \bar{R}'_\tau\mathcal{X},\, R'_\tau\mathcal{Y}\right),$$

for all $(x, y, \mathcal{X}, \mathcal{Y}) \in T^*\hat{\Sigma}$. Since $\mathbf{G}$ acts naturally on the right on $V = \mathbb{R}\mathbf{G}$, the map $\hat{h}$ induces an analytic symplectic diffeomorphism of $T^*\Sigma$.

Calculations show that $\langle\langle\phi_1(\bar{R}'_\tau\mathcal{X}), \phi_1(\bar{R}'_\tau\mathcal{X})\rangle\rangle = \langle\langle\phi_2(\mathcal{X}), \phi_2(\mathcal{X})\rangle\rangle$, $\langle\bar{R}_{\tau^{-1}}x, \hat{\sigma}\rangle = \langle x, \widehat{\sigma\tau}\rangle$ and $(R'_\tau\mathcal{Y})_\sigma = \mathcal{Y}_{\sigma\tau}$. Thus:

$$H_{\phi_1} \circ \hat{h} = \frac{1}{2}\langle\langle\phi_2(\mathcal{X}), \phi_2(\mathcal{X})\rangle\rangle + \sum_{\sigma\in\mathbf{G}} \exp(2b_{1,\sigma\tau^{-1}}\langle x, \hat{\sigma}\rangle)\mathcal{Y}_\sigma^{2b_{1,\sigma\tau^{-1}}},$$

where $b_{i,\sigma}$ is the integer b_σ defined for H_{ϕ_i}. To finish the proof, it suffices to show that $b_{2,\sigma} = b_{1,\sigma\tau^{-1}}$ for all $\sigma \in \mathbf{G}$.

Let ρ be the automorphism of the Dynkin diagram that is induced by μ, and let $r_{i,\sigma}$ be the unique root r that satisfies $\phi_i(\hat{\sigma}) = w_r r$. Then $b_{i,\sigma} = \omega/\omega_{r_{i,\sigma}}$ for all σ and i. Since $\bar{R}'_\tau\hat{\sigma} = \widehat{\sigma\tau^{-1}}$, one calculates that $\mu(w_{r_{2,\sigma}}r_{2,\sigma}) = w_{r_{1,\sigma\tau^{-1}}}r_{1,\sigma\tau^{-1}}$. Thus $\rho(r_{2,\sigma}) = r_{1,\sigma\tau^{-1}}$, so $b_{2,\sigma} = b_{1,\sigma\tau^{-1}}$.

Example 5. Let's apply Theorem 5 to a totally real, normal, quartic extension. In this case, the Galois group of $F/\mathbb{Q}$ is either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$. Let us examine the Klein 4-group first; $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is an example of such a totally real, normal, quartic number field with $\mathrm{Gal}(F/\mathbb{Q}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Units of S_o. Arguments similar to those of Example 4 show that $\mathcal{U}(S_o)$ is trivial.

Solutions to Equation (25). Let $\phi_1, \phi_2 \in \mathfrak{B}$. As above, if μ and $g = \bar{R}_r$ solve Equation (25), then it may be assumed that $r \in \mathcal{U}(S_o)^+$. On identifying $\mathbf{G}$, etc. with $\{1, 2, 3, 4\}$, the permutation representation of $\mathcal{U}(S_o)^+ \to S(\hat{\mathbf{G}})$ is naturally identified with the subgroup $V_4 = \langle(1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle$. This subgroup is normal in S_4. Repeating the arguments of the previous example shows that Equation (25) is solvable iff

$$\phi_1\phi_2^{-1}V_4 \cap \mathrm{Aut}(\Gamma(\Psi)) \neq \emptyset.$$

If $\Psi = A_3^{(1)}$, then $\mathrm{Aut}(\Gamma(\Psi)) \equiv D_4$ where D_4 is the dihedral group of order 8 generated by $(1\ 2\ 3\ 4)$ and V_4. Therefore, Equation (25) has a solution iff $\phi_1 \in D_4\phi_2$. Since $[S_4 : D_4] = 3$, it follows that there are 3 equivalence classes in $\mathfrak{B}$. The representative cosets are D_4, $D_4(1\ 2)$ and $D_4(1\ 3)$.

If $\Psi = B_3^{(1)}$, then $\mathrm{Aut}(\Gamma(\Psi)) \equiv \langle(1\ 4)\rangle =: C_2$. Since $C_2 \cap V_4, C_2 \cap V_4(1\ 4) \neq \emptyset$, and the remaining 4 cosets of V_4 intersect C_2 trivially, Equation (25) has a solution iff $\phi_1 \in (V_4 \cup V_4(1\ 4))\phi_2$. Thus $\mathfrak{B}$ is partitioned into 3 equivalence classes.

If $\Psi = C_3^{(1)}$, then $\mathrm{Aut}(\Gamma(\Psi)) \equiv \langle(1\ 2)(3\ 4)\rangle =: C_2$. Since $C_2 < V_4$, the remaining 5 cosets of V_4 intersect C_2 trivially, Equation (25) has a solution iff $\phi_1 \in V_4\phi_2$, so there are 6 equivalence classes.

Note that the normality of $\mathcal{U}(S_o)^+$ in the permutation group $S(\hat{\mathbf{G}})$ meant that each equivalence class of bijections has the same number of elements.

The next example shows that this fails in the case that $\mathcal{U}(S_o)^+$ is a non-normal subgroup of $S(\hat{\mathbf{G}})$.

Example 6. Let $F/\mathbb{Q}$ be a totally real, normal, quartic extension with $\mathrm{Gal}(F/\mathbb{Q}) = \mathbb{Z}_4$. An example of such a number field is given by $F = \mathbb{Q}(\zeta + \zeta^4 + \zeta^{-1} + \zeta^{-4})$ where ζ is a primitive 17-th root of unity.

Units of S_o. $\mathcal{U}(S_o)$ is trivial.

Solutions to Equation (25). Let $\phi_1, \phi_2 \in \mathfrak{B}$ and write $\alpha = \phi_1$ and $\beta = \phi_2\phi_1^{-1} \in S(\Omega)$. As above, if μ and $g = \bar{R}'_r$ solve Equation (25), then it may be assumed that $r \in \mathcal{U}(S_o)^+$. On identifying $\mathbf{G}$, etc. with $\{1, 2, 3, 4\}$, the permutation representation of $\mathcal{U}(S_o)^+ \to S(\hat{\mathbf{G}})$ is naturally identified with the subgroup $C_4 = \langle(1\ 2\ 3\ 4)\rangle$. This subgroup is normalized in S_4 by D_4 and has three conjugate subgroups: C_4, $(2\ 3)C_4(2\ 3)$ and $(3\ 4)C_4(3\ 4)$. Repetition of the arguments of the previous example along with the non-normality of $C_4 \equiv \mathcal{U}(S_o)^+$ shows that Equation (25) is solvable iff

$$\alpha C_4 \alpha^{-1} \cap \mathrm{Aut}(\Gamma(\Psi))\beta \neq \emptyset.$$

In case $\Psi = A_3^{(1)}$, then Equation (25) has a solution iff either (i) $\alpha, \beta \in D_4$; or (ii) $\alpha \in D_4(2\ 3)$ and $\beta \notin D_4(2\ 3)$ or $\alpha \in D_4(3\ 4)$ and $\beta \notin D_4(3\ 4)$. Thus, if $\phi_1 \in D_4$, then the equivalence class $[\phi_1]$ has 8 elements; if $\phi_1 \notin D_4$, then $[\phi_1]$ has 16 elements. Therefore, there are two equivalence classes: $[1] = D_4$ and $[(2\ 3)] = S_4 - D_4$.

If $\Psi = B_3^{(1)}$ or $\Psi = C_3^{(1)}$, then there are 3 equivalence classes, D_4, $D_4(2\ 3)$ and $D_4(3\ 4)$ each with 8 elements.

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