POSITIVE-ENTROPY INTEGRABLE SYSTEMS AND THE TODA LATTICE, II

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Abstract. This note constructs completely integrable convex Hamiltonians on the cotangent bundle of certain $\mathbb{T}^k$ bundles over $\mathbb{T}^l$. A central role is played by the Lax representation of a Bogoyavlenskij-Toda lattice. The classification of these systems, up to iso-energetic topological conjugacy, is related to the classification of abelian groups of Anosov toral automorphisms by their topological entropy function.

1. Introduction

Say that a smooth flow $\varphi : M \times \mathbb{R} \to M$ is *integrable* if there is an open dense subset $L \subset M$ such that L is fibred by b-dimensional tori and the smooth bundle coordinate charts $(I, \phi) : U \to \mathbf{D}^a \times \mathbb{T}^b$ conjugate φ to a smooth translation-type flow $t \cdot (I, \phi) = (I, \phi + t\xi(I))$ on the fibres of L. This local form, classically known as action-angle coordinates, suggests that integrable flows are dynamically uninteresting. The example of the geodesic flow of a compact 3-dimensional *Sol* manifold which is completely integrable and has positive topological entropy, due to Bolsinov and Taimanov [6], is proof that this is not the case. The present paper generalises the examples of [6, 11]. First, it shows how to construct integrable convex hamiltonian systems on cotangent bundles of certain solmanifolds in higher dimensions that are analogues to the *Sol* geometric 3-manifolds when the monodromy group is not $\mathbb{R}$-split; second, it shows that Lax representations of Bogoyavlenskij-Toda lattices are essential to construct these integrable systems, and moreover, the double-bracket Lax representations are essential to understand the dynamics on the singular set; third, the Lax map of a Bogoyavlenskij-Toda lattice and the 'momentum map' of a natural $\mathcal{F}$-structure on the solmanifold form a dual pair; and, finally, the topological classification of these integrable systems can be resolved by classifying abelian groups of Anosov toral automorphisms by the topological entropy function.

This appears to be a novel and interesting phenomenon: the construction of these integrable systems uses the machinery of Lax representations and R-matrices, while their dynamical classification uses machinery developed to understand hyperbolic dynamical systems.

Let us now sketch the constructions and results of the present paper.

1.1. **The *Sol*-manifolds.** Let A be a torsion-free, abelian group of diffeomorphisms of $\mathbb{T}^b$. The group A acts on $\mathbb{T}^b \times A_{\mathbb{R}}$, where $A_{\mathbb{R}} = A \otimes_{\mathbb{Z}} \mathbb{R}$, via the diagonal action

$$\forall \alpha \in A, y \in \mathbb{T}^b, x \in A_{\mathbb{R}} : \qquad \alpha \star (y, x) := (\alpha(y), x + \alpha \otimes 1). \tag{1.1}$$

This action is free and proper. The compact, smooth quotient is denoted by Σ or Σ_A. The fibring of Σ by the tori $\mathbb{T}^b$ equips Σ with a natural $\mathcal{F}$-structure.

Henceforth it is assumed that $A < \mathrm{GL}(b; \mathbb{Z})$ is an abelian group of semi-simple elements, hence contained in a Cartan subgroup of $\mathrm{GL}(b; \mathbb{C})$, and therefore an

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exponential subgroup of $\mathrm{GL}(b;\mathbb{C})$. When A is an exponential subgroup of $\mathrm{GL}(b;\mathbb{C})$, the universal cover $\tilde{\Sigma}$ of Σ admits the structure of a solvable Lie group as follows: When A is an exponential subgroup of $\mathrm{GL}(b;\mathbb{C})$, $A_{\mathbb{R}}$ is naturally identified with an abelian subgroup of $\mathrm{GL}(b;\mathbb{R})$. From this, there is a natural Lie group structure on $\mathbb{R}^b \star A_{\mathbb{R}} =: \mathbf{S}$ and $\mathbb{Z}^b \star A =: \mathbf{S}_{\mathbb{Z}}$ is a lattice subgroup of $\mathbf{S}$, *c.f.* 2.2. In the general case, A contains a finite-index exponential subgroup A_2 [26, Theorem 4.28]. The finite covering Σ_2 of Σ induced by A_2 has a universal cover with a solvable Lie group structure; in this case, the fundamental group of Σ need not embed as a subgroup in this universal cover, although it does act as a free and proper group of deck transformations [26, pp.s 70-71]. An elementary argument shows that if Γ is a finite group of deck transformations and φ is an integrable flow on M that is Γ-invariant, then the induced flow on M/Γ is integrable, also. So, to simplify the discussion in this introduction, without losing generality, it will be assumed that A is an exponential subgroup of $\mathrm{GL}(b;\mathbb{C})$.

1.2. **Integrable geodesic flows.** Let y_i be coordinates on $\mathbb{C}^n$ which diagonalise A. Define complex-valued differential 1-forms on Σ by

$$\nu_i = \exp(-\langle \ell_i, x\rangle)\, \mathrm{d}y_i, \qquad \text{and} \quad \eta_i = \mathrm{d}x_i \tag{1.2}$$

where $\ell_i \in \mathrm{Hom}(A_{\mathbb{R}};\mathbb{R})$ is the linear form which maps $x \in A_{\mathbb{R}}$ to the logarithm of the modulus of its i-th eigenvalue and $x_i = \langle \ell_i, x\rangle$. A riemannian metric on Σ can be defined by

$$\mathbf{g} = \sum\nolimits_{i,j} Q_{ij}\ \nu_i \cdot \nu_j + \sum\nolimits_{i,j} R_{ij}\ \eta_i \cdot \eta_j + \sum\nolimits_{i,j} S_{ij}\ \eta_i \cdot \nu_j, \tag{1.3}$$

where Q, R and S are constant, complex symmetric matrices chosen so that $\mathbf{g}$ is a real, symmetric, positive-definite $(0,2)$-tensor. The metric $\mathbf{g}$ is the general form of a left-invariant metric on $\mathbf{S} = \mathbb{R}^b \star A_{\mathbb{R}}$. When the off-diagonal term S vanishes, the subgroups $\mathbb{R}^b$ and $A_{\mathbb{R}}$ are orthogonal, totally geodesic and flat. By left-invariance of $\mathbf{g}$, each left translate of these two subgroups share these properties.

Question A. *Which metrics* $\mathbf{g}$ *have a completely integrable geodesic flow?*

Some answers are known. The examples of Bolsinov and Taimanov shows that when A is a cyclic group, then the geodesic flow is completely integrable for $\mathbf{g}$ with $S = 0$ and Q, R arbitrary [6, 7]. The present author showed that when A has rank $b-1$ (so A is $\mathbb{R}$-split), $Q_{ij} = \delta_{ij}\epsilon_i^2$, R is of a special form and $S = 0$, then the geodesic flow is completely integrable. To explain the special form of R, write the Hamiltonian of $\mathbf{g}$ in canonical coordinates:

$$2H_{\mathbf{g}} = \sum\nolimits_{ij} Q_*^{ij}\ \exp(\langle \ell_i + \ell_j, x\rangle) + \sum\nolimits_{ij} R^{ij}\ p_{x_i} p_{x_j}, \tag{1.4}$$

where $Q_*^{ij} = Q^{ij}\ p_{y_i} p_{y_j}$ (no sum). Because y_i is a cyclic variable, p_{y_i} is a first integral. $H_{\mathbf{g}}$ reduces to a family of Bogoyavlenskij-Toda-like Hamiltonians in the canonical variables (x, p_x). If one diagonalises Q, then the complete integrability of the Bogoyavlenskij-Toda Hamiltonian dictates the form of R. The introduction of [11] has an explicit example.[1] The work of Adler & Van Moerebeke and Kozlov & Treschev suggests that when $S = 0$ the only completely integrable Hamiltonians $H_{\mathbf{g}}$ arise from Bogoyavlenskij-Toda lattices or their deformations [3, 23, 22].

The preceding argument glosses over a subtlety: the cyclic variables p_{y_i} are defined only on the universal cover. In the above cases, one can construct smooth integrals that descend to the quotient; this is true in general, but the difficulty lies in choosing R. This is related to Lax representations.

[1] This is also referred to as the Toda lattice or the Kostant-Toda lattice, but Kostant in [21] attributes to Bogoyavlenskij [4] the recognition of the role played by root systems of semisimple Lie algebras.

1.3. **Lax Representations and momentum maps.** On the covering $\hat{\Sigma} = \mathbb{T}^b \times A_{\mathbb{R}}$, there is the obvious free action of $\mathbb{T}^b$. This induces an $\mathcal{F}$-structure on Σ, which one may think of as a locally-defined free action of $\mathbb{T}^b$. The momentum map $\hat{f}$ of the $\mathbb{T}^b$-action induces a map f by equivariance, as illustrated in the right-hand side of (1.5):

$$\begin{tikzcd}
\mathcal{L}_R^* \arrow[d] & T^*\hat{\Sigma} \arrow[l, "\hat{\mathbf{L}}"'] \arrow[r, "\hat{f}"] \arrow[d, "\hat{\Pi}"] & \mathrm{Lie}(\mathbb{T}^b)^* \arrow[d, "(\mathrm{mod}\ A)"'] \arrow[drr, "(\mathrm{mod}\ \sim)"] & & \\
\mathcal{L}_R^* & T^*\Sigma \arrow[l, "\mathbf{L}"'] \arrow[r] \arrow[rrr, bend right, "f"'] & \mathrm{Lie}(\mathbb{T}^b)^*/A \arrow[rr, "\mathrm{coll.}"] & & \mathrm{Lie}(\mathbb{T}^b)^*/\sim .
\end{tikzcd} \tag{1.5}$$

$\mathrm{Lie}(\mathbb{T}^b)^*/A$ is neither a smooth manifold nor a Hausdorff space but it does contain an open and dense subspace that is a smooth manifold. One can collapse the singular set of $\mathrm{Lie}(\mathbb{T}^b)^*/A$ to a single point to define a Hausdorff topological space $\mathrm{Lie}(\mathbb{T}^b)^*/\sim$, which is a smooth manifold outside of a single point, as illustrated in figure 1. Since the collapse is A-invariant, the map f is defined naturally. The map f is a first integral of $H_{\mathbf{g}}$ and one can loosely think of f as the momentum map of the locally-defined $\mathbb{T}^b$ action on $T^*\Sigma$.

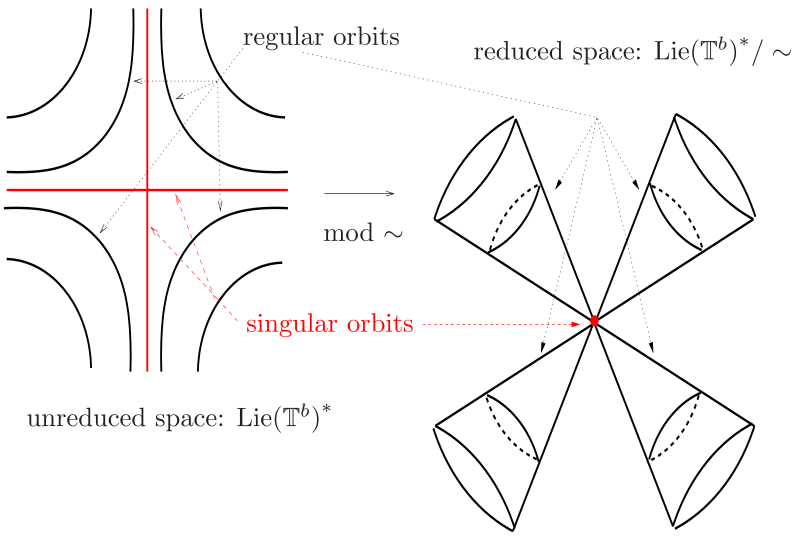


FIGURE 1. The quotient map from $\mathrm{Lie}(\mathbb{T}^b)^* \to \mathrm{Lie}(\mathbb{T}^b)^*/\sim$. The regular points are points with a non-zero component in each eigenspace of A; the singular set is the complement.

On the left of (1.5) is a map $\mathbf{L}$, called a Lax matrix, that is implicit in the identification of $H_{\mathbf{g}}$ with a Bogoyavlenskij-Toda Hamiltonian. The construction of a Poisson Lax map that Poisson commutes with f is the key difficulty in proving the complete integrability of $H_{\mathbf{g}}$.

Question B. *What conditions on A imply the existence of a Poisson Lax map $\mathbf{L}$ such that the $\mathbb{T}^b$-momentum map f and $\mathbf{L}$ form a dual pair?*

Implicit in the two papers of Bolsinov and Taimanov is the fact that if A is cyclic, then this question is trivially soluble. In [11, p. 529], the present author shows that there is a Poisson map that Poisson commutes with the $\mathbb{T}^b$-momentum map f when $A < \mathrm{GL}(b;\mathbb{Z})$ is $\mathbb{R}$-split and of finite index in its centraliser (the relation to Lax maps is hinted at in the remark on [11, p. 529]). To generalise that construction, it appears necessary to use the machinery of Lax representations.

1.3.1. *Positive topological entropy.* The geodesic flow of $\mathbf{g}$ must have positive topological entropy, since $\pi_1(\Sigma)$ has exponential word growth [14]. When $S = 0$, there is a direct proof of this: since $A_{\mathbb{R}}$ is flat and totally geodesic in $\tilde{\Sigma}$, as are all its left-translates, each curve $t \mapsto tv + y$ for $v \in A_{\mathbb{R}}$, $y \in \mathbb{R}^b$ is a geodesic. On Σ, for $v \in A$, the geodesic is periodic and one sees that the geodesic flow induces the return map on $\mathbb{T}^b$ defined by $y \mapsto v \cdot y$ – which is a partially hyperbolic, and generally Anosov, automorphism of $\mathbb{T}^b$ (see figure 2).

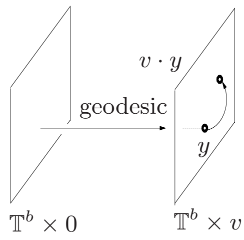


FIGURE 2. A return map.

The appearance of such 'subsystems' heavily constrains the topological conjugacy class of a completely integrable geodesic flow of the form of $\mathbf{g}$.

1.4. **Results.** Let us sketch the main theorems of this paper.

1.4.1. *Complete integrability.*

Definition 1. *A torsion-free abelian subgroup $A < \mathrm{GL}(b;\mathbb{Z})$ is* maximal *if its elements are semisimple and it is of finite index in its centraliser.*

Theorem 1. *If $A < \mathrm{GL}(b;\mathbb{Z})$ is a maximal subgroup, then there is a Poisson Lax map $\mathbf{L}$ such that (1.5) describes a dual pair, see (3.14). In particular, there is a reversible* Finsler *metric on Σ_A whose geodesic flow is completely integrable.*

If, in addition, an irreducible element in A has exactly r real eigenvalues and $2\mathsf{c}$ non-real eigenvalues, then the geodesic flow of the riemannian metric $\mathbf{g}$ (1.3) is completely integrable when its Hamiltonian is defined as in (3.16) with root system $\mathfrak{g}^{(m)}$ in the cases described by Table 1.

In all cases, the singular set is a real-analytic variety. Consequently, the integrable systems are semi-simple *in the sense of* [10].

r	c	$\mathfrak{g}^{(m)}$	r	c	$\mathfrak{g}^{(m)}$
$*$	0	$A_n^{(1)}, D_{n+1}^{(2)}$	2	1	$D_4^{(3)}$
0	$*$	$A_n^{(1)}, D_{n+1}^{(2)}$	2	$*$	$B_n^{(1)}, C_n^{(1)}$
1	$*$	$A_{2n}^{(2)}$	3	$*$	$A_{2n-1}^{(2)}$
			4	$*$	$D_n^{(1)}$

TABLE 1. Conditions on eigenvalues and root systems which produce a riemannian metric with integrable geodesic flow ($*$ is an arbitrary positive integer).

The first row in the upper left corner of the table summarises the result of [11]. In the cases not covered in the table, it is uncertain if Σ_A admits a riemannian metric with completely integrable geodesic flow – the construction here yields only completely integrable geodesic flows of reversible Finslers. If not, one would have the first example of a compact smooth manifold that admits a completely integrable reversible Finsler, but not a riemannian, geodesic flow.

1.4.2. *Topological Entropy and Iso-energetic Topological Conjugacy.* The tangent spaces to the $\mathbb{T}^b$-fibres of Σ form a sub-bundle $\mathbf{V} \subset T\Sigma$. Let $\mathbf{V}^\perp \subset T^*\Sigma$ be the annihilator of $\mathbf{V}$. The subspace $\mathbf{V}^\perp$ is invariant under the geodesic flow of Theorem 1 and that geodesic flow has positive topological entropy on $\mathbf{V}^\perp$. We prove

Theorem 2. *Let $\Sigma = \Sigma_A$ be as in Theorem 1 and let φ be the Hamiltonian flow on $T^*\Sigma$ induced by the Hamiltonian $\mathbf{H}$ defined in (3.16). Then*

(1) *$\mathbf{V}^\perp$ is a weakly normally hyperbolic invariant manifold. Its stable and unstable manifolds coincide and equal the pre-image of the equivalence class of 0 in $\mathrm{Lie}(\mathbb{T}^b)^*/\sim$ under the $\mathbb{T}^b$-momentum map f;*
(2) *the topological entropy of $\varphi|\mathbf{H}^{-1}(\frac{1}{2})$ equals that of $\varphi|\mathbf{V}^\perp \cap \mathbf{H}^{-1}(\frac{1}{2})$, when $\mathbf{H}$ is induced by the $A_n^{(1)}$ Bogoyavlenskij-Toda lattice;*
(3) *in all cases, the topological entropy of $\varphi|\mathbf{V}^\perp \cap \mathbf{H}^{-1}(\frac{1}{2})$ is calculable (see Table 5).*

The construction of the Lax map in theorem 1 is unique up to the action of a permutation group. If ϕ_1, ϕ_2 are two such permutations and φ_1, φ_2 are the resulting geodesic flows, an interesting question is whether these flows are topologically distinct. A topological invariant, namely the marked homology spectrum, does distinguish these flows in many cases even when topological entropy cannot. To explain, let

$$\mathrm{h}(v) = h_{top}(v) \qquad\qquad \mathrm{h} : A \to \mathbb{R} \tag{1.6}$$

be the entropy function, where $v \in A$ is viewed as a $\mathbb{T}^b$-automorphism.

Theorem 3. *If there is a topological conjugacy of φ_1, φ_2 on their respective unit sphere bundles, then there is an automorphism $f : A \to A$ such that*

$$\mathrm{h} \circ f = \mathrm{h}. \tag{1.7}$$

If the number-theoretic closure of A is a group of Anosov automorphisms, then f is induced by a Galois automorphism.

In many cases, the group of Galois automorphisms is trivial, which implies that each of the constructed Hamiltonian flows must be topologically non-conjugate. In general, one should not expect the number-theoretic closure of A to be a group of Anosov automorphisms, though.

Question C. *Which automorphisms of A fix the entropy function* h*?*

This leads to a further question, whose formulation is somewhat technical and is deferred to section 7, question F. Finally, theorem 7.3 provides information on the number of distinct topological conjugacy classes of integrable Hamiltonian flows provided by theorem 1.

Question C is a rigidity question: to what extent does the entropy of an action determine that action. An approach to this question is to ask which embeddings of $\mathbb{Z}^a \cong A$ into $\mathrm{GL}(b;\mathbb{Z})$ have equal entropies. Katok, Katok and Schmidt give examples of iso-entropic actions of $\mathbb{Z}^2$ on $\mathbb{T}^3$ by maximal subgroups of $\mathrm{GL}(3;\mathbb{Z})$ that are conjugate in $\mathrm{GL}(3;\mathbb{Q})$ but not conjugate in $\mathrm{GL}(3;\mathbb{Z})$ [20]. However, the suspension manifolds of these actions are not homotopy equivalent. Indeed, if $A' < \mathrm{GL}(b;\mathbb{Z})$ is not conjugate to A in $\mathrm{GL}(b;\mathbb{Z})$, then $\pi_1(\Sigma_{A'})$ is not isomorphic to $\pi_1(\Sigma_A)$. Thus, question C is somewhat finer than the iso-entropic rigidity problem examined in [20].

1.5. **Two additional questions.** Let Σ be a torus bundle over a torus of the type described in section 1.1. Σ is aspherical and the fundamental group of Σ is a poly-$\mathbb{Z}$ group[2], so Theorem 15B.1 of [29] implies that $\pi_1(\Sigma)$ determines Σ up to homeomorphism. The *standard* smooth structure on Σ is defined by the above construction. In general, the topological manifold Σ may admit several inequivalent smooth structures.

Question D. *Which smooth structures on the topological manifold Σ admit a riemannian or Finsler metric whose geodesic flow is completely integrable?*

This question is already quite interesting when $A = 1$ and Σ is a torus since [12] shows that the integrals cannot all be real-analytic if the smooth structure is non-standard. It is unknown if there are analogous obstructions when $A \neq 1$.

And, finally,

Question E. *What conditions on $A < \mathrm{GL}(b; \mathbb{Z})$ imply that Σ_A admits a riemannian or Finsler metric whose geodesic flow is completely integrable?*

Theorem 4.1 shows that there are natural examples of groups A that are not maximal, yet Σ_A admits a completely integrable Finsler. These examples are constructed using symmetries provided by number-theoretic considerations. The difficulty in the general case, where there are no obvious symmetries, is the construction of the Lax map $\mathbf{L}$ appears to break down.

2. Notation and Preliminary Definitions

2.1. **Integrability.** The present paper's definition of complete integrability follows that of [5, 11].

Let Σ be a real-analytic manifold. The set of smooth functions on the cotangent bundle of Σ, $C^\infty(T^*\Sigma)$, has two canonical algebraic structures: it is an abelian algebra when equipped with the natural operations of point-wise addition and multiplication; and, coupled with the canonical *Poisson bracket*, $\{,\}$, $(C^\infty(T^*\Sigma), \{,\})$ is a Lie algebra of derivations of the algebra $C^\infty(T^*\Sigma)$. A hamiltonian $H \in C^\infty(T^*\Sigma)$ induces a vector field $Y_H := \{\,, H\}$. For $\mathcal{A} \subset C^\infty(T^*\Sigma)$ and $P \in T^*\Sigma$, let $d\mathcal{A}_P = \operatorname{span}\{df_P \ : \ f \in \mathcal{A}\}$ and let $Z(\mathcal{A}) = \{f \in \mathcal{A} \ : \ \{\mathcal{A}, f\} \equiv 0\}$. Let $k = \sup_P \dim d\mathcal{A}_P$, $l = \sup_P \dim dZ(\mathcal{A})_P$. Let us say $P \in T^*\Sigma$ is $\mathcal{A}$-*regular* if there exist $f_1, \ldots, f_k \in \mathcal{A}$ such that P is a regular value for the map $F = (f_1, \ldots, f_k)$ and $f_1, \ldots, f_l \in Z(\mathcal{A})$; if P is not $\mathcal{A}$-regular then it is $\mathcal{A}$-*critical*. Let $L(\mathcal{A})$ be the set of $\mathcal{A}$-regular points. H is assumed to be proper.

Definition 2 (*c.f.* [5])**.** *$H \in C^\infty(T^*\Sigma)$ is* integrable *if there is a Lie subalgebra $\mathcal{A} \subset C^\infty(T^*\Sigma)$ such that:*

(1) *$H \in Z(\mathcal{A})$;*
(2) *$k + l = \dim T^*\Sigma$ and $L(\mathcal{A})$ is an open and dense subset of $T^*\Sigma$.*

If $k = l = dim\,\Sigma$, we will say that H is completely integrable.

Bolsinov and Jovanovic [5] introduced this definition of complete integrability. The standard definition of complete integrability (resp. non-commutative integrability) are special cases of Definition 2 with $\mathcal{A} = \operatorname{span}\{f_1, \ldots, f_k\}$ and $l = k$ (resp. $l \leq k$) and the regular-point set of $F = (f_1, \ldots, f_k)$ is dense. Definition 2 is both more intrinsic, and more suited to the examples of the present paper. Note that the present definition of integrability is equivalent to that of Dazord & Delzant [13].

[2] That is, there is a sequence of subgroups $0 = \mathrm{D}_m \lhd \mathrm{D}_{m-1} \lhd \cdots \lhd \mathrm{D}_0 = \mathrm{D}$ such that $\mathrm{D}_i/\mathrm{D}_{i+1} \cong \mathbb{Z}$ for all i.

2.2. **Construction of the solmanifolds and number theory.** There is a well-known correspondence between abelian subgroups of $\mathrm{GL}(b;\mathbb{Z})$ and groups of units in algebraic number fields of degree d dividing b [20]. The present paper exploits this correspondence extensively. The following section establishes notation that is used throughout. In terms of the terminology in the introduction, we use the following translation table:

abelian $A < \mathrm{GL}(b;\mathbb{Z})$	$\rightarrow$	a group of units in the field generated by the eigenvalues of all $a \in A$;
$\mathbb{Z}^b$	$\rightarrow$	a direct sum of copies of a subgroup of the integers of a number field.

2.2.1. *Preliminaries.* Let

$$\mathbb{Q} \subset F \overset{\iota}{\subset} E$$

be an inclusion of algebraic number fields. For a field extension E/F let the set of embeddings of E into $\mathbb{C}$ which fix F be denoted by $\mathbf{G}_{E/F}$; we adopt the convention $/\mathbb{Q}$ is omitted. Define vector spaces

$$\mathrm{W}_E = \sum_{\sigma \in \mathbf{G}_E} \mathbb{C}\sigma, \tag{2.1}$$

and

$$\mathrm{V}_E = \{x \in \mathrm{W}_E \ : \ x_{\bar\sigma} = \bar{x}_\sigma \ \forall \sigma \in \mathbf{G}_E \ \}, \tag{2.2}$$

where $^-$ denotes complex conjugation and $\bar\sigma$ is the embedding σ *followed* by complex conjugation. We also define

$$\mathrm{V}_{o,E} = \{x \in \mathrm{V}_E \ : \ \sum_{\sigma \in \mathbf{G}_E} x_\sigma = 0, \ \& \ x_\sigma = x_{\bar\sigma} \ \forall \sigma \in \mathbf{G}_E \ \}. \tag{2.3}$$

$\mathbf{G}_E$ is a basis of V_E which induces the dual basis $\mathbf{G}_E^*$ of V_E^*. An element in the dual basis shall be denoted by $\hat\sigma$ for $\sigma \in \mathbf{G}_E$. The basis and dual basis establish a linear isomorphism between V_E and V_E^* which shall be denoted by the circumflex operator, $\mathrm{V}_E \to \mathrm{V}_E^* : \ x \mapsto \hat{x}$, whose inverse is $\mathrm{V}_E^* \to \mathrm{V}_E : \ x \mapsto \check{x}$.

One obtains a basis of $\mathrm{V}_{o,E}^*$ as follows: note that $\hat{\mathrm{t}} = \frac{1}{|\mathbf{G}_E|} \sum_{\sigma \in \mathbf{G}_E} \hat\sigma$ and $\hat\sigma - \hat{\bar\sigma}$ vanish on $\mathrm{V}_{o,E}$ for all $\sigma \in \mathbf{G}_E$. If one defines $\mathbf{G}_E^r$ to be the set of real embeddings of E and $\mathbf{G}_E^c$ to be one-half of the non-real embeddings such that $\mathbf{G}_E^c$ is disjoint from its complex conjugate, then one observes that

$$\begin{aligned} \mathrm{V}_{o,E}^\perp &= \mathbb{R} \cdot \hat{\mathrm{t}} \oplus \sum_{\sigma \in \mathbf{G}_E^c} \mathbb{R} \cdot (\hat\sigma - \hat{\bar\sigma}), \\ \mathrm{V}_{o,E}^* &= \sum_{\sigma \in \mathbf{B}_F} \mathbb{R} \cdot \hat\sigma|_{\mathrm{V}_{o,E}} \end{aligned} \tag{2.4}$$

where $\mathbf{B}_E = \mathbf{G}_E^r \cup \mathbf{G}_E^c$.

The inclusion $F \overset{\iota}{\subset} E$ induces

$$\begin{array}{rcclcl} & \mathrm{V}_E \overset{\iota^*}{\twoheadrightarrow} \mathrm{V}_F & \quad \text{where} & \iota^*(\sigma) & = & \sigma|_F, \\ \text{and} & \mathrm{V}_E^* \overset{\iota}{\hookleftarrow} \mathrm{V}_F^* & \quad \text{where} & \iota(\hat\tau) & = & \displaystyle\sum_{\sigma \in \mathbf{G}_E, \sigma|_F = \tau} \hat\sigma. \end{array} \tag{2.5}$$

Finally, define a map $\mathrm{V}_E^* \overset{\alpha}{\twoheadrightarrow} \mathrm{V}_{o,F}^*$ by

$$\alpha = \jmath^* \, \hat\iota^*, \quad \hat\sigma \mapsto \hat\tau|_{\mathrm{V}_{o,F}} \text{ where } \tau = \sigma|_F, \tag{2.6}$$

$\jmath^*$ is the adjoint of the inclusion map $\mathrm{V}_{o,F} \overset{\jmath}{\subset} \mathrm{V}_F$ and $\hat{\iota}^* = \hat{\ }\iota^{*\check{}}$. This allows one to define a pairing between V_E^* and $\mathrm{V}_{o,F}$, denoted as follows

$$\begin{aligned} \langle \hat{\sigma}, x \rangle &:= \langle \alpha(\hat{\sigma}), x \rangle && \forall \sigma \in \mathbf{G}_E, x \in \mathrm{V}_{o,F}, \\ &= \langle \hat{\tau}, x \rangle && \text{where } \tau = \sigma|_F. \end{aligned} \tag{2.7}$$

Since $x = \sum_{\tau \in \mathbf{G}_F} x_\tau \cdot \tau$, it is apparent that

$$\langle \hat{\sigma}, x \rangle = x_{(\sigma|_F)} \qquad \forall \sigma \in \mathbf{G}_E, x \in \mathrm{V}_{o,F}, \tag{2.8}$$

so the notation is natural.

2.3. **An embedding of $\mathcal{O}_E$ in V_E.** Let $\mathcal{O}_E$ be the ring of integers of E, and let $\mathcal{U}_E$ be the group of multiplicative units of $\mathcal{O}_E$. Define a map $\eta : \mathcal{O}_E \to \mathrm{V}_E$ by

$$\eta(\alpha) := \sum_{\sigma \in \mathbf{G}_E} \sigma(\alpha) \cdot \sigma, \tag{2.9}$$

for each $\alpha \in \mathcal{O}_E$.

Lemma 2.1. *The map η is an embedding whose image—call it N_E—is a discrete, cocompact subgroup of V_E.*

Proof. This is standard. □

Let $\mathbf{T}_E = \mathrm{V}_E/\mathrm{N}_E$ be the resulting torus. $\mathbf{T}_E$ is equipped with a canonical affine structure from V_E and the group $\mathcal{U}_E$ acts by automorphisms of $\mathbf{T}_E$ defined by

$$u \cdot \mathsf{y} = \sum_{\sigma \in \mathbf{G}_E} \sigma(u) \cdot \mathsf{y}_\sigma \cdot \sigma \ + \ \mathrm{N}_E, \tag{2.10}$$

where $\mathsf{y} = \sum_{\sigma \in \mathbf{G}_E} \mathsf{y}_\sigma \cdot \sigma \ + \ \mathrm{N}_E$ is an element in $\mathbf{T}_E$ and $u \in \mathcal{U}_E$. The action in (2.10) is well-defined since N_E is mapped to itself by $\mathcal{U}_E$. *A fortiori*, equation (2.10) also defines an action of $\mathcal{U}_F \subset \mathcal{U}_E$ as an abelian group of automorphisms of $\mathbf{T}_E$.

2.4. **An embedding of $\mathcal{U}_F^+$ in $\mathrm{V}_{o,F}$.** Define a map $\ell : \mathcal{U}_F \to \mathrm{V}_F$ by

$$\ell(u) = \sum_{\sigma \in \mathbf{G}_F} \ln|\sigma(u)| \cdot \sigma. \tag{2.11}$$

Since $\bar{\sigma}$ is σ followed by complex conjugation, it is clear that $\mathfrak{L}_F =: \mathrm{im}\,\ell \subset \mathrm{V}_{o,F}$. Dirichlet's theorem on the group of units of an algebraic number field characterises the image of ℓ as a discrete, cocompact subgroup of $\mathrm{V}_{o,F}$, while $\ker \ell =: \mathcal{R}_F$ is the set of units all of whose conjugates lie on the unit circle. Stated otherwise, there is an unnatural splitting of $\mathcal{U}_F$ via a commutative diagram

$$\begin{tikzcd} \mathcal{R}_F \arrow[r, hook] \arrow[d, "="] & \mathcal{U}_F \arrow[r, two heads] \arrow[d, "="] & \mathcal{U}_F/\mathcal{R}_F \arrow[r, "\ell", "\cong"'] \arrow[d, "\cong"] & \mathfrak{L}_F, \arrow[d, "\cong"] \\ \mathcal{R}_F \arrow[r, hook] & \mathcal{R}_F \oplus \mathcal{U}_F^+ \arrow[r, two heads] & \mathcal{U}_F^+ \arrow[r, "\cong"] & \mathbb{Z}^{r+c-1}, \end{tikzcd}$$

where r (resp. $2c$) is the number of real (resp. non-real) embeddings of F. When F has a real embedding, which one may take to be the identity embedding $F \subset \mathbb{C}$, then $\mathcal{R}_F = \{\pm 1\}$ and $\mathcal{U}_F^+$ may be taken to be the multiplicative group of positive units in $\mathcal{U}_F$ — hence the notation. To summarise

Lemma 2.2. *The image of the map $\ell : \mathcal{U}_F^+ \to \mathrm{V}_{o,F}$ — call it $\mathfrak{L}_F$ — is a discrete, cocompact subgroup of $\mathrm{V}_{o,F}$ isomorphic to $\mathcal{U}_F^+$.*

2.5. **An action of $\mathcal{U}_F^+$ on $\mathbf{T}_E \times \mathrm{V}_{o,F}$.** For $\mathsf{y} \in \mathbf{T}_E$, $\mathsf{x} \in \mathrm{V}_{o,F}$ and $u \in \mathcal{U}_F^+$ define

$$u \cdot (\mathsf{y}, \mathsf{x}) := (u \cdot \mathsf{y}, \mathsf{x} + \ell(u)). \tag{2.12}$$

This action is clearly free and proper. Let Σ denote the compact manifold obtained by quotienting $\mathbf{T}_E \times \mathrm{V}_{o,F}$ by this action of $\mathcal{U}_F^+$.

Lemma 2.3. *There is a commutative diagram of natural maps*

$$\begin{tikzcd}
\mathrm{V}_E \times \mathrm{V}_{o,F} \arrow[r, two heads] \arrow[d, hook, "="] & \mathbf{T}_E \times \mathrm{V}_{o,F} \arrow[r, two heads] \arrow[d, hook, "="] & (\mathbf{T}_E \times \mathrm{V}_{o,F})/\mathcal{U}_F^+ \arrow[d, hook, "="] \\
\tilde{\Sigma} \arrow[r, two heads, "\tilde{\pi}"] & \hat{\Sigma} \arrow[r, two heads, "\hat{\pi}"] & \Sigma.
\end{tikzcd} \tag{2.13}$$

Therefore, $\pi_1(\Sigma)$ is naturally isomorphic to the semi-direct product $\Delta = \mathcal{U}_F^+ \star \mathcal{O}_E$, while there is a natural fibring of Σ by tori over a torus

$$\begin{tikzcd} \mathbf{T}_E \arrow[r, hook] & \Sigma \arrow[r, two heads, "\mathrm{p}"] & \mathbf{T}_{o,F}, \end{tikzcd} \tag{2.14}$$

where $\mathbf{T}_{o,F} = \mathrm{V}_{o,F}/\mathfrak{L}_F$.

Proof. Naturality of the construction implies the lemma. □

2.6. **The cotangent bundle $T^*\Sigma$.** The vector space structures on V_E and $\mathrm{V}_{o,F}$ give a tautological trivialisation of their cotangent bundles. Lemma 2.3 therefore implies that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{V}_E^* \times \mathrm{V}_E \times \mathrm{V}_{o,F}^* \times \mathrm{V}_{o,F} \arrow[r, two heads] \arrow[d, hook, "="] & \mathrm{V}_E^* \times \mathbf{T}_E \times \mathrm{V}_{o,F}^* \times \mathrm{V}_{o,F} \arrow[r, two heads] \arrow[d, hook, "="] & (\mathrm{V}_E^* \times \mathbf{T}_E \times \mathrm{V}_{o,F}^* \times \mathrm{V}_{o,F})/\mathcal{U}_F^+ \arrow[d, hook, "="] \\
T^*\tilde{\Sigma} \arrow[r, two heads, "\tilde{\Pi}"] & T^*\hat{\Sigma} \arrow[r, two heads, "\tilde{\Pi}"] & T^*\Sigma,
\end{tikzcd} \tag{2.15}$$

where $\hat{\Pi}$ is the covering map induced by $\hat{\pi}$, etc. Let us introduce coordinates on $T^*\hat{\Sigma}$ by

$$P \in T^*\hat{\Sigma} \iff P = (\mathsf{Y}, \mathsf{y} + \mathrm{N}_E, \mathsf{X}, \mathsf{x}) \in \mathrm{V}_E^* \times \mathbf{T}_E \times \mathrm{V}_{o,F}^* \times \mathrm{V}_{o,F}.$$

The action of $\mathcal{U}_F^+$ on $T^*\hat{\Sigma}$ is the natural lift of the action on Σ

$$u \cdot P = (u \cdot \mathsf{Y}, u \cdot \mathsf{y} + \mathrm{N}_E, \mathsf{X}, \mathsf{x} + \ell(u)) \tag{2.16}$$

where $u \cdot \mathsf{y}$ is defined in equation (2.10) and $u \cdot \mathsf{Y} = \sum_{\sigma \in \mathbf{G}_E} \mathsf{Y}_\sigma \cdot \sigma(u)^{-1} \cdot \hat{\sigma}$ is the induced contragredient action.

2.7. **Functions on $T^*\Sigma$.** The function $P \mapsto \mathsf{X}$ is $\mathcal{U}_F^+$-invariant, so one may view X as a submersion $T^*\Sigma \twoheadrightarrow \mathrm{V}_{o,F}^*$.

Fix a positive integer b_σ for each $\sigma \in \mathbf{G}_E$ and define the function

$$\gamma_\sigma(P) := \exp(b_\sigma \cdot \langle \hat{\sigma}, \mathsf{x} \rangle) \times |\mathsf{Y}_\sigma|^{b_\sigma} \tag{2.17}$$

where the pairing $\langle \hat{\sigma}, \mathsf{x} \rangle$ is defined in equation (2.7).

Lemma 2.4. *The function γ_σ is $\mathcal{U}_F$-invariant and it is real-analytic if b_σ is even.*

Proof. From equation (2.16), we know that for each $u \in \mathcal{U}_F$

$$\gamma_\sigma(u \cdot P) = \gamma_\sigma(P) \times \exp(b_\sigma \ln|\sigma(u)|) \times |\sigma(u)|^{-b_\sigma} = \gamma_\sigma(P). \tag{2.18}$$

It is clear that $\exp(b_\sigma \cdot \langle \hat{\sigma}, \mathsf{x} \rangle)$ is real-analytic, and $|\mathsf{Y}_\sigma|^{b_\sigma}$ is real-analytic if b_σ is a positive even integer. □

Remark 2.1. Fix even integers b_σ as in lemma 2.4. One may define a *momentum-like* map $\lambda : T^*\Sigma \to \mathrm{V}^*_{o,F} \oplus \mathrm{V}^*_E$ by

$$\lambda(P) = \mathsf{X} \oplus \sum_{\sigma \in \mathbf{G}_E} \gamma_\sigma(P) \cdot \hat{\sigma}, \qquad \forall P \in T^*\Sigma. \tag{2.19}$$

When the universal cover $\tilde{\Sigma}$ of Σ admits the structure of a solvable Lie group with Δ as a lattice subgroup, $\mathrm{V}^*_{o,F} \oplus \mathrm{V}^*_E$ – as the dual of a Lie algebra – admits a canonical Poisson structure. In this case, the map λ is left-invariant and Poisson and therefore mimics the properties of the classical momentum map.

3. Lax Representations

3.1. **Real split affine Lie algebras.** Let us briefly recall the construction underlying the Lax representation of periodic Bogoyavlenskij-Toda lattices. This discussion follows that in [27, 1, 2]. Let $\mathfrak{g}$ be a simple real Lie algebra with the real split Cartan sub-algebra $\mathfrak{h}$; $\mathfrak{g}$ is also known as the real normal form of the simple complex Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. The Cartan-Killing form of $\mathfrak{g}$ is denoted by $\langle\langle , \rangle\rangle$ when viewed as a bilinear form on $\mathfrak{g}$, and it is denoted by κ when viewed as a linear isomorphism of $\mathfrak{g}$ with $\mathfrak{g}^*$. Recall that $\langle\langle , \rangle\rangle$ is non-degenerate on $\mathfrak{h}$. As $\mathfrak{h}$ is a real split Cartan sub-algebra, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h} + \sum_{r \in \Psi_*} \mathfrak{g}_r \tag{3.1}$$

where $\Psi_* \subset \mathfrak{h}^*$ is the set of roots and $\mathfrak{g}_r$ is the root space associated with r, $\mathfrak{g}_r = \{x \in \mathfrak{g} \ : \ \mathrm{ad}_h x = \langle r, h\rangle \ x \quad \forall h \in \mathfrak{h}\}$. There is a set of simple roots $\Psi_0 \subset \Psi_*$ such that every root is an integer linear combination of the roots in Ψ_0 with entirely non-negative or non-positive coefficients. The height of a root is the sum of these coefficients; there is a unique root, η, of minimal height. Let Ψ be $\Psi_0 \cup \{\eta\}$.

Define $\mathcal{L}$ to be the set of Laurent polynomials in the variable λ with coefficients in $\mathfrak{g}$; $\mathcal{L}$ inherits an obvious Lie algebra structure. Let $\mathrm{d} = \lambda \frac{\partial}{\partial\lambda}$ be a derivation; define $[\mathrm{d}, x \cdot \lambda^n] = nx \cdot \lambda^n$ for all integers n and $x \in \mathfrak{g}$. Then $\hat{\mathfrak{g}} = \mathcal{L} + \mathbb{R} \cdot \mathrm{d}$ is a real split Lie algebra with Cartan sub-algebra $\hat{\mathfrak{h}} = \mathfrak{h} + \mathbb{R} \cdot \mathrm{d}$. The Cartan sub-algebra induces a weight-space decomposition of $\mathcal{L}$ as

$$\mathcal{L} = \mathfrak{h} + \sum_{\mathbf{r} \in \boldsymbol{\Psi}_*} \mathcal{L}_{\mathbf{r}} \tag{3.2}$$

where $\boldsymbol{\Psi}_* = \left\{ \mathbf{r} \in \hat{\mathfrak{h}}^* : \ \mathbf{r}|_{\mathfrak{h}} \in \Psi_* \cup \{0\}, \langle \mathbf{r}, \mathrm{d}\rangle \in \mathbb{Z}, \mathbf{r} \neq 0 \right\}$. The weight set $\boldsymbol{\Psi}_*$ has a basis of simple weights $\boldsymbol{\Psi} = \Psi \cup \{\boldsymbol{\eta}\}$, where $\boldsymbol{\eta}|_{\mathfrak{h}} = \eta$ and $\langle \boldsymbol{\eta}, \mathrm{d}\rangle = 1$. Each $\mathbf{r} \in \boldsymbol{\Psi}_*$ is an integer linear combination of roots in $\boldsymbol{\Psi}$. By defining the height of $\mathbf{r}$ as the sum of these coefficients one obtains the principal grading

$$\mathcal{L} = \sum_{n \in \mathbb{Z}} \mathcal{L}_n, \tag{3.3}$$

where $\mathcal{L}_0 = \mathfrak{h}$, $\mathcal{L}_n = \sum_{\mathrm{ht}(\mathbf{r})=n} \mathcal{L}_{\mathbf{r}}$ otherwise, and $[\mathcal{L}_n, \mathcal{L}_m] \subseteq \mathcal{L}_{m+n}$ for all m, n. It is observed that

$$\mathcal{L}_{\pm 1} = \mathfrak{g}_\eta \lambda^{\pm 1} + \sum_{r \in \Psi} \mathfrak{g}_{\pm r} = \sum_{\mathbf{r} \in \boldsymbol{\Psi}} \mathbb{R} \cdot \mathrm{e}_{\pm \mathbf{r}} \tag{3.4}$$

the same sign appearing throughout, and $\mathrm{e}_{\mathbf{r}}$ is a vector normalised so that $\kappa \cdot [\mathrm{e}_{\mathbf{r}}, \mathrm{e}_{-\mathbf{r}}] \in \Psi_0$. The sub-algebras $\mathcal{L}_+ = \sum_{n \geq 0} \mathcal{L}_n, \mathcal{L}_- = \sum_{n<0} \mathcal{L}_n$ permit the definition of a second Lie algebra structure on $\mathcal{L}$, defined by

$$[x, y]_R := [x_+, y_+] - [x_-, y_-] \tag{3.5}$$

for $x = x_+ + x_+, y = y_- + y_+ \in \mathcal{L}_- \oplus \mathcal{L}_+$. The Cartan-Killing form κ allows one to identify $\mathcal{L}^*_n = \mathcal{L}_{-n}$ for all n, in such a way that $\kappa(\mathrm{e}_{\mathbf{r}}) = \mathrm{e}_{-\mathbf{r}}$ or $\langle\langle \mathrm{e}_{\mathbf{r}}, \mathrm{e}_{-\mathbf{s}} \rangle\rangle = \delta_{\mathbf{r},\mathbf{s}}$.

Indeed, note that $\mathrm{e}_{\mathbf{r}} = \mathrm{e}_r \cdot \lambda^n$ where $r = \mathbf{r}|_{\mathfrak{h}}$ and $n = \langle \mathbf{r}, \mathrm{d}\rangle$ for all roots $\mathbf{r} \in \boldsymbol{\Psi}$, so it suffices to find a suitable basis of $\mathfrak{g}$ in order to define the vectors $\mathrm{e}_{\mathbf{r}}$. One also knows that

Lemma 3.1. *For each $\mu \in \mathcal{L}_{-1}^*$, the affine subspace $\mu + \mathcal{L}_0^* + \mathcal{L}_1^*$ is a Poisson subspace of $\mathcal{L}_R^*$. The Casimirs of $\mathcal{L}^*$ are in involution on $\mathcal{L}_R^*$.*

Proof. See references [27, 1]. □

3.2. **A second splitting.** Let ${}_0\mathcal{L} = \mathfrak{h} + \sum_{\mathbf{r}\in\boldsymbol{\Psi}_*} \mathbb{R}\cdot \mathrm{e}_{\mathbf{r}}$, a sub-algebra of the loop algebra $\mathcal{L}$ on which the Cartan-Killing form is non-degenerate. One can distinguish two sub-algebras ${}_0\mathcal{L}_\pm$ such that ${}_0\mathcal{L} = {}_0\mathcal{L}_- \oplus {}_0\mathcal{L}_+$ as a vector space:

$$\begin{aligned} &{}_0\mathcal{L}_- = \textstyle\sum_{\mathbf{r}\in\boldsymbol{\Psi}_+} \mathbb{R}\cdot(\mathrm{e}_{\mathbf{r}} - \mathrm{e}_{-\mathbf{r}}), \quad {}_0\mathcal{L}_+ = \mathfrak{h} + \textstyle\sum_{\mathbf{r}\in\boldsymbol{\Psi}_+} \mathbb{R}\cdot \mathrm{e}_{\mathbf{r}}, \quad \text{so} \\ &{}_0\mathcal{L}_-^* \equiv {}_0\mathcal{L}_+^\perp = \textstyle\sum_{\mathbf{r}\in\boldsymbol{\Psi}_+} \mathbb{R}\cdot \mathrm{e}_{\mathbf{r}}, \quad {}_0\mathcal{L}_+^* \equiv {}_0\mathcal{L}_-^\perp = \mathfrak{h} + \textstyle\sum_{\mathbf{r}\in\boldsymbol{\Psi}_+} \mathbb{R}\cdot(\mathrm{e}_{\mathbf{r}} + \mathrm{e}_{-\mathbf{r}}), \end{aligned} \tag{3.6}$$

where $\boldsymbol{\Psi}_+ \subset \boldsymbol{\Psi}_*$ is the set of positive roots. One can define a grading on both ${}_0\mathcal{L}_\pm$ by defining the height of a root $\mathbf{r} \in \boldsymbol{\Psi}_+$ to be $\mathrm{ht}(\mathbf{r}) = \mathrm{ht}(r) + (1+k)\langle \mathbf{r}, \mathrm{d}\rangle$ where k is the height of the maximal root of $\mathfrak{g}$. With this grading, a basis of ${}_0\mathcal{L}_{+1}$ (resp. ${}_0\mathcal{L}_{-1}$) is $\{\mathrm{e}_{\mathbf{r}} : \mathbf{r} \in \boldsymbol{\Psi}\}$ (resp. $\{\mathrm{e}_{\mathbf{r}} - \mathrm{e}_{-\mathbf{r}} : \mathbf{r} \in \boldsymbol{\Psi}\}$) while a basis of ${}_0\mathcal{L}_{+1}^*$ (resp. ${}_0\mathcal{L}_{-1}^*$) is $\{\mathrm{e}_{\mathbf{r}} + \mathrm{e}_{-\mathbf{r}} : \mathbf{r} \in \boldsymbol{\Psi}\}$ (resp. $\{\mathrm{e}_{-\mathbf{r}} : \mathbf{r} \in \boldsymbol{\Psi}\}$). One therefore knows that ${}_0\mathcal{L}$ admits an R-bracket analogous to that defined in (3.5) and that Lemma 3.1 also holds for ${}_0\mathcal{L}_R$.

Remark 3.1. If α is an automorphism of the graded Lie algebra $\mathcal{L}$ that fixes $\mathfrak{h}$, then the fixed point set of α is a sub-algebra that inherits a grading, splitting and a root space decomposition from $\mathcal{L}$. The constructions of both subsections 3.1 and 3.2 are applicable in this case, too. The automorphism α satisfies $\alpha(x\cdot\lambda^n) = \alpha(x)\cdot(\epsilon\lambda)^n$ for all $x \in \mathfrak{g}$ and n, where ϵ is a primitive order(α) root of unity. This construction yields the so-called twisted loop algebras. The twisted loop algebra is traditionally denoted by $\mathfrak{g}^{(m)}$ where m is the order of the automorphism α; when $m = 1$, one has the usual loop algebra $\mathcal{L}$.

3.3. **Examples.** Let $\mathfrak{g} = A_2 = sl(3;\mathbb{R})$. For $\mathfrak{h}$ one can take the sub-algebra of trace zero diagonal matrices and for the basis of positive roots of $\mathfrak{g}$ one can take the roots r_1 and r_2 with the minimal root η:

$$r_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad r_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \eta = -r_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.7}$$

which satisfy the linear relation $r_1 + r_2 + \eta = 0$. A root $\mathbf{r} \in \boldsymbol{\Psi}$ may be written formally as $\mathbf{r} = \pm r_i + n$ where $n = \langle \mathbf{r}, \mathrm{d}\rangle$. The height of $\mathbf{r}$ is then computed to be $3n \pm 1$ for $i = 1, 2$ and $3n \pm 2$ for $i = 3$. From this, one can see that the graded pieces of $\mathcal{L}$, as in (3.3), are $\mathcal{L}_0 = \mathfrak{h}$ and

$$\begin{aligned} &\mathcal{L}_{+1} = \left\{ \begin{bmatrix} \lambda a_1 & \alpha_1 & 0 \\ 0 & \lambda a_2 & \alpha_2 \\ \lambda\alpha_3 & 0 & \lambda a_3 \end{bmatrix} \right\}, \qquad \mathcal{L}_{+2} = \left\{ \begin{bmatrix} \lambda^2 a_1 & 0 & \alpha_3 \\ \lambda\alpha_1 & \lambda^2 a_2 & 0 \\ 0 & \lambda\alpha_2 & \lambda^2 a_3 \end{bmatrix} \right\}, \\ &\mathcal{L}_{-1} = \left\{ \begin{bmatrix} \lambda^{-1} a_1 & 0 & \lambda^{-1}\alpha_3 \\ \alpha_1 & \lambda^{-1} a_2 & 0 \\ 0 & \alpha_2 & \lambda^{-1} a_3 \end{bmatrix} \right\}, \quad \mathcal{L}_{-2} = \left\{ \begin{bmatrix} \lambda^{-2} a_1 & \lambda^{-1}\alpha_1 & 0 \\ 0 & \lambda^{-2} a_2 & \lambda^{-1}\alpha_2 \\ \alpha_3 & 0 & \lambda^{-2} a_3 \end{bmatrix} \right\}, \end{aligned} \tag{3.8}$$

where a_i, α_i are real numbers and $\sum a_i = 0$.

The splitting in (3.6) of ${}_0\mathcal{L}$ implies that the root spaces of height ± 1 and their duals are

$$
\begin{aligned}
{}_0\mathcal{L}_{-1} &= \left\{ \begin{bmatrix} 0 & \alpha_1 & -\alpha_3\lambda^{-1} \\ -\alpha_1 & 0 & \alpha_2 \\ \alpha_3\lambda & -\alpha_2 & 0 \end{bmatrix} \right\}, &
{}_0\mathcal{L}^*_{-1} &= \left\{ \begin{bmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \\ \alpha_3\lambda & 0 & 0 \end{bmatrix} \right\}, \\
{}_0\mathcal{L}_{+1} &= \left\{ \begin{bmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \\ \alpha_3\lambda & 0 & 0 \end{bmatrix} \right\}, &
{}_0\mathcal{L}^*_{+1} &= \left\{ \begin{bmatrix} 0 & \alpha_1 & \alpha_3\lambda^{-1} \\ \alpha_1 & 0 & \alpha_2 \\ \alpha_3\lambda & \alpha_2 & 0 \end{bmatrix} \right\}, \qquad (3.9)
\end{aligned}
$$

where the α_i are real.

3.4. **Bijections.** Assume that $F/\mathbb{Q}$ is an algebraic field of degree m with r real embeddings and $2c$ non-real embeddings such that $r + c = n$, and that $\mathfrak{g}$ is a real split affine Lie algebra of rank $n-1$. Since $\mathbf{B}_F$, the set of real embeddings of F plus one-half the set of complex embeddings of F, has n elements and the simple roots of $\mathcal{L}$, $\mathbf{\Psi}$, have n elements, the sets are isomorphic.

Definition 3. *Let $\mathfrak{B}$ be the set of bijections $\mathbf{B}_F \to \mathbf{\Psi}$.*

Each $\rho \in \mathfrak{B}$ can be extended to a map $\mathbf{G}_F \to \mathbf{\Psi}$ by $\rho(\bar\sigma) := \rho(\sigma)$ for all $\sigma \in \mathbf{B}_F$. This extension shall be understood throughout.

Additionally, each $\rho \in \mathfrak{B}$ naturally induces a linear isomorphism $\phi = \phi_\rho : \mathrm{V}^*_{o,F} \to \mathfrak{h}^*$. To define ϕ let us recall two things. First, note that the projection $\hat{\mathfrak{h}}^* \twoheadrightarrow \mathfrak{h}^*$ that is dual to the inclusion $\mathfrak{h} \hookrightarrow \hat{\mathfrak{h}}$ induces the bijection $\mathbf{\Psi} \cong \Psi : \mathbf{r} \mapsto r = \mathbf{r}|_{\mathfrak{h}}$. Second, there are unique positive integers ω_r such that

$$\sum_{r \in \Psi} \omega_r r = 0, \qquad \gcd(\omega_r \ : \ r \in \Psi) = 1. \tag{3.10}$$

For each $\tau \in \mathbf{B}_F$, define n_τ to be 1 if τ is a real embedding; and 2 if not. Then, define

$$\phi(\hat\tau|_{\mathrm{V}_{o,F}}) = n_\tau^{-1}\, \omega_r r \qquad \text{where } \mathbf{r} = \rho(\tau). \tag{3.11}$$

Since $\hat\tau$ equals $\hat{\bar\tau}$ when restricted to $\mathrm{V}_{o,F}$, the sole linear dependence relation amongst the set $\left\{ \hat\tau|_{\mathrm{V}_{o,F}} \ : \ \tau \in \mathbf{B}_F \right\}$ is the relation

$$\sum_{\tau \in \mathbf{B}_F} n_\tau\, \hat\tau|_{\mathrm{V}_{o,F}} = \sum_{\tau \in \mathbf{G}_F} \hat\tau|_{\mathrm{V}_{o,F}} = 0.$$

Thus, equation (3.10) implies that ϕ extends to a linear isomorphism.

3.5. **Lax representations.** Fix $\rho \in \mathfrak{B}$ and let $\phi = \phi_\rho$ be the induced linear isomorphism. Let $\Phi : \mathrm{V}^*_{o,F} \to \mathfrak{h}^*$ be a linear map and let $g_\pm : \mathrm{V}^*_E \times \mathbf{G}_E \to \mathcal{L}^*_\pm$ be smooth maps. Define a map $\mathbf{L} = \mathbf{L}_{\rho,\Phi} : T^*\tilde\Sigma \to \mathcal{L}^*$ by

$$\kappa^{-1} \cdot \mathbf{L}(P) = \sum_{\sigma \in \mathbf{G}_E} g_{-,\sigma}(\mathsf{Y}) \cdot \mathrm{e}_{\mathbf{r}} + \Phi(\mathsf{X}) + \sum_{\sigma \in \mathbf{G}_E} g_{+,\sigma}(\mathsf{Y}) \cdot \exp(b_\sigma \cdot \langle \hat\sigma, \mathsf{x} \rangle) \cdot \mathrm{e}_{-\mathbf{r}} \tag{3.12}$$

where it is understood that $\mathbf{r} = \rho(\sigma|_F)$ in the sums and $\mathcal{L}^*$ is identified with $\mathcal{L}$ via the Cartan-Killing form κ.

There are several choices of Lax representation that are useful. The first is (in all cases, $\mathbf{r} = \rho(\sigma|_F)$ is understood)

$$\kappa^{-1} \cdot \mathbf{L}(P) = \sum_{\sigma \in \mathbf{G}_E} |\mathsf{Y}_\sigma|^{b_\sigma} \cdot \mathrm{e}_{\mathbf{r}} + \Phi(\mathsf{X}) + \frac{1}{2} \times \sum_{\sigma \in \mathbf{G}_E} \exp(b_\sigma \cdot \langle \hat\sigma, \mathsf{x} \rangle) \cdot \mathrm{e}_{-\mathbf{r}}, \tag{3.13}$$

while the second is

$$\kappa^{-1} \cdot \mathbf{L}(P) = \sum_{\sigma \in \mathbf{G}_E} \mathrm{e}_{\mathbf{r}} + \Phi(\mathsf{X}) + \frac{1}{2} \times \sum_{\sigma \in \mathbf{G}_E} |\mathsf{Y}_\sigma|^{b_\sigma} \cdot \exp(b_\sigma \cdot \langle \hat\sigma, \mathsf{x} \rangle) \cdot \mathrm{e}_{-\mathbf{r}}, \tag{3.14}$$

and a third is

$$\kappa^{-1} \cdot \mathbf{L}(P) = \Phi(\mathsf{X}) \; + \; \frac{1}{\sqrt{2}} \times \sum_{\sigma \in \mathbf{G}_E} |\mathsf{Y}_\sigma|^{\frac{1}{2}b_\sigma} \cdot \exp(\frac{1}{2} b_\sigma \cdot \langle \hat{\sigma}, \mathsf{x} \rangle)(\mathrm{e}_{\mathbf{r}} + \mathrm{e}_{-\mathbf{r}}). \tag{3.15}$$

Note that the Lax representations in (3.13–3.14) are related to the splitting of the loop algebra in section 3.1; the final Lax representation in (3.15) is related to the splitting in section 3.2. In all cases, the pullback of the Casimir $x \mapsto \frac{1}{2} \times \kappa(x, x)$ on $\mathcal{L}^*$ by any of the three Lax matrices in (3.13–3.15) $\mathbf{L}$ is equal to

$$\mathbf{H} := \frac{1}{2} \times \langle \mathcal{Q} \cdot \mathsf{X}, \mathsf{X} \rangle + \frac{1}{2} \sum_{\sigma \in \mathbf{G}_E} |\mathsf{Y}_\sigma|^{b_\sigma} \cdot \exp(b_\sigma \cdot \langle \hat{\sigma}, \mathsf{x} \rangle), \tag{3.16}$$

where $\mathcal{Q} : \mathrm{V}^*_{o,F} \to \mathrm{V}_{o,F}$ is defined by $\mathcal{Q} = \Phi^* \kappa \Phi$. This Hamiltonian is fibre-wise quadratic — hence, induced by a riemannian metric — iff $b_\sigma = 2$ for all σ; in all cases, it is fibre-wise convex. The next theorem implies that there are constraints on F if $\mathbf{H}$ is fibre-wise quadratic.

As a second step, recall that $\mathrm{sl}_2\mathbb{R}$ has a basis $h, \mathrm{e}_+, \mathrm{e}_-$ such that $[h, \mathrm{e}_\pm] = \pm \mathrm{e}_\pm$, and $[\mathrm{e}_+, \mathrm{e}_-] = 2h$. The Cartan-Killing form identifies the dual basis as $h, \mathrm{e}_-, \mathrm{e}_+$. For each $\sigma \in \mathbf{G}_E$, let $\mathrm{sl}_2\mathbb{R}_\sigma$ be a copy of $\mathrm{sl}_2\mathbb{R}$ and let $h_\sigma, \mathrm{e}_{\pm,\sigma}$ be copies of $h, \mathrm{e}_\pm$. Define $\mathbf{L}_1 : T^*\tilde{\Sigma} \to \mathfrak{g}_1^*$, $\mathfrak{g}_1 = \sum_{\sigma \in \mathbf{G}_E} \mathrm{sl}_2\mathbb{R}_\sigma$ by

$$\mathbf{L}_1(P) = \sum_{\sigma \in \mathbf{G}_E} \mathsf{Y}_\sigma \cdot h_\sigma + \exp(\langle \hat{\sigma}, \mathsf{y} \rangle) \cdot \mathrm{e}_{-,\sigma}. \tag{3.17}$$

Theorem 3.2. $\mathbf{L}_1$ *is a Poisson map.* $\mathbf{L} = \mathbf{L}_{\rho,\Phi} : T^*\tilde{\Sigma} \to \mathcal{L}_R^*$ *is a Poisson map iff there is a* $c \in \frac{1}{2}\mathbb{Z}^+$ *such that:*

(1) *for all* $\sigma \in \mathbf{G}_E$ *and* $\mathbf{r} \in \mathbf{\Psi}$ *with* $\rho(\sigma|_F) = \mathbf{r}$*, one has* $n^{-1}_{(\sigma|_F)} b_\sigma \omega_r = c$*; and*
(2) $\Phi = c^{-1} \times \phi_\rho$*.*

The map $\mathbf{L}_2 = \mathbf{L} + \mathbf{L}_1 : T^*\tilde{\Sigma} \to \mathcal{L}^* + \mathfrak{g}_1^*$ *is a Poisson embedding if either* g_+ *or* g_- *is an embedding and* $E = F$*.*

Proof. The proof shall assume that $\mathbf{L}$ is defined by equation (3.13); the remaining cases are not significantly different. To prove that $\mathbf{L}_1$ is a Poisson map, one needs to prove that

$$\{f \circ \mathbf{L}_1, g \circ \mathbf{L}_1\}_{T^*\tilde{\Sigma}} = \{f, g\}_{\mathfrak{h}_1^*} \circ \mathbf{L}_1, \tag{3.18}$$

for all smooth functions f, g on $\mathfrak{h}_1^*$. It suffices to verify equation (3.18) holds for linear functions f, g, for a single copy of $\mathrm{sl}_2\mathbb{R}$, and a single pair of conjugate variables Y and y. For $f = h$ and $g = \mathrm{e}_+$ one sees that

$$\{h, \mathrm{e}_+\}_{\mathrm{sl}_2\mathbb{R}^*} \circ \mathbf{L}_1 = -\langle \langle \mathbf{L}_1, [h, \mathrm{e}_+] \rangle \rangle = -e^{\mathsf{y}}, \tag{3.19}$$

while

$$\{h \circ \mathbf{L}_1, \mathrm{e}_+ \circ \mathbf{L}_1\}_{T^*\tilde{\Sigma}} = \{\mathsf{Y}, e^{\mathsf{y}}\} = -e^{\mathsf{y}}. \tag{3.20}$$

Since h and e_+ are functionally independent at almost all points on almost all co-adjoint orbits, this proves that $\mathbf{L}_1$ is a Poisson map.

To prove the claim concerning $\mathbf{L}$, one needs to prove that

$$\{f \circ \mathbf{L}, g \circ \mathbf{L}\}_{T^*\tilde{\Sigma}} = \{f, g\}_{\mathcal{L}_R^*} \circ \mathbf{L}, \tag{3.21}$$

for all $f, g \in C^\infty(\mathcal{L}^*)$. As above, it suffices to verify equation (3.21) holds for all $f, g \in \mathcal{L}_R$. Given the bracket relations on $\mathcal{L}_R$, it suffices to prove the equation for all $f, g \in \mathcal{L}_{-1} + \mathcal{L}_0 + \mathcal{L}_{+1}$. Let us break this into cases:

(1) If $f, g \in \mathcal{L}_0$ or $f \in \mathcal{L}_{-1}$ and $g \in \mathcal{L}_{+1}$ or $f \in \mathcal{L}_0$ and $g \in \mathcal{L}_{-1}$, then $[f, g]_R = 0$ so $\{f, g\}_{\mathcal{L}_R^*} \circ \mathbf{L} = 0$. On the other hand, $\{f \circ \mathbf{L}, g \circ \mathbf{L}\}_{T^*\tilde{\Sigma}} = 0$ since functions of X alone, or a function of Y and a function of x, or a function of Y and a function of X alone Poisson commute.

(2) If $f, g \in \mathcal{L}_{-1}$ or $f, g \in \mathcal{L}_{+1}$, then $[f, g]_R \in \mathcal{L}_{\pm 2}$. Therefore,
$$\{f, g\}_{\mathcal{L}_R^*} \circ \mathbf{L} = -\langle\langle \mathbf{L}, [f, g]_R \rangle\rangle = 0 \tag{3.22}$$
since $\mathbf{L}$ lies in $\mathcal{L}_{-1} + \mathcal{L}_0 + \mathcal{L}_{+1}$. On the other hand, $f \circ \mathbf{L}$ and $g \circ \mathbf{L}$ are either functions of Y or x alone. In either case, they Poisson commute on $T^*\tilde{\Sigma}$.

(3) If $f \in \mathcal{L}_0$ and $g \in \mathcal{L}_{+1}$, then it suffices to assume that $g = \mathrm{e}_{\mathbf{r}}$ for some $\mathbf{r} \in \mathbf{\Psi}$. In this case,
$$\begin{aligned} \{f, g\}_{\mathcal{L}_R^*} \circ \mathbf{L} &= -\langle \mathbf{L}, [f, \mathrm{e}_{\mathbf{r}}]_R \rangle = -\langle \mathbf{r}, f \rangle \times \langle\langle \mathbf{L}, \mathrm{e}_{\mathbf{r}} \rangle\rangle \\ &= -\langle r, f \rangle \times \sum_{\sigma \in \mathbf{G}_E \text{ s.t. } \rho(\sigma|_F) = \mathbf{r}} \exp(b_\sigma \langle \hat{\sigma}, \mathsf{x} \rangle). \end{aligned} \tag{3.23}$$

On the other hand,
$$f \circ \mathbf{L} = \langle \mathsf{X}, \Phi^* f \rangle, \tag{3.24}$$
$$g \circ \mathbf{L} = \sum_{\sigma \in \mathbf{G}_E \text{ s.t. } \rho(\sigma|_F) = \mathbf{r}} \exp(b_\sigma \langle \hat{\sigma}, \mathsf{x} \rangle), \tag{3.25}$$
so the Poisson bracket of these functions is
$$\{f \circ \mathbf{L}, g \circ \mathbf{L}\}_{T^*\tilde{\Sigma}} = -\sum_{\sigma \in \mathbf{G}_E \text{ s.t. } \rho(\sigma|_F) = \mathbf{r}} \exp(b_\sigma \langle \hat{\sigma}, \mathsf{x} \rangle) \times b_\sigma \langle \hat{\sigma}, \Phi^* f \rangle \tag{3.26}$$
Because ρ is a bijection of $\mathbf{B}_F$ with $\mathbf{\Psi}$, there is a unique $\tau \in \mathbf{B}_F$ such that $\rho(\tau) = \mathbf{r}$. Therefore, due to the way that ρ is extended to $\mathbf{G}_F$, the σ's involved in the above summations all satisfy $\sigma|_F = \tau$ or $\bar{\tau}$. Therefore, $\langle \hat{\sigma}, \mathsf{x} \rangle = \langle \hat{\tau}, \mathsf{x} \rangle$ for all $\sigma \in \mathbf{G}_E$ such that $\rho(\sigma|_F) = \mathbf{r}$.

This fact about the σ's also implies that $\langle \hat{\sigma}, \Phi^* f \rangle = \left\langle \Phi(\hat{\tau}|_{\mathrm{V}_{o,F}}), f \right\rangle$. Since $\hat{\tau}$ will only appear when it acts on $\mathrm{V}_{o,F}$, the notation $|_{\mathrm{V}_{o,F}}$ will be suppressed.

Therefore, if the right-hand sides of equations (3.23) and (3.26) are equated for all $f \in \mathcal{L}_0$, then one concludes that
$$\sum_{\sigma \in \mathbf{G}_E \text{ s.t. } \rho(\sigma|_F) = \mathbf{r}} \exp(b_\sigma \langle \hat{\tau}, \mathsf{x} \rangle) \times [b_\sigma \Phi(\hat{\tau}) - r] = 0. \tag{3.27}$$
The functions $u \mapsto e^{au}, e^{bu}$ are linearly independent if $a \neq b$. If the b_σ's in the above sum are not constant, then the sum in equation (3.27) contains two linearly independent functions. Therefore, the coefficients on these two functions must vanish. But this forces $\Phi(\hat{\tau})$ to equal two different multiples of r. Absurd. Therefore, the b_σ's in the sum must be constant. This implies that b_σ is determined by $\mathbf{r}$ alone, or equivalently, by τ alone.

As cases (1—3) are the only independent cases to be considered, one concludes that if $\mathbf{L}$ is a Poisson map, then there are integers b_τ, $\tau \in \mathbf{B}_F$, such that the integers b_σ, $\sigma \in \mathbf{G}_E$, satisfy
$$b_\sigma = b_\tau \qquad \text{where } \sigma|_F = \tau \text{ or } \bar{\tau}. \tag{3.28}$$
Moreover, equation (3.27) implies that if $\tau \in \mathbf{B}_F$ and $\rho(\tau) = \mathbf{r}$, then
$$\Phi(\hat{\tau}) = b_\tau^{-1} r. \tag{3.29}$$

Summing over $\tau \in \mathbf{G}_F$, and using the fact that ρ is a bijection of $\mathbf{B}_F$ and $\mathbf{\Psi}$,

$$\sum_{\tau\in\mathbf{G}_F} \Phi(\hat{\tau}) = \sum_{\tau\in\mathbf{B}_F} \Phi(n_\tau\,\hat{\tau}) = \sum_{r\in\Psi} n_\tau\, b_\tau^{-1} r, \qquad \text{(where } \rho(\tau) = \mathbf{r}\text{)}. \tag{3.30}$$

The left-hand side vanishes because $\sum_{\tau\in\mathbf{G}_F} \hat{\tau}|_{\mathrm{V}_{o,F}} = 0$. Therefore

$$\sum_{r\in\Psi} n_\tau\, b_\tau^{-1} r = 0, \tag{3.31}$$

while the unique linear dependence relation in equation (3.10) implies that there must be a constant c such that $n_\tau^{-1} b_\tau \omega_r = c$ for all $r \in \Psi$. The constant c is a positive integer, or one-half such, since b_τ and ω_r are positive integers and $n_\tau = 1$ or 2. This implies part (1) of the Theorem.

The equation that Φ must satisfy is, for all $\tau \in \mathbf{B}_F$,

$$\Phi(\hat{\tau}) = \frac{1}{cn_\tau} \times \omega_r r \qquad \text{where } \mathbf{r} = \rho(\tau). \tag{3.32}$$

Comparison with equation (3.11) shows that $\Phi = c^{-1} \times \phi_\rho$, which is part (2) of the Theorem.

The claim that $\mathbf{L}_2 = \mathbf{L} + \mathbf{L}_1$ is an embedding is obvious.

□

Remark 3.2. Theorem (3.2) exploits the naturality of the constructions. In cases where the group A is not of finite index in $\mathcal{U}_F$, one encounters the problem that there is no obvious Lax map. This is what makes question D difficult.

Remark 3.3. Condition (1) implies that b_σ depends only on $\sigma|_F$. Condition (2) implies that $2c$ is divisible by all ω_r, hence by their lcm, ω. Therefore, there is a unique choice of Φ and integers b_σ if one insists that c be as small as possible and the b_σ be even. In case F is totally real, condition (1) implies that c is divisible by $\omega = \mathrm{lcm}(\omega_r : r \in \Psi)$. When c is chosen to be 2ω – so that $b_\sigma = 2\omega/\omega_r$ is even –, condition (2) implies that $\Phi(\hat{\sigma}|_F) = \frac{1}{2} w_r r$, where $w_r = \omega_r/\omega$, and $\rho(\sigma|_F) = \mathbf{r}$. This condition is stated in [11, Lemma 7], except for the factor of $\frac{1}{2}$ in Φ.[3] This discrepancy is due to the choice of a slightly different Poisson structure in [11, equation (9-10)]. With these choices, the Hamiltonian $\mathbf{H}$ in equation (3.16) is equal to that in [11, equation 15] when $E = F$ is totally real. In particular (*c.f.* equation 3.16),

$$\mathcal{Q} = \phi_\rho^* \cdot \kappa \cdot \phi_\rho. \tag{3.33}$$

In the event that F has no real embeddings, this smallest choice is $c = \omega$ and $b_\sigma = 2\omega/\omega_r$; in other events, the solution is somewhat more involved to state, as it depends on the bijection ρ (see tables 2–4 in section 4.1.3 for examples). It is possible to state

Proposition 3.1. *Let* r *be the number of real embeddings of* F*, and* $2\mathsf{c}$ *the number of non-real embeddings. If the Hamiltonian* $\mathbf{H}$ *is fibre-wise quadratic, then table* (1) *is true. In particular, if* $\mathsf{r} > 4$ *and* $\mathsf{c} > 0$*, then none of the Hamiltonians* $\mathbf{H}$ *are fibre-wise quadratic.*

Proof. From equation (3.16), it is clear that $\mathbf{H}$ is fibre-wise quadratic iff $b_\sigma = 2$ for all σ. Since, for all roots r, $\omega_r = n_\tau c/2$ where $\tau = \sigma|_F$ and $\rho(\tau) = \mathbf{r}$, one has

$\tau \in \mathbf{G}_F^c$:	$\tau \in \mathbf{G}_F^r$:
$\omega_r = c$	$\omega_r = c/2.$

[3] The coefficients b_σ in [11] are one-half those in the present paper.

This shows that each weight is 1 or 2. If $\mathsf{c} = 0$, then $c = 2$ and $\omega_r = 1$ for all roots. If $\mathsf{r} = 0$, then $c = 1$ and $\omega_r = 1$ for all roots. If $\mathsf{r}, \mathsf{c} > 0$, then $c = 2$ and $\mathbf{\Psi}$ has r roots with weight 1 and c roots with weight 2. Inspection of the root systems in figures (8–9) completes the proof. □

3.6. **Quotients of the Lax Representations.**

Lemma 3.3. *There is a natural action of* $\Delta = \mathcal{U}_F^+ \star \mathcal{O}_F$*—which factors through* $\mathcal{U}_F^+$*—on* $\mathcal{L}_R^*$ *such that the map defined in equation (3.13) is* Δ*-equivariant, hence induces a Poisson map*

$$\begin{array}{ccc} T^*\tilde{\Sigma} & \xrightarrow{\mathbf{L}} & \mathcal{L}_R^* \\ \Big\downarrow{\scriptstyle\tilde{\Pi}} & & \Big\downarrow \\ T^*\Sigma & \xrightarrow{\mathbf{L}} & \mathcal{L}_R^*/\mathcal{U}_F^+. \end{array}$$

The action of $\mathcal{U}_F^+$ *on* $\operatorname{im}\mathbf{L} \subset \mathcal{L}_R^*$ *is free and proper.*

Proof. Define the action of $g = (u, \alpha) \in \mathcal{U}_F^+ \star \mathcal{O}_F$ on $\mathcal{L}_R^*$ by

$$g \cdot \mathrm{e}_{\mathbf{r}} = \begin{cases} |\sigma(u)|^{-b_\sigma} \cdot \mathrm{e}_{\mathbf{r}} & \text{if} \quad \mathbf{r} \in \mathbf{\Psi}, \rho(\sigma) = \mathbf{r} \\ |\sigma(u)|^{b_\sigma} \cdot \mathrm{e}_{\mathbf{r}} & \text{if} \quad -\mathbf{r} \in \mathbf{\Psi}, \rho(\sigma) = \mathbf{r} \\ \mathrm{e}_r & \qquad\quad \text{otherwise,} \end{cases} \tag{3.34}$$

and $g|_{\mathcal{L}_0} = 1$. It is straightforward to see that $\mathbf{L}(g \cdot P) = g \cdot \mathbf{L}(P)$ for all g and $P \in T^*\tilde{\Sigma}$.

Since the coefficients of $\mathrm{e}_{\mathbf{r}}$, $-\mathbf{r} \in \mathbf{\Psi}$, do not vanish on $\operatorname{im}\mathbf{L}$, one sees that the action of $\mathcal{U}_F^+$ is semi-conjugate to its action on $\mathrm{V}_{o,F}$. Hence, it is free and proper. □

Remark 3.4. The preceding lemma implies that $\mathbf{H}$ and all the "spectral" integrals of $\mathbf{H}$ descend, but with some additional work. The alternative Lax matrix, equation (3.14), gives us a simple proof of this fact.

Lemma 3.4. *The map defined in equation (3.14) is* $\mathcal{U}_F^+ \star \mathcal{O}_F$*-invariant, hence it induces a Poisson map*

$$\begin{tikzcd} T^*\tilde{\Sigma} \arrow[r, "\mathbf{L}"] \arrow[d, "\tilde{\Pi}"] & \mathcal{L}_R^* \\ T^*\Sigma. \arrow[ur, "\mathbf{L}"] & \end{tikzcd}$$

Consequently, if h *is a Casimir of* $\mathcal{L}^*$*, then* $h \circ \mathbf{L}$ *Poisson commutes with* $\mathbf{H}$.

Proof. By equation (2.17), one can write

$$\mathbf{L}(P) = \sum_{\sigma \in \mathbf{G}_E} \mathrm{e}_{\mathbf{r}} \; + \; \Phi(\mathsf{X}) \; + \; \frac{1}{2} \times \sum_{\sigma \in \mathbf{G}_E} \gamma_\sigma \cdot \mathrm{e}_{-\mathbf{r}}. \tag{3.35}$$

By Lemma 2.4, each function γ_σ is $\mathcal{U}_F^+$-invariant, hence $\mathcal{U}_F^+ \star \mathcal{O}_F$-invariant. □

3.7. **Additional Integrals.** A consequence of Theorem 3.2 is that the function $P \mapsto \mathsf{Y} \,:\, T^*\hat{\Sigma} \to \mathrm{V}_E^*$ is a first integral of any function on $\mathcal{L}_R^*$ pulled-back to $T^*\hat{\Sigma}$ by the Lax matrix $\mathbf{L}$ (equation (3.13)). Unfortunately, this map is not $\mathcal{U}_F^+$-invariant. However, one is able to construct a map, $\mathfrak{f}$, from Y which is $\mathcal{U}_F^+$-invariant. Naively, one might try to define $\mathfrak{f}$ by means of equivariance. That is the task of this section.

For each $\tau \in \mathbf{G}_F$, let the subspace of V_E spanned by $\{\sigma \,:\, \sigma|_F = \tau\}$ be denoted by $\mathrm{V}_{\tau,E}$. One may define

$$\mathsf{Y}_\tau = \sum_{\sigma|_F = \tau} \mathsf{Y}_\sigma \cdot \hat{\sigma}, \tag{3.36}$$

for all $\tau \in \mathbf{G}_F$. Since $\mathsf{Y}_{\bar\sigma} = \bar{\mathsf{Y}}_\sigma$ for all σ, it is clear that complex conjugation induces a real linear isomorphism between $\mathrm{V}^*_{\tau,E}$ and $\mathrm{V}^*_{\bar\tau,E}$. This linear isomorphism maps $\mathsf{Y}_\tau \to \mathsf{Y}_{\bar\tau}$, which implies that as real vector spaces

$$\mathrm{V}^*_E \cong \sum_{\tau\in\mathbf{B}_F} \mathrm{V}^*_{\tau,E}. \tag{3.37}$$

In the sequel, this natural isomorphism is understood.

The group $\mathcal{U}^+_F$ acts on V^*_E by $u\cdot\hat\sigma = \sigma(u)^{-1}\cdot\hat\sigma$. Since $u \in F$, this action is (*c.f.* equation 2.16)

$$u\cdot\mathsf{Y} = \sum_{\tau\in\mathbf{B}_F} \tau(u)^{-1}\cdot\mathsf{Y}_\tau. \tag{3.38}$$

Lemma 3.5. *Let*

$$\mathrm{V}^*_{E,0} = \left\{\mathsf{Y}\in\mathrm{V}^*_E \ :\ \forall\tau\in\mathbf{B}_F, \mathsf{Y}_\tau \neq 0\right\}. \tag{3.39}$$

The set $\mathrm{V}^*_{E,0}$ *is* $\mathcal{U}^+_F$*-invariant and* $\mathrm{V}^*_{E,0}/\mathcal{U}^+_F$ *is a smooth manifold of dimension* $dim\,\mathrm{V}^*_E$.

Proof. Inspection of equation (3.38) shows the invariance of $\mathrm{V}^*_{E,0}$. To prove that the action of $\mathcal{U}^+_F$ is free and proper, define $\hat{\mathfrak{q}} : \mathrm{V}^*_{E,0} \to \mathrm{V}_{o,F}/\mathfrak{L}_F$ by

$$\hat{\mathfrak{q}}(\mathsf{Y}) = \sum_{\tau\in\mathbf{G}_F} \ln|\mathsf{Y}_\tau|\cdot\tau - \sum_{\tau\in\mathbf{G}_F} \ln|\mathsf{Y}_\tau|\cdot \mathrm{t} \qquad \mathrm{mod}\,\mathfrak{L}_F, \tag{3.40}$$

where

$$\mathrm{t} = \frac{1}{|\mathbf{G}_F|}\sum_{\tau\in\mathbf{G}_F}\tau.$$

From equation (3.38), one sees that $u^*|\mathsf{Y}_\tau| = -\ln|\tau(u)| + |\mathsf{Y}_\tau|$, so $\hat{\mathfrak{q}}$ is $\mathcal{U}^+_F$-invariant, hence it defines a continuous map $\mathfrak{q} : \mathrm{V}^*_{E,0}/\mathcal{U}^+_F \to \mathrm{V}_{o,F}/\mathfrak{L}_F$. The action of $\mathcal{U}^+_F$ is therefore both free and proper, since $\mathfrak{q}$ maps cosets onto cosets. □

Define a function $g_\tau : T^*\hat\Sigma \to \mathbb{R}$ by

$$g_\tau(P) = |\mathsf{Y}_\tau|^2 = \sum_{\sigma|_F=\tau} |\mathsf{Y}_\sigma|^2. \tag{3.41}$$

These functions are first integrals of $\mathbf{H}$ (see below), but they are not $\mathcal{U}^+_F$-invariant. However, their product is invariant:

$$\mathrm{k} = \prod_{\tau\in\mathbf{G}_F} g_\tau = \prod_{\tau\in\mathbf{G}_F}\sum_{\sigma|_F=\tau}|\mathsf{Y}_\sigma|^2. \tag{3.42}$$

From k one obtains the important subspaces

$$\mathfrak{U} = \left\{P\in T^*\Sigma \ :\ \mathrm{k}(P)\neq 0\right\}, \qquad \mathfrak{Z} = \left\{P\in T^*\Sigma \ :\ \mathrm{k}(P) = 0\right\}. \tag{3.43}$$

It is clear from the definition of k that $\mathfrak{Z}$ is the union $\cup_{\tau\in\mathbf{G}_F}\mathfrak{Z}_\tau$, where $\mathfrak{Z}_\tau = g_\tau^{-1}(0)$ (although g_τ is not $\mathcal{U}^+_F$-invariant, its zero set is). It is also clear that $\mathfrak{U}$ is an open and dense analytic submanifold of $T^*\Sigma$, $\mathfrak{Z} = \Sigma\times\mathrm{V}^*_{o,F}\times\mathrm{V}^*_{E,0}$, and that $\mathfrak{Z}$ is an analytic sub-variety.

Lemma 3.6. *Define the map* $\mathfrak{f} : \mathfrak{U} \to \mathrm{V}^*_{E,0}/\mathcal{U}^+_F$ *by, for all* $P\in\mathfrak{U}$,

$$\mathfrak{f}(P) = \mathsf{Y}\cdot\mathcal{U}^+_F \tag{3.44}$$

where the action of $\mathcal{U}^+_F$ *is given by equation (3.38). Then* $\mathfrak{f}$ *is an analytic submersion.*

Proof. This is clear. □

Remark 3.5. (1) $\mathfrak{U}$ is a union of regular Liouville tori and singular tori (see below). The singular set $\mathfrak{Z}$ has co-dimension equal to $[E : F]$. Therefore, when $F \subsetneq E$, the co-dimension is two or more. In this case, the set of regular Liouville tori is connected. (2) The topological structure of $\mathrm{V}^*_{E,0}/\mathcal{U}^+_F$ is interesting. The map $\mathfrak{q}$ is a submersion whose typical fibre is diffeomorphic to the Cartesian product of the unit spheres in $\mathrm{V}^*_{\tau,E}$, for $\tau \in \mathbf{B}_F$, with the positive real numbers. The bundle is generally non-trivial, since the action of $\mathcal{U}^+_F$ twists the fibres. Indeed, one sees that the map $\mathfrak{q} \times \mathrm{k}$ is a proper submersion with

$$\prod_{\tau\in\mathbf{B}_F} S^{d_\tau - 1} \hookrightarrow \mathrm{V}^*_{E,0}/\mathcal{U}^+_F \xrightarrow{\mathfrak{q}\times\mathrm{k}} \mathrm{V}_{o,F}/\mathfrak{L}_F \times \mathbb{R}^+ \tag{3.45}$$

where we identify k with a function defined on V^*_E and $d_\tau = [E : F]$. This also exhibits $\mathrm{V}^*_{E,0}/\mathcal{U}^+_F$ as a compact manifold times $\mathbb{R}^+$. The compact manifold is something like a torus bundle over a torus. In particular, the ends of $\mathrm{V}^*_{E,0}/\mathcal{U}^+_F$ are quite uncomplicated. (3) Let us relate the preceding discussion to that in the introduction, *c.f.* diagram (1.5) and figure 1. Let $\sim$ be the equivalence relation on V^*_E that is generated by defining $\mathsf{Y} \sim 0$ if $\mathsf{Y}_\tau = 0$ for some $\tau \in \mathbf{B}_F$ and $\mathsf{Y} \sim u \cdot \mathsf{Y}$ for all $u \in \mathcal{U}^+_F$. The topological space $\mathrm{V}^*_E/\sim$ is a quotient of $\mathrm{V}^*_E/\mathcal{U}^+_F$ where one collapses the set $\{\mathsf{Y} : \prod_{\tau\in\mathbf{B}_F} \mathsf{Y}_\tau = 0\}$ to a point. We have the following commutative diagram:

$$\begin{tikzcd}
 & \hat{\mathfrak{U}} \arrow[rr, "\mathsf{Y}"] \arrow[dl, "\text{incl.}"'] \arrow[dd, "/\mathcal{U}^+_F"] & & \mathrm{V}^*_{E,0} \arrow[dl, "\text{incl.}"'] \arrow[dd, "/\mathcal{U}^+_F"] & \\
T^*\hat{\Sigma} \arrow[rr, "\hat{f}=\mathsf{Y}"] \arrow[dd, "/\mathcal{U}^+_F"'] & & \mathrm{V}^*_E \arrow[dd, "/\mathcal{U}^+_F"] & & \\
 & \mathfrak{U} \arrow[rr, "\mathfrak{f}"] \arrow[dl, "\text{incl.}"'] & & \mathrm{V}^*_{E,0}/\mathcal{U}^+_F \arrow[dl, "\text{incl.}"'] \arrow[dr, "\text{incl.}"] & \\
T^*\Sigma \arrow[rr] \arrow[rrrr, bend right, "f"'] & & \mathrm{V}^*_E/\mathcal{U}^+_F \arrow[rr, "\text{collapse}"] & & \mathrm{V}^*_E/\sim
\end{tikzcd} \tag{3.46}$$

in (1.5) and $\hat{f} = \mathsf{Y}$ is the momentum-map of the torus $\mathrm{V}_E/\mathrm{N}_E$ acting on $T^*\hat{\Sigma}$. One can see that $\mathrm{V}^*_{E,0}/\mathcal{U}^+_F$ is the complement of the coset of 0 in $\mathrm{V}^*_E/\sim$ and that the first-integral map f is the natural extension of the map $\mathfrak{f}$ from $\mathfrak{U}$ to $T^*\Sigma$. From the diagram (3.45), one can see that $\mathrm{V}^*_E/\sim$ is homeomorphic to the cone on $\mathbb{R}^+\backslash\mathrm{V}^*_{E,0}/\mathcal{U}^+_F$, where $\mathbb{R}^+$ acts by scalar multiplication. The diagram (3.45) also shows that when $F = E$, the fibres of $\mathfrak{q} \times \mathrm{k}$ are disconnected, so that $\mathrm{V}^*_E/\sim$ is a union of disjoint cones pinched at the cone point as in figure (1). When F is a proper subfield of E, then the fibres of $\mathfrak{q} \times \mathrm{k}$ are connected and $\mathrm{V}^*_E/\sim$ is a cone on a connected space. (4) There are globally-defined, $\mathcal{U}^+_F$-invariant functions on V^*_E. The most natural construction is a generalisation of the quadratic Casimir from the 3-dimensional *Sol* manifolds and the Casimir in the totally real case [6, 11]. Each copy of $\mathrm{V}_{\tau,E}$ may be naturally identified with $\mathrm{V}_{E/F}$,[4] and similarly for the dual spaces. One can therefore define the map

$$\mathfrak{k}(\mathsf{Y}) = \prod_{\tau\in\mathbf{G}_F} \mathsf{Y}_\tau, \qquad \mathfrak{k} : \mathrm{V}^*_E \to S^d(\mathrm{V}_{E/F}) \tag{3.47}$$

[4] Note that $\mathrm{V}_{E/E} = \mathbb{R}$.

where $d = [E : F]$ and $S^*(\mathrm{V}_{E/F})$ is the vector space of polynomial functions on the vector space $\mathrm{V}_{E/F}$. It is clear that $\mathfrak{k}$ is $\mathcal{U}_F^+$ invariant. A simple computation shows that $\mathfrak{q} \times \mathfrak{k}$ is a submersion on $\mathrm{V}_{E,0}^*/\mathcal{U}_F^+$. (5) One might want to use the map

$$\mathsf{Y} \mapsto \sum_{\tau \in \mathbf{B}_F} \frac{\mathsf{Y}_\tau}{|\mathsf{Y}_\tau|}, \qquad \mathrm{V}_{E,0}^* \to \prod_{\tau \in \mathbf{B}_F} S^{d_\tau - 1} \subset \sum_{\tau \in \mathbf{B}_F} \mathrm{V}_{\tau,E}^*$$

to "split" the fibre bundle in (3.45). In general, however, the map induced by equivariance is not well-defined. Rather, to obtain a well-defined map by equivariance, the set $U_\tau = \left\{\tau(u)/|\tau(u)| \ : \ u \in \mathcal{U}_F^+\right\}$ needs to be finite for all $\tau \in \mathbf{G}_F$; if one of these sets is not finite, then the co-domain of the induced map is not a manifold; if all the sets are finite, then the induced map's co-domain is a product of lens spaces so it does not split the fibre bundle, but it does split a suitable finite covering. Finiteness fails in many important cases: if F possesses a unit of infinite order on the unit circle, for example. (6) The map $\mathfrak{f}$ induces a sub-algebra of $C^\infty(T^*\Sigma)$ by

$$\mathsf{R} = \left\{\mathfrak{f}^* h \ : \ h \in C^\infty(\mathrm{V}_{E,0}^*/\mathcal{U}_F^+), \ h \text{ has compact support } \right\}. \tag{3.48}$$

The sub-algebra R is the substitute on $T^*\Sigma$ for the momentum map $P \mapsto \mathsf{Y}$ on the level of algebras of functions.

4. Complete Integrability

Let $Z^\infty(\mathcal{L}^*)$ be the set of smooth Casimirs of $\mathcal{L}^*$ with its standard Poisson bracket. This section proves that

Theorem 4.1. *Let $h \in Z^\infty(\mathcal{L}^*)$ be a Casimir and let $\mathbf{L} : T^*\Sigma \to \mathcal{L}_R^*$ be the Lax matrix of equation (3.13)*

$$\mathbf{L}(P) = \sum_{\sigma \in \mathbf{G}_E} \mathrm{e}_{\mathbf{r}} \ + \ \Phi(\mathsf{X}) \ + \ \sum_{\sigma \in \mathbf{G}_E} |\mathsf{Y}_\sigma|^{b_\sigma} \cdot \exp(b_\sigma \cdot \langle \hat{\sigma}, \mathsf{x} \rangle) \cdot \mathrm{e}_{-\mathbf{r}},$$

where $\Phi : \mathrm{V}_{o,F}^* \to \mathfrak{h}^*$ satisfies the conclusions of Theorem (3.2). Then, the following are true

(1) *$H := \mathbf{L}^* h$ is a completely integrable Hamiltonian with smooth integrals;*
(2) *the algebras $\mathsf{L} := \mathbf{L}^* Z^\infty(\mathcal{L}^*)$ and R form a dual pair;*
(3) *the singular set is an analytic variety.*

Proof. (1-2) Let $\hat{\mathsf{R}} = \hat{\Pi}^*\mathsf{R}$ be the pullback of R to $T^*\hat{\Sigma}$. By the construction of R, $\hat{\mathsf{R}} \subset \mathsf{Y}^* C^\infty(\mathrm{V}_E^*)$ and their functional dimension is equal on $\hat{\mathfrak{U}}$.

A Casimir h of $\mathcal{L}^*$ is, *a fortiori*, invariant under the co-adjoint action of $\mathcal{L}_0$. Therefore, $h|_{\mathcal{L}_{-1}^* + \mathcal{L}_0^* + \mathcal{L}_{+1}^*}$ must be functionally dependent on the co-adjoint invariants of $\mathcal{L}_0$, $\mathrm{e}_{\mathbf{r}} \cdot \mathrm{e}_{-\mathbf{r}}$, $\mathbf{r} \in \mathbf{\Psi}$ and $x \in \mathcal{L}_0$. From the formula for $\mathbf{L}$, the function $H = \mathbf{L}^* h$ must therefore be a function of $\gamma_\sigma = |\mathsf{Y}_\sigma|^{b_\sigma} \exp(b_\sigma \langle \hat{\sigma}, \mathsf{x} \rangle)$ and X. These functions, and therefore H, are involution with $\hat{\mathsf{R}}$.

This proves that L and R are commuting algebras of functions whose sum $\mathsf{L} + \mathsf{R}$ is also abelian.

Let $\mathfrak{R} \subset \mathrm{e} + \mathcal{L}_0^* + \mathcal{L}_{+1}^*$ be the set of regular points of the algebra $Z^\infty(\mathcal{L}^*)$ restricted to the subspace $\mathrm{e} + \mathcal{L}_0^* + \mathcal{L}_{+1}^*$, where $\mathrm{e} = \sum_{\mathbf{r} \in \mathbf{\Psi}} \mathrm{e}_{\mathbf{r}}$. This regular-point set is an open and dense real-analytic subset of $\mathrm{e} + \mathcal{L}_0^* + \mathcal{L}_{+1}^*$. Since $\mathbf{L}|_{\mathfrak{U}}$ is an analytic submersion whose image is open in $\mathrm{e} + \mathcal{L}_0^* + \mathcal{L}_{+1}^*$, $\mathbf{L}^{-1}(\mathfrak{R})$ is an open and dense analytic subset of $\mathfrak{U}$.

Therefore, for all $P \in \mathbf{L}^{-1}(\mathfrak{R})$,

$$\dim \mathrm{d}\mathsf{L}_P = \operatorname{rank} \Psi = \dim \mathrm{V}_{o,F}, \qquad \dim \mathrm{d}\mathsf{R}_P = \dim \mathrm{V}_E,$$

while it is clear that

$$\mathrm{d}\mathsf{L}_P \cap \mathrm{d}\mathsf{R}_P = \{0\}.$$

Since $\dim\Sigma = \dim \mathrm{V}_{o,F} + \dim \mathrm{V}_E$, this proves (1-2).
(3) The singular set of $\mathsf{R}+\mathsf{L}$ is the union of $\mathbf{L}^{-1}(\mathfrak{R}^c)$ and $\mathfrak{Z} = \mathrm{k}^{-1}(0)$. Both are real-analytic subsets of $T^*\Sigma$, hence their union is, too.

□

Theorem 1. Let A be maximal in the sense of definition 1. This implies that $\mathbb{Z}^b$ is an irreducible A-module. Let $T \in \mathrm{GL}(b;\mathbb{C})$ be a matrix that conjugates A to a subgroup of the set of diagonal matrices in $\mathrm{GL}(b;\mathbb{C})$ and let $\Gamma = T^{-1}AT$ and $M = T^{-1}\mathbb{Z}^b$. Let F be the extension field of $\mathbb{Q}$ that is generated by the $(1,1)$-entries of $\gamma \in \Gamma$; since A is maximal, $F/\mathbb{Q}$ has degree b. The map δ, defined for each $\gamma \in \Gamma$ by,

$$\delta(\gamma) := \gamma_{11} \qquad\qquad \delta : \Gamma \to \mathcal{U}_F$$

is a group homomorphism. Indeed, the maps $\delta_j(\gamma) = \gamma_{jj}$ are group homomorphisms into the group of units of the j-th conjugate of F.

It is clear that the first column of the matrix T can be supposed to have entries in $\mathcal{O}_F$ and the j-th column of T can be supposed to be the j-th conjugate of the first column. It is claimed that $\det T = q \cdot d$ where q is a non-zero integer q and d is the different of F. By definition, $d = \det U$ where the entries of the first column of U form a $\mathbb{Z}$-basis of $\mathcal{O}_F$ and the remaining columns are the conjugates of the first column. Let v be the first column of T. If the entries of v do not rationally span F, then there is a non-zero $t \in \mathbb{Z}^b$ such that $\langle t, v\rangle = 0$. One can take the conjugates of this linear equation and conclude that t is orthogonal to each column of T and therefore $t = 0$. Absurd. One concludes that the entries of v generate a finite index subgroup of $\mathcal{O}_F$. The index of this subgroup is $\det T/d$. This proves the claim. Therefore, for all $m \in M$, the j-th entry of $qd \times m$ lies in the j-th conjugate of $\mathcal{O}_F$. Define the map δ for each $m \in M$ by

$$\delta(m) := qd \cdot m_1 \qquad\qquad \delta : M \to \mathcal{O}_F,$$

where m_1 is the first entry of m. It is clear that δ is a morphism of modules that faithfully intertwines the representation of Γ on M with that of $\delta(\Gamma)$ on $\delta(M)$; or, δ extends to a group embedding $M \star \Gamma \hookrightarrow \mathcal{O}_F \star \mathcal{U}_F$ whence there is a group embedding $\mathbb{Z}^b \star A \hookrightarrow \mathcal{O}_F \star \mathcal{U}_F$.

Because $\mathbb{Z}^b$ is an irreducible A-module, the degree of $F/\mathbb{Q}$ is b so $\mathbb{Z}^b$ is embedded as a finite index subgroup of $\mathcal{O}_F$. Since A is maximal, A is embedded as a torsion-free, finite-index subgroup of $\mathcal{U}_F$. Since $\delta(\Gamma)$ is torsion-free, there is a choice of $\mathcal{U}_F^+$ such that $\delta(\Gamma) \subset \mathcal{U}_F^+$. Therefore, one has obtained an embedding $\mathbb{Z}^b \star A \hookrightarrow \mathcal{O}_F \star \mathcal{U}_F^+$ which is of finite index. This proves that Σ_A is a finite covering of the manifold Σ constructed in lemma (2.3) with $E = F$.

The proof of the theorem follows now by virtue of theorem 4.1 and the fact that the covering map $T^*\Sigma_A \to T^*\Sigma$ is a local symplectomorphism. □

4.1. **Examples.** Let us illustrate the results of this section with two examples.

4.1.1. *A non-normal cubic extension.* To illustrate the construction behind Theorem 1 take the case where $A \lhd \mathrm{GL}(3;\mathbb{Z})$ is the group generated by

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \qquad\qquad \mathcal{A}_2 = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -2 & 1 \\ 1 & 2 & -2 \end{bmatrix}. \tag{4.1}$$

A is conjugate by a $T \in \mathrm{SL}(3;\mathbb{R})$ to the group Γ generated by

$$\mathcal{B}_1 = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}, \qquad \mathcal{B}_2 = \begin{bmatrix} \alpha_4 & 0 & 0 \\ 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_6 \end{bmatrix}, \tag{4.2}$$

where α_j for $j = 1, 2, 3$ are the roots of the cubic $f(x) = x^3 - 5x - 1$ and $\alpha_j = \alpha_{j-3} - 2$ for $j = 4, 5, 6$. For definiteness, one can take T to be the matrix

$$T = \begin{bmatrix} 4 & 3\alpha_3 + \alpha_2 & 3\alpha_3 + \alpha_2 \\ 6 & 5\alpha_2 + \alpha_1 & 5\alpha_2^2 + \alpha_1^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{bmatrix}, \tag{4.3}$$

whence $\det T = \sqrt{473}$, which is the different of f and the number field $F = \mathbb{Q}[\alpha_1]$. Let $M = T^{-1}(\mathbb{Z}^3)$ and $\Delta = M \star \Gamma$ so that $T^*\Sigma_A = T^*(\Delta\backslash\mathbb{R}^3 \times \mathbb{R}^2)$.

To define the Lax matrix in (3.14), it is convenient to embed A by

$$A \xrightarrow{\log \circ \mathrm{Ad}_{T^{-1}}} \mathfrak{h}, \qquad \mathcal{A}_{i+1}^2 \longmapsto \begin{bmatrix} \log|\alpha_{3i+1}| & & \\ & \log|\alpha_{3i+2}| & \\ & & \log|\alpha_{3i+3}| \end{bmatrix}$$

for $i = 0, 1$, where $\mathfrak{h} \cong \mathbb{R}^2$ is the Cartan subalgebra of $\mathrm{SL}(3;\mathbb{R})$ consisting of trace zero diagonal 3×3 matrices. This embeds A as a lattice in $\mathfrak{h}$. One can define the coordinates for $P = (\mathsf{Y}, \mathsf{y}, \mathsf{X}, \mathsf{x}) \in T^*\tilde{\Sigma} = T^*\mathbb{R}^3 \times T^*\mathfrak{h}$ and thereby obtain the Lax matrix

$$\mathbf{L}(P) = \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \mathsf{X}_1 & 0 & 0 \\ 0 & \mathsf{X}_2 & 0 \\ 0 & 0 & \mathsf{X}_3 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \\ \lambda\delta_3 & 0 & 0 \end{bmatrix} \tag{4.4}$$

where $\delta_i = |\mathsf{Y}_i|^2 \exp(2\mathsf{x}_i - 2\mathsf{x}_{i+1})$ and $\sum \mathsf{X}_i = \sum \mathsf{x}_i = 0$. One obtains the two Poisson-commuting functions

$$\mathbf{H} = \frac{1}{2} \times \mathrm{Tr}(\mathbf{L}^2) = \frac{1}{2} \times \left(\mathsf{X}_1^2 + \mathsf{X}_2^2 + \mathsf{X}_3^2\right) + \frac{1}{2} \times (\delta_1 + \delta_2 + \delta_3) \tag{4.5}$$

$$\mathbf{F} = \frac{1}{3} \times \mathrm{Tr}(\mathbf{L}^3) \equiv \frac{1}{3} \times \left(\mathsf{X}_1^3 + \mathsf{X}_2^3 + \mathsf{X}_3^3\right) - \frac{1}{2} \times (\delta_1\mathsf{X}_3 + \delta_2\mathsf{X}_1 + \delta_3\mathsf{X}_2) \tag{4.6}$$

that are in involution with Y.

One may permute the indices i; it is clear that a cyclic permutation yields the same $\mathbf{H}$ and it is not difficult to see that transpositions yield equivalent hamiltonians (remark 7.2).

4.1.2. *A non-normal cubic extension and* $\mathbb{Z}^6$. To illustrate the construction behind Theorem 1 take the case where $A \lhd \mathrm{GL}(6;\mathbb{Z})$ is the group generated by

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 2 & -4 & 0 & 1 & -2 \\ 0 & 0 & 0 & -2 & 2 & 1 \\ -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} -2 & 2 & -4 & 0 & 1 & -2 \\ 0 & -2 & 0 & -2 & 2 & 1 \\ -1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & -1 & -2 & 0 & -2 \\ 0 & 1 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}. \tag{4.7}$$

A is conjugate by a $T \in \mathrm{SL}(6;\mathbb{R})$ to the group Γ generated by

$$\mathcal{B}_1 = \begin{bmatrix} \alpha_1 I_2 & 0 & 0 \\ 0 & \alpha_2 I_2 & 0 \\ 0 & 0 & \alpha_3 I_2 \end{bmatrix}, \qquad \mathcal{B}_2 = \begin{bmatrix} \alpha_4 I_2 & 0 & 0 \\ 0 & \alpha_5 I_2 & 0 \\ 0 & 0 & \alpha_6 I_2 \end{bmatrix}, \tag{4.8}$$

where I_2 is the 2×2 identity matrix and α_j for $j = 1, 2, 3$ are the roots of the cubic $f(x) = x^3 - 5x - 1$ and $\alpha_j = \alpha_{j-3} - 2$ for $j = 4, 5, 6$. One notes that the matrix $\mathcal{A}_1$

is the matrix of the root α_1 acting on the integers of $\mathcal{O}_E$, where $E = \mathbb{Q}[\alpha_1, \sqrt{473}]$ is the normal closure of the field F of the previous example. There is not a simple expression for such a matrix T, because unlike the previous example $\mathcal{A}_1$ is not conjugate over $\mathbb{Z}$ to its companion matrix. In all events, let $M = T^{-1}(\mathbb{Z}^6)$ and $\Delta = M \star \Gamma$ so that $T^*\Sigma_A = T^*(\Delta\backslash\mathbb{R}^6 \times \mathbb{R}^2)$.

To define the Lax matrix in (3.14), it is convenient to embed A by

$$A \xrightarrow{\log \circ \mathrm{Ad}_{T^{-1}}} \mathfrak{h}, \qquad \mathcal{A}_{i+1} \longmapsto \begin{bmatrix} \log|\alpha_{3i+1}|I_2 & & \\ & \log|\alpha_{3i+2}|I_2 & \\ & & \log|\alpha_{3i+3}|I_2 \end{bmatrix}$$

consisting of trace zero diagonal 6×6 matrices. This embeds A as a lattice in the subspace $\mathfrak{h}_2 \subset \mathfrak{h}$ consisting of matrices which are of the form $B \otimes I_2$ for a 3×3 diagonal, trace zero matrix B. One can define the coordinates for $P = (\mathsf{Y}, \mathsf{y}, \mathsf{X}, \mathsf{x}) \in T^*\tilde{\Sigma} = T^*\mathbb{R}^6 \times T^*\mathfrak{h}_2$ and thereby obtain the Lax matrix

$$\mathbf{L}(P) = \begin{bmatrix} 0 & 0 & \lambda^{-1}I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} + \begin{bmatrix} \mathsf{X}_1 I_2 & 0 & 0 \\ 0 & \mathsf{X}_2 I_2 & 0 \\ 0 & 0 & \mathsf{X}_3 I_2 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 0 & \delta_1 I_2 & 0 \\ 0 & 0 & \delta_2 I_2 \\ \lambda\delta_3 I_2 & 0 & 0 \end{bmatrix} \tag{4.9}$$

where $\delta_i = |\mathsf{Y}_i|^2 \exp(2\mathsf{x}_i - 2\mathsf{x}_{i+1})$ and $\sum \mathsf{X}_i = \sum \mathsf{x}_i = 0$. One obtains the two Poisson-commuting functions

$$\mathbf{H} = \frac{1}{4} \times \mathrm{Tr}(\mathbf{L}^2) = \frac{1}{2} \times \left(\mathsf{X}_1^2 + \mathsf{X}_2^2 + \mathsf{X}_3^2\right) + \frac{1}{2} \times (\delta_1 + \delta_2 + \delta_3) \tag{4.10}$$

$$\mathbf{F} = \frac{1}{6} \times \mathrm{Tr}(\mathbf{L}^3) \equiv \frac{1}{3} \times \left(\mathsf{X}_1^3 + \mathsf{X}_2^3 + \mathsf{X}_3^3\right) - \frac{1}{2} \times (\delta_1\mathsf{X}_3 + \delta_2\mathsf{X}_1 + \delta_3\mathsf{X}_2) \tag{4.11}$$

that are in involution with Y ($\equiv$ indicates equality modulo functions of Y).

4.1.3. *A non-normal quartic extension.* To illustrate the construction behind Theorem 1 take the case where $A \lhd \mathrm{GL}(4;\mathbb{Z})$ is the group generated by

$$\mathcal{A}_1 = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathcal{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix}. \tag{4.12}$$

A is conjugate by a $T \in \mathrm{SL}(4;\mathbb{R})$ to the group Γ generated by

$$\mathcal{B}_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_4), \qquad \mathcal{B}_2 = \mathrm{diag}(\alpha_5, \ldots, \alpha_8), \tag{4.13}$$

where α_j for $j = 1, \ldots, 4$ are the roots of the palindromic quartic $f(x) = x^4 - 2x^3 + x^2 - 2x - 1$ and $\alpha_j = \alpha_{j-4}^3 - \alpha_{j-4}^2 - 1$ for $j = 5, \ldots, 8$. The roots α_j equal $\frac{1}{2} \times \left(1 + s\sqrt{2} + t\sqrt{(1+s\sqrt{2})^2 - 4}\right)$ where $s, t \in \{\pm 1\}$, $j = 1, \ldots, 4$. This gives two real reciprocal roots that are approximately 1.883 and 0.531 and two conjugate complex roots on the unit circle that are approximately $0.207 \pm 0.978\sqrt{-1}$. Since f is $\mathbb{Q}$-irreducible, the complex roots are not roots of unity, which also implies that the largest positive root is a Salem number. One notes that the matrix $\mathcal{A}_1$ is the matrix of the root α_1 acting on the integers of $\mathcal{O}_F$, where $F = \mathbb{Q}[\alpha_1]$.

As with example 4.1.1, one can compute a straightforward representation of T

$$\begin{bmatrix} -1 & \alpha_4 - \alpha_3 + \alpha_2 - 2\,\alpha_1 & \alpha_4^2 - \alpha_3^2 + \alpha_2^2 - 2\,\alpha_1^2 & \alpha_4^3 - \alpha_3^3 + \alpha_2^3 - 2\,\alpha_1^3 \\ 0 & \alpha_4 - \alpha_3 + \alpha_2 - \alpha_1 & \alpha_4^2 - \alpha_3^2 + \alpha_2^2 - \alpha_1^2 & \alpha_4^3 - \alpha_3^3 + \alpha_2^3 - \alpha_1^3 \\ 1 & -\alpha_4 + \alpha_3 + \alpha_1 & -\alpha_4^2 + \alpha_3^2 + \alpha_1^2 & -\alpha_4^3 + \alpha_3^3 + \alpha_1^3 \\ 2 & \alpha_3 + \alpha_1 & \alpha_3^2 + \alpha_1^2 & \alpha_3^3 + \alpha_1^3 \end{bmatrix} \tag{4.14}$$

and one can verify that $\det T = -8\sqrt{-7}$, which is the different of F. In all events, let $M = T^{-1}(\mathbb{Z}^4)$ and $\Delta = M \star \Gamma$ so that $T^*\Sigma_A = T^*(\Delta\backslash\mathbb{R}^4 \times \mathbb{R}^2)$.

To define a Lax matrix as in (3.14), it is convenient to embed A into the Cartan subalgebra $\mathfrak{h} \cong \mathbb{R}^2$ of the real symplectic group of 4×4 matrices

$$\mathfrak{h} = \{\mathrm{diag}(a, b, -a, -b) : \ a, b \in \mathbb{R}\}$$

by the embedding

$$A \xrightarrow{\log \circ \mathrm{Ad}_{T^{-1}}} \mathfrak{h}, \qquad \mathcal{A}_{i+1} \longmapsto \mathrm{diag}(\log|\alpha_{4i+1}|, \log|\alpha_{4i+2}|, \log|\alpha_{4i+4}|, \log|\alpha_{4i+3}|)$$

for $i = 0, 1$. To make this an embedding, one must stipulate that the roots α_j and α_{5-j} must be reciprocals for $j = 1, 2$; it is also supposed that α_1 (resp. α_2) has positive imaginary part (resp. is the largest real root of f). This embeds A as a lattice in $\mathfrak{h}$. One can define the coordinates for $P = (\mathsf{Y}, \mathsf{y}, \mathsf{X}, \mathsf{x}) \in T^*\tilde{\Sigma} = T^*\mathbb{R}^4 \times T^*\mathfrak{h}$ and thereby obtain the Lax matrix $\mathbf{L}(P)$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda^{-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \overbrace{\begin{bmatrix} a_1\mathsf{X}_1 & 0 & 0 & 0 \\ 0 & a_2\mathsf{X}_2 & 0 & 0 \\ 0 & 0 & -a_1\mathsf{X}_1 & 0 \\ 0 & 0 & 0 & -a_2\mathsf{X}_2 \end{bmatrix}}^{\Phi(\mathsf{X})} + \frac{1}{2} \times \begin{bmatrix} 0 & 0 & \lambda\delta_3 & 0 \\ \delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_1 \\ 0 & \delta_2 & 0 & 0 \end{bmatrix} \tag{4.15}$$

where δ_i, a_i are determined in Table 2. One obtains the two Poisson-commuting functions

$$\mathbf{H} = \frac{1}{4} \times \mathrm{Tr}(\mathbf{L}^2) = \frac{1}{2} \times \left(a_1^2\mathsf{X}_1^2 + a_2^2\mathsf{X}_2^2\right) + \frac{1}{2} \times (\delta_1 + \frac{1}{2}\delta_2 + \frac{1}{2}\delta_3) \tag{4.16}$$

$$\mathbf{F} = \det \mathbf{L} \equiv \frac{\delta_2\delta_3}{4} + \frac{\delta_1^2}{4} - \delta_1 a_1 a_2 \mathsf{X}_1\mathsf{X}_2 + \frac{a_2^2\mathsf{X}_2^2\delta_3}{2} + \frac{a_1^2\mathsf{X}_1^2\delta_2}{2} + a_1^2a_2^2\mathsf{X}_1^2\mathsf{X}_2^2 \tag{4.17}$$

that are in involution with Y ($\equiv$ indicates equality modulo functions of Y).

To explain the following choices for the functions δ_i, one defines the embeddings of the number field F by $\tau_i(\alpha_1) = \alpha_i$, so that $\mathbf{B}_F = \{\tau_1, \tau_2, \tau_3\}$ and $\tau_4 = \bar{\tau}_1$. A bijection $\rho : \mathbf{B}_F \to \mathbf{\Psi}$ is identified as a permutation s of $\{1, 2, 3\}$ under the convention that $\rho(\tau_i) = \mathbf{r}_{s(j)}$. Only three choices are listed since the remaining three are obtained by permuting Y_2 and Y_3 in the formulae below (these unlisted choices are also conjugate to the listed choices, since this permutation induces an analytic symplectomorphism of $T^*\Sigma$).

c	ρ	b_τ	a_1, a_2	δ_1	δ_2	δ_3
2	(1)	$2,2,2$	$1,1$	$2\lvert\mathsf{Y}_1\rvert^2 e^{2\mathsf{x}_1 - 2\mathsf{x}_2}$	$\lvert\mathsf{Y}_2\rvert^2 e^{4\mathsf{x}_2}$	$\lvert\mathsf{Y}_3\rvert^2 e^{-4\mathsf{x}_1}$
4	(2 1)	$8,2,4$	$\frac{1}{2}, \frac{1}{4}$	$\lvert\mathsf{Y}_2\rvert^2 e^{4\mathsf{x}_1 - 8\mathsf{x}_2}$	$2\lvert\mathsf{Y}_1\rvert^8 e^{16\mathsf{x}_2}$	$\lvert\mathsf{Y}_3\rvert^4 e^{-8\mathsf{x}_1}$
4	(3 1)	$8,4,2$	$\frac{1}{4}, \frac{1}{2}$	$\lvert\mathsf{Y}_3\rvert^2 e^{8\mathsf{x}_1 - 4\mathsf{x}_2}$	$\lvert\mathsf{Y}_2\rvert^4 e^{8\mathsf{x}_2}$	$2\lvert\mathsf{Y}_1\rvert^8 e^{-16\mathsf{x}_1}$

TABLE 2. Choices for the Lax matrix $\mathbf{L}$; y_i (Y_i) is a coordinate on the α_i-eigenspace with $\mathsf{y}_1 = \bar{\mathsf{y}}_4$ ($\mathsf{Y}_1 = \bar{\mathsf{Y}}_4$). See Theorem 3.2.

With these choices of δ_i, (4.15) gives a Lax representation of the hamiltonian vector field of $\mathbf{H}$ (4.16) with the integral $\mathbf{F}$ (4.17). Although fibrewise convex for all choices, the hamiltonian $\mathbf{H}$ is only fibre-wise quadratic for the first choice.

Additional Lax Representations. One can define additional Lax representations with the aid of the remaining rank 2 affine Kac-Moody algebras.

$A_2^{(1)}$. Embed A into the Cartan subalgebra $\mathfrak{h} \cong \mathbb{R}^2$ of $\mathrm{SL}(3;\mathbb{R})$ via

$$A \xrightarrow{\log \circ \mathrm{Ad}_{T^{-1}}} \mathfrak{h}, \qquad \mathcal{A}_{i+1} \longmapsto \mathrm{diag}(2\log|\alpha_{4i+1}|, \log|\alpha_{4i+2}|, \log|\alpha_{4i+3}|)$$

where the roots α_j are labelled as above. This embeds A as a lattice in $\mathfrak{h}$. One can define the coordinates for $P = (\mathsf{Y}, \mathsf{y}, \mathsf{X}, \mathsf{x}) \in T^*\tilde{\Sigma} = T^*\mathbb{R}^4 \times T^*\mathfrak{h}$ and thereby obtain the Lax matrix

$$\mathbf{L}(P) = \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} a_1\mathsf{X}_1 & 0 & 0 \\ 0 & a_2\mathsf{X}_2 & 0 \\ 0 & 0 & a_3\mathsf{X}_3 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \\ \lambda\delta_3 & 0 & 0 \end{bmatrix} \tag{4.18}$$

where $\sum a_i \mathsf{X}_i = \sum a_i^{-1}\mathsf{x}_i = 0$ and δ_i is defined below. One obtains the two Poisson-commuting functions $\mathbf{H} = \frac{1}{2} \times \mathrm{Tr}(\mathbf{L}^2)$ and $\mathbf{F} = \det \mathbf{L}$ where

$$\mathbf{H} = a_1^2\mathsf{X}_1^2 + a_1\mathsf{X}_1 a_2\mathsf{X}_2 + a_2^2\mathsf{X}_2^2 + \frac{1}{2} \times (\delta_1 + \delta_2 + \delta_3) \tag{4.19}$$

$$\mathbf{F} \equiv -a_1\mathsf{X}_1 a_2^2\mathsf{X}_2^2 - a_1^2\mathsf{X}_1^2 a_2\mathsf{X}_2 + \frac{1}{2}\, a_1\mathsf{X}_1\,(\delta_1 - \delta_2) + \frac{1}{2}\, a_2\mathsf{X}_2\,(\delta_1 - \delta_3)\,, \tag{4.20}$$

where the functions δ_i are determined in table 3, following the conventions in table 2.

c	ρ	b_τ	a_1, a_2	δ_1	δ_2	δ_3
2	(1)	2, 2, 4	$\frac{1}{2}, 1$	$2\lvert\mathsf{Y}_1\rvert^2 e^{2\mathsf{x}_2 - 4\mathsf{x}_1}$	$\lvert\mathsf{Y}_2\rvert^2 e^{-4\mathsf{x}_2 - 4\mathsf{x}_1}$	$\lvert\mathsf{Y}_3\rvert^4 e^{2\mathsf{x}_2 + 8\mathsf{x}_1}$
2	(1 2)	2, 4, 2	$1, \frac{1}{2}$	$\lvert\mathsf{Y}_2\rvert^4 e^{4\mathsf{x}_2 - 2\mathsf{x}_1}$	$2\lvert\mathsf{Y}_1\rvert^2 e^{-8\mathsf{x}_2 - 2\mathsf{x}_1}$	$\lvert\mathsf{Y}_3\rvert^2 e^{4\mathsf{x}_2 + 4\mathsf{x}_1}$
2	(1 3)	4, 2, 2	$1, 1$	$\lvert\mathsf{Y}_3\rvert^2 e^{2\mathsf{x}_2 - 2\mathsf{x}_1}$	$\lvert\mathsf{Y}_2\rvert^2 e^{-4\mathsf{x}_2 - 2\mathsf{x}_1}$	$2\lvert\mathsf{Y}_1\rvert^4 e^{2\mathsf{x}_2 + 4\mathsf{x}_1}$

TABLE 3. Choices for the Lax matrix $\mathbf{L}$; y_i (Y_i) is a coordinate on the α_i-eigenspace with $\mathsf{y}_1 = \bar{\mathsf{y}}_4$ ($\mathsf{Y}_1 = \bar{\mathsf{Y}}_4$). See Theorem 3.2.

$G_2^{(1)}$. One proceeds as above and obtains the hamiltonian

$$\mathbf{H} = \frac{1}{24} \times \left(a_1^2\mathsf{X}_1^2 + 3a_1\mathsf{X}_1 a_2\mathsf{X}_2 + 3a_2^2\mathsf{X}_2^2\right) + 16 \times (3\delta_1 + \delta_2 + \delta_3) \tag{4.21}$$

where δ_i is defined by

c	ρ	b_τ	a_1, a_2	δ_1	δ_2	δ_3
12	(1)	8, 6, 12	$\frac{1}{4}, \frac{1}{3}$	$2\lvert\mathsf{Y}_1\rvert^8 e^{16\mathsf{x}_1 - 6\mathsf{x}_2}$	$\lvert\mathsf{Y}_2\rvert^6\ e^{12\mathsf{x}_2 - 24\mathsf{x}_1}$	$\lvert\mathsf{Y}_3\rvert^{12} e^{-6\mathsf{x}_2}$
12	(3 2)	8, 12, 6	$\frac{1}{4}, \frac{1}{3}$	$2\lvert\mathsf{Y}_1\rvert^8 e^{16\mathsf{x}_1 - 12\mathsf{x}_2}$	$\lvert\mathsf{Y}_3\rvert^6\ e^{24\mathsf{x}_2 - 24\mathsf{x}_1}$	$\lvert\mathsf{Y}_2\rvert^{12} e^{-12\mathsf{x}_2}$
24	(3 2 1)	24, 4, 6	$\frac{1}{3}, \frac{1}{12}$	$\lvert\mathsf{Y}_3\rvert^6 e^{48\mathsf{x}_1 - 4\mathsf{x}_2}$	$2\lvert\mathsf{Y}_1\rvert^{24} e^{8\mathsf{x}_2 - 72\mathsf{x}_1}$	$\lvert\mathsf{Y}_2\rvert^4\ e^{-4\mathsf{x}_2}$
12	(2 1)	6, 2, 6	$1, \frac{1}{3}$	$\lvert\mathsf{Y}_2\rvert^2 e^{12\mathsf{x}_1 - 2\mathsf{x}_2}$	$2\lvert\mathsf{Y}_1\rvert^6\ e^{4\mathsf{x}_2 - 18\mathsf{x}_1}$	$\lvert\mathsf{Y}_3\rvert^6\ e^{-2\mathsf{x}_2}$
12	(2 3 1)	6, 6, 2	$\frac{1}{3}, 1$	$\lvert\mathsf{Y}_2\rvert^6 e^{12\mathsf{x}_1 - 6\mathsf{x}_2}$	$\lvert\mathsf{Y}_3\rvert^2\ e^{12\mathsf{x}_2 - 18\mathsf{x}_1}$	$2\lvert\mathsf{Y}_1\rvert^6\ e^{-6\mathsf{x}_2}$
24	(3 1)	24, 6, 4	$\frac{1}{2}, \frac{1}{3}$	$\lvert\mathsf{Y}_3\rvert^4 e^{48\mathsf{x}_1 - 6\mathsf{x}_2}$	$\lvert\mathsf{Y}_2\rvert^6\ e^{12\mathsf{x}_2 - 72\mathsf{x}_1}$	$2\lvert\mathsf{Y}_1\rvert^{24} e^{-6\mathsf{x}_2}$

TABLE 4. Choices for the Lax matrix $\mathbf{L}$; y_i (Y_i) is a coordinate on the α_i-eigenspace with $\mathsf{y}_1 = \bar{\mathsf{y}}_4$ ($\mathsf{Y}_1 = \bar{\mathsf{Y}}_4$) and x_i is the coordinate on $\mathfrak{h}$ induced by the simple coroots [17, p. 346].

5. The singular set and gradient flows

Two prefatory comments: first, the fibre bundle structure

$$\mathrm{V}_E/\mathcal{O}_E \hookrightarrow \Sigma \xrightarrow{\mathrm{p}} \mathrm{V}_{o,F}/\boldsymbol{\mathfrak{L}}_F$$

induces the sub-bundle $\mathbf{V} = \ker \mathrm{dp} \subset T\Sigma$ and its annihilator $\mathbf{V}^\perp \subset T^*\Sigma$. The sub-bundle $\mathbf{V}^\perp$ is naturally isomorphic to $\Sigma \times \mathrm{V}_{o,F}^*$. Second, recall that the *stable manifold* of a point p is the set of points whose orbits converge to that of p's as time goes to ∞; the *unstable manifold* is defined symmetrically as time goes to $-\infty$; the stable and unstable manifolds of a set are the union of the stable and unstable manifolds of each point in the set. In this section it is shown that

Theorem 5.1. $\mathbf{V}^\perp$ *is an invariant set for the Hamiltonian flow of* $\mathbf{H}$ *(equation 3.16). The stable and unstable manifolds of* $\mathbf{V}^\perp$, $\mathrm{W}^\pm(\mathbf{V}^\perp)$, *coincide and*

$$\mathrm{W}^\pm(\mathbf{V}^\perp) = \mathfrak{Z} \qquad (= \mathrm{k}^{-1}(0)). \tag{5.1}$$

Before proceeding with the proof, let us explain why theorem 5.1 is natural from the perspective of Bogoyavlenskij-Toda lattices. It is a well-known result that the open Bogoyavlenskij-Toda lattices undergo scattering: the particles interact over some time interval and then separate and proceed off to infinity. The net result of the interaction is that the momenta of the particles may be permuted from $t = -\infty$ to $t = \infty$; in terms of the Lax matrix, $\mathbf{L}(-\infty)$ and $\mathbf{L}(\infty)$ are diagonal matrices which differ by the action of some element in the Weyl group. Since the open Bogoyavlenskij-Toda lattices are obtained from the periodic Bogoyavlenskij-Toda lattices by turning off the potential term associated to the root $\boldsymbol{\eta}$, it is plausible that when other potential terms are turned off, the system should still exhibit such scattering behaviour. To confirm this, one must develop the double-bracket or gradient representation of these systems.

5.1. Double-bracket and gradient representations.

Let us recall the constructions of [9], where it is demonstrated that the open Bogoyavlenskij-Toda lattices may be viewed as gradient flows. Let $\mathfrak{g}$ be a semi-simple Lie algebra with Cartan-Killing form $\kappa = \langle\langle , \rangle\rangle$. For $x \in \mathfrak{g}^*$ let $\mathcal{O}_x$ denote the co-adjoint orbit of x, let $\mathfrak{g}_x$ be the stabiliser algebra of x and let $\mathfrak{g}_x^\perp$ be the κ-orthogonal complement of $\mathfrak{g}_x$. The map $v \mapsto \mathrm{ad}_v x$ is a linear isomorphism of $\mathfrak{g}_x^\perp$ with $T_x\mathcal{O}_x$.

Definition 5.1. *The* normal metric, $\mathbf{n}$, *on* $\mathcal{O}_x$ *is defined at* $T_x\mathcal{O}_x$ *by*

$$\forall u, v \in \mathfrak{g}_x^\perp : \qquad \mathbf{n}(\mathrm{ad}_u x, \mathrm{ad}_v x) = \langle\langle u, v\rangle\rangle \tag{5.2}$$

Lemma 5.2. *If* $H \in C^\infty(\mathfrak{g}^*)$, *then the gradient vector field of* $H|\mathcal{O}_x$ *at* x *is*

$$\overset{\mathbf{n}}{\nabla} H(x) = -[x, [x, y]] \in T_x\mathcal{O}_x \tag{5.3}$$

where $y = \nabla H(x)$ *is the* κ*-gradient of* H.

For a proof, see [9].

5.2. Bogoyavlenskij-Toda lattices and double brackets.

Let us specialise the construction of the previous section. The semi-simple Lie algebra is the loop algebra $\mathcal{L}$ or its twisted counterpart of section 3.1. Let $\boldsymbol{\Psi}_0 \subsetneq \boldsymbol{\Psi}$ be a proper subset obtained by removing a single root from $\boldsymbol{\Psi}$. Let

$$x = h + \sum_{\mathbf{r}\in\boldsymbol{\Psi}_0} x_{\mathbf{r}}(\mathrm{e}_{\mathbf{r}} + \mathrm{e}_{-\mathbf{r}}) \in \mathcal{L}^*, \quad m = \sum_{\mathbf{r}\in\boldsymbol{\Psi}_0} x_{\mathbf{r}}(\mathrm{e}_{\mathbf{r}} - \mathrm{e}_{-\mathbf{r}}), \quad X(x) = [x, m], \tag{5.4}$$

where $h \in \mathfrak{h}$. The vector field X is a Bogoyavlenskij-Toda-like vector field associated to the splitting of ${}_0\mathcal{L} \subset \mathcal{L}$ as in section 3.2.

Lemma 5.3. *X is a gradient vector field relative to the normal metric, hence X is tangent to $\mathcal{O}_x$.*

Proof. It suffices to determine a $y \in \mathfrak{h}$ such that $X = \overset{\mathbf{n}}{\nabla} H$ where $H(x) = \langle\langle x, y\rangle\rangle$. To do so, it suffices to determine y such that $m = -[x, y]$. This reduces to the solubility of the equations

$$\forall \mathbf{r} \in \mathbf{\Psi} : \qquad x_{\mathbf{r}} = x_{\mathbf{r}} \langle r, y\rangle . \tag{5.5}$$

Since at least one of the $x_{\mathbf{r}}$ vanishes, and any subset of $\mathbf{\Psi}$ of cardinality $\#\mathbf{\Psi} - 1$ restricts to a basis of $\mathfrak{h}^*$, there is always a solution to (5.5). □

The vector field X is equivalent to the differential equations

$$-\dot{h} = \sum_{\mathbf{r} \in \mathbf{\Psi}_0} 2x_{\mathbf{r}}^2 \, h_{\mathbf{r}}, \qquad \text{and} \qquad \forall \mathbf{r} \in \mathbf{\Psi} : \quad \dot{x}_{\mathbf{r}} = x_{\mathbf{r}} \langle r, h\rangle , \tag{5.6}$$

where $h_{\mathbf{r}} = [\mathrm{e}_{\mathbf{r}}, \mathrm{e}_{-\mathbf{r}}]$. In particular, X is tangent to $x_{\mathbf{r}} = 0$ for any $\mathbf{r}$. It is also clear that X vanishes at x iff

$$\forall \mathbf{s} \in \mathbf{\Psi} : \qquad x_{\mathbf{s}} \langle s, h\rangle = 0 \quad \text{and} \quad \sum_{\mathbf{r} \in \mathbf{\Psi}_0} 2x_{\mathbf{r}}^2 \, \langle\langle s, r\rangle\rangle = 0, \tag{5.7}$$

where the identity $\langle s, h_{\mathbf{r}}\rangle = \langle\langle s, r\rangle\rangle$ has been used. Since the matrix $[\langle\langle s, r\rangle\rangle]_{r,s \in \mathbf{\Psi}_0}$ has full rank, the second part of (5.7) implies that $x_{\mathbf{r}} = 0$ for all $\mathbf{r} \in \mathbf{\Psi}_0$ and therefore for all $\mathbf{r} \in \mathbf{\Psi}$. This proves that

Lemma 5.4. *X vanishes at x iff $x \in \mathfrak{h}$.*

It remains to prove that all orbits of X limit onto $\mathfrak{h}$. Since $\dot{H} = \langle\langle y, -[x, [x, y]]\rangle\rangle = \langle\langle \mathrm{ad}_y x, \mathrm{ad}_y x\rangle\rangle$, and $\mathrm{ad}_y x = -\sum_{\mathbf{r} \in \mathbf{\Psi}} x_{\mathbf{r}} \, \langle r, y\rangle \, (\mathrm{e}_{\mathbf{r}} - \mathrm{e}_{-\mathbf{r}})$ one concludes from (5.5)that

$$\dot{H} = -2 \sum_{\mathbf{r} \in \mathbf{\Psi}} x_{\mathbf{r}}^2 \le 0 \tag{5.8}$$

with equality iff $X = 0$. Thus, the ω-limit set of every point x lies in $\mathfrak{h}$, hence $\mathcal{O}_x \cap \mathfrak{h}$. The latter is a finite set and since X is a gradient vector field on $\mathcal{O}_x$, the ω-limit set is a single point. Let $h_0 \in \mathcal{O}_x \cap \mathfrak{h}$ be this point and let $\mathbf{\Psi}_1 = \{\mathbf{r} \ : \ x_{\mathbf{r}} = 0\}$. Let us linearise X about h_0 subject to the condition that $x_{\mathbf{r}} = 0$ for all $\mathbf{r} \in \mathbf{\Psi}_1$:

$$-\delta\dot{h} = 0, \qquad \text{and} \qquad \forall \mathbf{r} \notin \mathbf{\Psi}_1 : \quad \delta\dot{x}_{\mathbf{r}} = \delta x_{\mathbf{r}} \langle r, h_0\rangle , \tag{5.9}$$

where $\delta x, \delta h$ denote variations. It is clear that a necessary condition for stability of h_0 is that $\langle r, h_0\rangle \le 0$ for all $\mathbf{r} \notin \mathbf{\Psi}_1$. A simple argument involving the transitivity of the action of the Weyl group on the Weyl chambers, shows that such an h_0 must exist. This proves

Lemma 5.5. *For each x of the form in equation (5.4), the ω-limit set of x under the gradient flow of $X = \overset{\mathbf{n}}{\nabla} H$ is a point $h_0 \in \mathcal{O}_x \cap \mathfrak{h}$ that satisfies $\langle r, h_0\rangle \le 0$ for all $\mathbf{r} \in \mathbf{\Psi}_1$.*

A similar statement is true for the α-limit set, too. It should be observed that while $\mathfrak{h}$ contains the ω-limit set of every point x, $\mathfrak{h}$ is not a normally hyperbolic manifold. One can see this from (5.9): when $\langle h_0, r\rangle = 0$, one loses hyperbolicity.

Theorem 5.1. For each $\tau \in \mathbf{G}_F$ the Hamiltonian vector field of $\mathbf{H}$ in (3.16), when restricted to the invariant set $g_\tau^{-1}(0)$ (equation 3.41), is semi-conjugate to a vector field of the form of X in (5.4). The semi-conjugacy is provided by the Lax representation in equation (3.15). Lemma 5.5 implies that the ω-limit set of a point $P \in g_\tau^{-1}(0)$ lies in $\mathbf{V}^\perp$. Similarly for the α-limit set of P. Since $\mathrm{k}^{-1}(0) = \cup_{\tau \in \mathbf{G}_F} g_\tau^{-1}(0)$, this proves the theorem. □

6. Uniqueness up to Energy-Preserving Topological Conjugacy

6.1. **Marked Homology Spectrum of a Flow.** Two flows $\phi : M \times \mathbb{R} \to M$ and $\varphi : N \times \mathbb{R} \to N$ are topologically conjugate if there is a homeomorphism $h : M \to N$ such that $h\phi_t = \varphi_t h$ for all $t \in \mathbb{R}$. Let P_ϕ be the set of periodic points of the flow ϕ. For each periodic orbit γ of ϕ, let the homology class of γ be denoted by $\bar{\gamma}$ and its period by $\mathrm{Period}(\gamma)$. Let $\mathrm{P}_{\phi,\bar{\gamma},T}$ denote the union of periodic orbits of ϕ whose homology class is $\bar{\gamma}$ and period is T. The number of connected components of $\mathrm{P}_{\phi,\bar{\gamma},T}$ is denoted by $\beta_{\phi,\bar{\gamma},T}$. The following two definitions originate in Schwartzman's work [28].

Definition 4. *Let* $\mathcal{M}_\phi = \{(\bar{\gamma}, \mathrm{Period}(\gamma), \beta_{\phi,\bar{\gamma},\mathrm{Period}(\gamma)}) \ : \ \gamma \in \mathrm{P}_\phi\}$. *We call* $\mathcal{M}_\phi$ *the* marked homology spectrum *of* ϕ.

The marked homology spectrum is a subset of $H_1(M;\mathbb{Z}) \times \mathbb{R} \times \mathbb{N}$ that is an invariant of topological conjugacy in the following sense: if ϕ and φ are topologically conjugate then

$$(h_* \times id_{\mathbb{R}} \times id_{\mathbb{N}})(\mathcal{M}_\phi) = \mathcal{M}_\varphi,$$

where $h_* : H_1(M;\mathbb{Z}) \to H_1(N;\mathbb{Z})$ is the obvious isomorphism.

Example 6.1. Let $v \in \mathrm{V}_{o,F}$ and define the flow $\phi^v : \Sigma \times \mathbb{R} \to \Sigma$ by

$$\phi^v_t(\mathsf{y},\mathsf{x}) = (\mathsf{y}, \mathsf{x} + tv) \bmod \Delta. \tag{6.1}$$

A point $(\mathsf{y},\mathsf{x}) \in \Sigma$ is periodic of period T for ϕ^v iff $Tv = \ell(u)$ for some $u \in \mathcal{U}^+_F$ and $u \cdot \mathsf{y} = \mathsf{y} \bmod \mathrm{N}_E$.

The map $u : \mathrm{V}_E/\mathrm{N}_E \to \mathrm{V}_E/\mathrm{N}_E$ is a toral automorphism. The number of fixed points of u is, up to sign, the degree of the map $u-1$. The latter is $\det(u-1) = \prod_{\sigma \in \mathbf{G}_E} \sigma(u-1)$, which is also the norm of $u - 1 \in E$. But since $u - 1 \in F$, this norm equals $N_F(u-1)^{[E:F]}$.

Thus,

$$\mathcal{M}_{\phi^v} = \Big\{(\ell(u), T, |N_F(u-1)|^{[E:F]}) \ : \ \forall u \in \mathcal{U}^+_F \ \& \ T \in \mathbb{R}^+ \text{ s.t. } Tv = \ell(u)\Big\} \tag{6.2}$$

Example 6.2. Let $\mathcal{Q} : \mathrm{V}^*_{o,F} \to \mathrm{V}_{o,F}$ be a linear isomorphism and $M = T^*(\mathrm{V}_{o,F}/\mathfrak{L}_F) = \mathrm{V}^*_{o,F} \times \mathrm{V}_{o,F}/\mathfrak{L}_F$. Let $\phi_t(\mathsf{X},\mathsf{x}) = (\mathsf{X}, \mathsf{x} + t\mathcal{Q}\cdot\mathsf{X} \bmod \mathfrak{L}_F)$. Clearly, $\eta^\pm(\mathsf{X},\mathsf{x}) = \{\pm\mathcal{Q}\cdot\mathsf{X}\}$ for all $(\mathsf{X},\mathsf{x}) \in M$.

Let $\mathcal{V}_1 = \{(\mathsf{X},\mathsf{x}) \in M \ : \ \langle \mathcal{Q}\cdot\mathsf{X}, \mathsf{X}\rangle = 1\}$ be the unit-sphere bundle, $\phi^1 = \phi|\mathcal{V}_1$ and $|m|_\mathcal{Q} = \sqrt{|\langle \mathcal{Q}^{-1}m, m\rangle|}$ for all $m \in \mathrm{V}_{o,F}$. The marked homology spectrum of ϕ^1 is easily seen to equal

$$\mathcal{M}_{\phi^1} = \{(\ell(u), |\ell(u)|_\mathcal{Q}, 1) \ : \ u \in \mathcal{U}^+_F\}. \tag{6.3}$$

Example 6.3. The fibre-bundle structure $\mathrm{V}_E/\mathrm{N}_E \hookrightarrow \Sigma \overset{\mathrm{p}}{\twoheadrightarrow} \mathrm{V}_{o,F}/\mathfrak{L}_F$ allows one to pullback the unit-sphere bundle $\mathcal{V}_1$ and the flow ϕ^1 of the previous example. Let φ^1 be the pulled-back flow on $\mathrm{p}^*\mathcal{V}_1$. The previous two examples show that the marked homology spectrum of φ^1 is

$$\mathcal{M}_{\phi^1} = \{(\ell(u), |\ell(u)|_\mathcal{Q}, |N_F(u-1)|^{[E:F]}) \ : \ u \in \mathcal{U}^+_F\}. \tag{6.4}$$

The marked homology spectrum is especially interesting because it contains information about both the quadratic form restricted to the Dirichlet lattice, and it contains information about the periodic points of the toral automorphisms $u : \mathrm{V}_E/\mathrm{N}_E \to \mathrm{V}_E/\mathrm{N}_E$ for $u \in \mathcal{U}^+_F$. In [11], this extra information about the fixed points of the toral automorphisms was not noticed. It turns out that this information is extremely important.

6.2. **Asymptotic Homology of a Flow.** Let $\pi : \hat{M} \to M$ be the universal abelian covering of M. The flow ϕ is covered by a flow $\hat{\phi} : \hat{M} \times \mathbb{R} \to \hat{M}$. Let $F \subset \hat{M}$ be a fundamental domain for the group of deck transformations $\mathrm{Deck}(\pi)$. For each $p \in M$, choose $\hat{p} \in F \cap \pi^{-1}(p)$. For each t there is a $g \in \mathrm{Deck}(\pi)$ such that $\hat{\phi}_t(p) \in g.F$; let $g_t(p)$ be one such element and let $\frac{1}{t} g_t(p) \in \mathrm{Deck}(\pi) \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that $\mathrm{Deck}(\pi) \otimes_{\mathbb{Z}} \mathbb{R} \simeq H_1(M; \mathbb{R})$.

Definition 5. *Let*

$$\eta_\phi(p) := \bigcap_{T \geq 0} \overline{\left\{ \frac{1}{t} g_t(p) \ : \ t \geq T \right\}}$$

be the asymptotic homology *of $p \in M$. Let $\eta^\pm_\phi = \eta_{\phi^\pm}$ where $\phi^\pm_t = \phi_{\pm t}$.*

One can show that $\eta_\phi(p)$ is independent of the choice of representatives and if M is compact then $\eta_\phi(p)$ is non-empty for all p. It is also clear that if there is a semi-conjugacy h with $h \circ \phi = \varphi \circ h$, then $h_* \eta^\pm_\phi(p) = \eta^\pm_\varphi(h(p))$.

Lemma 6.1. *Let $\mathbf{H}$ be a Hamiltonian defined by equation 3.16, and let $\varphi : T^*\Sigma \times \mathbb{R} \to T^*\Sigma$ be its Hamiltonian flow. Let $\mathfrak{U}_\tau = \{g_\tau \neq 0\}$ for each $\tau \in \mathbf{G}_F$. If $P \in \mathfrak{U}_\tau$, then*

$$\langle \eta^\pm_\varphi(P), \hat{\tau} \rangle \leq 0.$$

Remark. This lemma is very close in spririt to lemma 5.5.

Proof. Let $\hat{P} = (\mathsf{Y}, \mathsf{y} + \mathrm{N}_E, \mathsf{X}, \mathsf{x}) \in \hat{\mathfrak{U}}_\sigma$ and let $P = \Pi(\hat{P})$, *c.f.* (2.15). Since $g_\tau(\hat{P}) \neq 0$, $\mathsf{Y}_\tau \neq 0$. If $v \in \eta^\pm_\varphi(P)$, then there is a sequence $T_k \to \pm\infty$ such that

$$v = \lim_{k \to \infty} \frac{1}{|T_k|} (\mathsf{x}(T_k) - \mathsf{x}(0)),$$

where $\hat{\varphi}_t(\mathsf{y} + \mathrm{N}_E, \mathsf{Y}, \mathsf{X}, \mathsf{x}) = (\mathsf{Y}(t), \mathsf{y}(t) + \mathrm{N}_E, \mathsf{X}(t), \mathsf{x}(t))$ and $\hat{\varphi}_t$ is the lift of φ_t to $T^*\hat{\Sigma}$. Thus:

$$\langle v, \hat{\tau} \rangle = \lim_{k \to \infty} \frac{1}{|T_k|} \langle \mathsf{x}(T_k), \hat{\tau} \rangle.$$

On the other hand $\hat{\mathbf{H}}$ and g_τ are first integrals of $\hat{\varphi}_t$. Inspection of equation 3.16 shows that $\hat{\mathbf{H}}(\hat{P}) \geq g_\tau^{b_\tau/2} \exp(b_\sigma \langle \mathsf{x}(T), \hat{\tau} \rangle)$ for all T. Since $b_\sigma, b_\tau > 0$ and $g_\tau \neq 0$, this inequality implies that

$$\frac{1}{|T_k|} \langle \mathsf{x}(T_k), \hat{\tau} \rangle \leq \frac{1}{|T_k| b_\sigma} \left(\ln \hat{\mathbf{H}} - \frac{b_\tau}{2} \ln g_\tau \right) \stackrel{k \to \infty}{\longrightarrow} 0.$$

Since $v \in \eta^\pm_\varphi(P)$ was arbitrary, this proves the lemma. $\square$

As noted above, the fibre bundle structure $\mathrm{V}_E/\mathrm{N}_E \hookrightarrow \Sigma \stackrel{\mathrm{p}}{\twoheadrightarrow} \mathrm{V}_{o,F}/\mathfrak{L}_F$ of Σ induces the sub-bundle $\mathbf{V} = \ker \mathrm{dp} \subset T\Sigma$ and its annihilator $\mathbf{V}^\perp \subset T^*\Sigma$. The sub-bundle $\mathbf{V}^\perp$ is the intersection of $\mathfrak{Z}_\tau = g_\tau^{-1}(0)$ over all $\tau \in \mathbf{B}_F$; it is also isomorphic to $\Sigma \times \mathrm{V}^*_{o,F}$.

Lemma 6.2. *Let $\mathbf{H}_1, \mathbf{H}_2$ be defined by equation 3.16 with root bases Ψ_1, Ψ_2 . If $h : T^*\Sigma \to T^*\Sigma$ conjugates their Hamiltonian flows, then*

$$h(\mathbf{V}^\perp) = \mathbf{V}^\perp.$$

Proof. Let U be the set of points in $\mathbf{V}^\perp$ that are mapped out of $\mathbf{V}^\perp$ under h. Since $P \notin \mathbf{V}^\perp$ iff $\exists \tau \in \mathbf{B}_F$ such that $g_\tau(P) \neq 0$, one sees that

$$U = h^{-1}(\cup_{\tau \in \mathbf{B}_F} \mathfrak{U}_\tau) \cap \mathbf{V}^\perp.$$

It suffices to prove that U is empty, since a symmetric argument applies to h^{-1}. Therefore, it suffices to prove that $U_\tau = h^{-1}(\mathfrak{U}_\tau) \cap \mathbf{V}^\perp$ is empty for all τ. Since $\mathfrak{U}_\tau$ is open, U_τ is an open subset of $\mathbf{V}^\perp$, so to prove that it is empty, it suffices to show that U_τ is nowhere dense. As noted above, $\mathbf{V}^\perp$ is naturally isomorphic to $\Sigma \times V_o^*$. Let $\pi_o : \mathbf{V}^\perp \to V_o^*$ denote the projection onto the second factor. Clearly, π_o is an open map and $\pi_o(P) = \mathsf{X}$ where $P = \Pi(0, \mathsf{y}, \mathsf{X}, \mathsf{x}) \in \mathbf{V}^\perp$. It suffices to show that $\pi_o(U_\tau)$ lies in a hyper-plane to prove the lemma.

Let φ^i be the Hamiltonian flow of $\mathbf{H}_i$, and $\mathcal{Q}_i$ the quadratic form used to define $\mathbf{H}_i$ (Equation 3.16). If $P \in U_\tau$, then $P \in \mathbf{V}^\perp$ so

$$\eta^\pm_{\varphi^1}(P) = \{\pm \mathcal{Q}_1 \cdot \mathsf{X}\},$$

while $h(P) \in \mathfrak{U}_\tau$, so from the previous lemma

$$\langle \eta^\pm_{\phi^2}(h(P)), \hat{\tau} \rangle \leq 0.$$

Since $\varphi^2_t h = h \varphi^1_t$,

$$\eta^\pm_{\varphi^2}(h(P)) = h_* \eta^\pm_{\varphi^1}(P)$$

which implies that

$$\pm \langle h_* \mathcal{Q}_1 \mathsf{X}, \hat{\tau} \rangle \leq 0.$$

Therefore, $\langle h_* \mathcal{Q}_1 \mathsf{X}, \hat{\tau} \rangle$ vanishes. Since $h_* \mathcal{Q}_1$ is non-degenerate, $\mathsf{X} = \pi_o(P)$ lies in a fixed hyper-plane. Thus, $\pi_o(U_\tau)$ lies in a hyper-plane. Since π_o is an open map, U_τ is empty. ☐

Remark 6.1. Lemmas 6.1 and 6.2 can be reformulated and shown to hold in much greater generality. Let Σ_A be defined as in 1.1 and let $H : T^*\Sigma_A \to \mathbb{R}$ be a smooth, fibre-wise convex hamiltonian that is left-invariant. Left-invariance implies that H enjoys the integral f (1.5). In particular, if one defines the function $\gamma_i(P) = |p_{y_i} \exp(\langle \ell_i, x \rangle)|$, then the properness of H implies that there is a function $c = c(H)$ such that $0 \leq \gamma_i(P) \leq c(H(P))$ for all $P \in T^*\Sigma_A$. The proof of lemma 6.2 applies to show that if $\gamma_i(P) \neq 0$, then $\langle \ell_i, v \rangle \leq 0$ for all $v \in \eta^\pm(P)$. This implies that the asymptotic homology of a point P with $\prod_i \gamma_i(P) \neq 0$ is trivial and that a topological conjugacy of two such hamiltonian flows must map $\mathbf{V}^\perp$ to itself.

Definition 6. *A homeomorphism $h : T^*\Sigma \to T^*\Sigma$ is* energy-preserving *if $h(\{\mathbf{H}_1 = \frac{1}{2}\}) = \{\mathbf{H}_2 = \frac{1}{2}\}$.*

We use the notation of Lemma 6.2 and its proof:

Theorem 6.3. *Let $\mathbf{H}_1, \mathbf{H}_2$ be defined by Equation 3.16 corresponding to root bases Ψ_1, Ψ_2. If $h \in \mathrm{Homeo}(T^*\Sigma)$ is an energy-preserving conjugacy of φ^1 with φ^2, then*

(1) *$h_* : H_1(T^*\Sigma) \to H_1(T^*\Sigma)$ induces automorphisms of $\mathfrak{L}_F$ and $\mathcal{U}_F^+$ such that the following commutes*

$$\begin{array}{ccc} \mathcal{U}_F^+ & \xrightarrow{\alpha} & \mathcal{U}_F^+ \\ \downarrow \ell & & \downarrow \ell \\ \mathfrak{L}_F & \xrightarrow{f} & \mathfrak{L}_F; \end{array} \qquad (*)$$

(2) *f is an isometry of $(\mathfrak{L}_F, \mathcal{Q}_2)$ with $(\mathfrak{L}_F, \mathcal{Q}_1)$;*

(3) *α preserves the number of fixed points of $u \in \mathcal{U}_F^+$ acting on $\mathrm{V}_E/\mathrm{N}_E$:*

$$|N_F(\alpha(u) - 1)| = |N_F(u - 1)| \qquad \forall u \in \mathcal{U}_F^+.$$

Proof. (1) The map h_* on H_1 induces an automorphism f of $\mathfrak{L}_F$. The isomorphism ℓ allows the definition of α as an automorphism of $\mathcal{U}_F^+$ and shows that (*) commutes.
(2) Let $\mathcal{V}_i = \mathbf{V}^\perp \cap \mathbf{H}_i^{-1}(\frac{1}{2})$. Since h is energy preserving, Lemma 6.1 implies that $h(\mathcal{V}_1) = \mathcal{V}_2$. Let $\varphi^i|\mathcal{V}_i$ be denoted by Φ^i and let $h|\mathcal{V}_1$ continue to be denoted by h. Examples 6.1 and 6.1 show that

$$\mathcal{M}_{\Phi^i} = \{(\ell(u), |\ell(u)|_{\mathcal{Q}_i}, |N_F(u-1)|^{[E:F]}) \ : \ u \in \mathcal{U}_F^+, u \neq \pm 1\}$$

for $i = 1, 2$.
(3) Finally, by hypothesis $h\Phi^1$ equals $\Phi^2 h$, so from the identity $\mathcal{M}_{\Phi^2} = (h_* \times id_{\mathbb{R}} \times id_{\mathbb{N}})\mathcal{M}_{\Phi^1}$ one sees that

$$|\ell(u)|_{\mathcal{Q}_1} = |f \circ \ell(u)|_{\mathcal{Q}_2} = |\ell(\alpha(u))|_{\mathcal{Q}_2}, \tag{6.5}$$

$$|N_F(u-1)| = |N_F(\alpha(u) - 1)| \tag{6.6}$$

for all $u \in \mathcal{U}_F^+$. Equation (6.5) shows that f is an isometry, while equation (6.6) shows that α preserves the number of fixed points.

□

Let us dualise Theorem 6.3. Let ϕ_i be a linear isomorphism $\mathrm{V}_{o,F}^* \to \mathfrak{h}_i^*$ induced by a bijection $\rho_i : \mathbf{B}_F \to \Psi$ (see Definition 3). The norms $|\cdot|_{\mathcal{Q}_i}$ on $\mathfrak{L}_F$ are equivalent modulo $\mathrm{Aut}(\mathfrak{L}_F)$ iff the dual norms $|\cdot|_{\mathcal{Q}_i}^*$ on $\mathfrak{L}_F^*$ are equivalent modulo $\mathrm{Aut}(\mathfrak{L}_F^*)$. Since, by Theorem 3.2, there is a $c_i \in \mathbb{N}$ such that $|\mathsf{X}|_{\mathcal{Q}_i}^* = c_i^{-1}\sqrt{\langle\langle \phi_i(\mathsf{X}), \phi_i(\mathsf{X})\rangle\rangle_i}$, Theorem 6.3 implies

Corollary 6.4. *If φ^1 and φ^2 are topologically conjugate by an energy-preserving homeomorphism, then there exists $\mu \in \mathrm{Isom}(\mathfrak{h}_2^*; \mathfrak{h}_1^*)$ and $g = f^* \in \mathrm{Aut}(\mathfrak{L}_F^*)$ such that*

$$\mu = \frac{c_2}{c_1} \times \phi_1 g \phi_2^{-1}. \tag{6.7}$$

Remark 6.2. One might attempt to use Corollary 6.4 to try to determine the topological conjugacy classes of Hamiltonian flows. This is the approach taken in [11]. However, this approach leads to some very delicate and long-outstanding issues in transcendence and algebraic-independence theory. This paper skirts those difficulties by employing all the information in the marked homology spectrum.

6.3. **Periodic points of toral automorphisms.** Part (3) of Theorem 6.3 has a useful corollary: the number of period-k periodic points of the automorphisms u and $\alpha(u)$ of the torus $\mathrm{V}_E/\mathrm{N}_E$ are equal for all k. Therefore, their asymptotic rates of growth are equal. Define the function $\mathrm{h} : \mathrm{V}_{o,F} \to \mathbb{R}$ by

$$\mathrm{h}(v) = \sum_{\tau \in \mathbf{B}_F} n_\tau \ \langle \hat{\tau}, v \rangle^+ \tag{6.8}$$

for all $v \in \mathrm{V}_{o,F}$, where $\bullet^+ = \max(\bullet, 0)$. Since the growth rate of the number of period-k periodic points of $u \in \mathcal{U}_F^+$ is $[E:F] \times \mathrm{h}(\ell(u))$, this proves

Lemma 6.5. *Under the hypotheses of Theorem 6.3, the automorphism $f : \mathfrak{L}_F \to \mathfrak{L}_F$ satisfies*

$$\mathrm{h} = \mathrm{h} \circ f.$$

The function h is piecewise linear. One can characterise the sets on which h is linear as follows. For $J \subset \mathbf{B}_F$, let

$$\mathrm{V}_{o,F}^J := \{ v \in \mathrm{V}_{o,F} \ : \ \forall \tau \in J, \ \langle \hat{\tau}, v \rangle > 0 \ \& \ \forall \tau \notin J, \ \langle \hat{\tau}, v \rangle < 0 \ \}. \tag{6.9}$$

Note that if $J = \emptyset$ or $J = \mathbf{B}_F$, then $\mathrm{V}_{o,F}^J$ is empty; otherwise $\mathrm{V}_{o,F}^J$ is an open set that is closed under addition and multiplication by positive scalars. Since $\mathrm{V}_{o,F}^J$ is open and it is closed under positive dilations, it contains balls of arbitrarily large diameter and hence it contains points in $\mathfrak{L}_F$. Therefore,

$$\mathfrak{L}_F^J := \mathfrak{L}_F \cap \mathrm{V}_{o,F}^J \tag{6.10}$$

is a non-empty subset of $\mathfrak{L}_F$, for all $J \subset \mathbf{B}_F$ with $J \neq \emptyset$ and $\mathbf{B}_F$.

To return to h: for all $J \subset \mathbf{B}_F$, define

$$r_J := \sum_{\tau \in J} n_\tau \, \hat{\tau}. \tag{6.11}$$

Lemma 6.6. *The following is true:*

1. *if $v \in \mathrm{V}_{o,F}^J$, then $\mathrm{h}(v) = \langle r_J, v \rangle$;*
2. *if $v \in \mathfrak{L}_F^J$ and $f(v) \in \mathfrak{L}_F^I$, then $r_J = f^* r_I$;*
3. *for each $J \subset \mathbf{B}_F$ with $J \neq \emptyset$ and $\mathbf{B}_F$, there is a unique $I \subset \mathbf{B}_F$ such that $f(\mathfrak{L}_F^J) \subset \mathfrak{L}_F^I$;*
4. *f induces a permutation $\boldsymbol{\pi}$ of the power set $\mathbf{2}^{\mathbf{B}_F}$ that satisfies*
   (a) *$\boldsymbol{\pi}(\emptyset) = \emptyset$ and $\boldsymbol{\pi}(\mathbf{B}_F) = \mathbf{B}_F$;*
   (b) *$\boldsymbol{\pi}(J) = I$ iff $f(\mathfrak{L}_F^J) \subset \mathfrak{L}_F^I$.*

Proof. (1) h may be characterised as: $\mathrm{h}(v) = \max_{I \subset \mathbf{B}_F} \langle r_I, v \rangle$. On the set $\mathrm{V}_{o,F}^J$, this maximum is achieved uniquely at $I = J$. This proves that $\mathrm{h} = r_J$ on $\mathrm{V}_{o,F}^J$.
(2) Let $v \in \mathfrak{L}_F^J$ and $f(v) \in \mathfrak{L}_F^I$. Lemma (6.5) implies that

$$\langle r_J, v \rangle = \mathrm{h}(v) = \mathrm{h}(f(v)) = \langle f^* r_I, v \rangle .$$

It is clear that the set $\mathfrak{L}_F^J \cap f^{-1}(\mathfrak{L}_F^I)$ is an intersection of Zariski dense subsets of $\mathrm{V}_{o,F}$, hence is Zariski dense since it is non-empty. Therefore r_J must equal $f^* r_I$ on $\mathrm{V}_{o,F}$.
(3) Let $v_i \in \mathfrak{L}_F^J$ and assume that $f(v_i) \in \mathfrak{L}_F^{I_i}$. Therefore, from the previous step $f^* r_{I_1} = r_J = f^* r_{I_2}$. Since f is an automorphism $r_{I_1} = r_{I_2}$. Since the map $I \mapsto r_I \ : \ \mathbf{2}^{\mathbf{B}_F} \to \mathrm{V}_{o,F}^*$ is injective except at $\emptyset$ and $\mathbf{B}_F$ (both are sent to 0), one concludes that $I_1 = I_2$.
(4) From step (3), the properties (a-b) uniquely define a map $\boldsymbol{\pi} : \mathbf{2}^{\mathbf{B}_F} \to \mathbf{2}^{\mathbf{B}_F}$ because $\mathrm{V}_{o,F}^J \neq \emptyset$—hence $\mathfrak{L}_F^J \neq \emptyset$ — for all $J \subset \mathbf{B}_F$, $J \neq \emptyset, \mathbf{B}_F$. This map $\boldsymbol{\pi}$ is invertible because f is induced by the homeomorphism h: one can equally start with h^{-1}, get f^{-1} and define $\boldsymbol{\pi}'$ thusly. Step (3) shows that $\boldsymbol{\pi}' = \boldsymbol{\pi}^{-1}$.

□

Let us be more precise about the nature of f. Lemma 6.7 should be compared with [11, Theorem 7], where the Gel'fond conjecture [19] is invoked to obtain the weaker conclusion that $f^* \in \mathrm{Aut}(\mathfrak{L}_F^*) \cap \mathrm{Aut}(\mathrm{V}_{o,F,\mathbb{Q}}^*)$.

Lemma 6.7. *Let $\mathrm{V}_{o,F,\mathbb{Z}}^*$ be the $\mathbb{Z}$-module spanned by $\left\{ n_\tau \hat{\tau}|_{\mathrm{V}_{o,F}} : \tau \in \mathbf{B}_F \right\}$ and $\mathfrak{L}_F^* = \mathrm{Hom}(\mathfrak{L}_F, \mathbb{Z})$. Then*

$$f^* \in \mathrm{Aut}(\mathfrak{L}_F^*) \cap \mathrm{Aut}(\mathrm{V}_{o,F,\mathbb{Z}}^*) \tag{6.12}$$

Proof. Note that $\boldsymbol{\pi}$ is defined by $r_J = f^* r_{\boldsymbol{\pi}(J)}$ for all J. If $J = \boldsymbol{\pi}^{-1}\{\tau\}$, then

$$f^*(n_\tau \hat{\tau}) = r_J \in \mathrm{V}_{o,F,\mathbb{Z}}^*, \tag{6.13}$$

since $r_{\{\tau\}} = n_\tau \hat{\tau}$. On the other hand, if $J = \{\tau\}$, then

$$(f^*)^{-1}(n_\tau \hat{\tau}) = r_{\boldsymbol{\pi}(J)} \in \mathrm{V}_{o,F,\mathbb{Z}}^*.$$

This proves that $f^* \in \mathrm{Aut}(\mathrm{V}_{o,F,\mathbb{Z}}^*)$, and since $f \in \mathrm{Aut}(\mathfrak{L}_F)$, the lemma is proven.

□

Lemma 6.8. *Let $\boldsymbol{\pi} : \mathbf{2}^{\mathbf{B}_F} \to \mathbf{2}^{\mathbf{B}_F}$ be the permutation defined in Lemma 6.6. If $I, J \subset \mathbf{B}_F$ are disjoint sets, then*

$$\boldsymbol{\pi}(I \sqcup J) = \boldsymbol{\pi}(I) \sqcup \boldsymbol{\pi}(J),$$

$\sqcup =$ disjoint union. Consequently, $\boldsymbol{\pi}$ is induced by a permutation of $\mathbf{B}_F$.

Proof. Since $I \cap J = \emptyset$, $r_{I \sqcup J} = r_I + r_J$. Therefore

$$r_{\boldsymbol{\pi}(I \sqcup J)} = (f^*)^{-1} r_{I \sqcup J} = (f^*)^{-1}(r_I + r_J) = r_{\boldsymbol{\pi}(I)} + r_{\boldsymbol{\pi}(J)}. \tag{6.14}$$

Assume that $\boldsymbol{\pi}(I)$ and $\boldsymbol{\pi}(J)$ are not disjoint. Then, there is a $\tau \in \boldsymbol{\pi}(I) \cap \boldsymbol{\pi}(J)$. The coefficient on $\hat{\tau}$ in the right-hand side of (6.14) is therefore $2n_\tau$. The coefficient on $\hat{\tau}$ in the left-hand side of (6.14) is at most n_τ, however. Absurd. Therefore $\boldsymbol{\pi}(I) \cap \boldsymbol{\pi}(J)$ must be empty.

Consider the $\#\mathbf{B}_F + 1$ subsets of $\mathbf{B}_F$ that contain at most 1 element. This is the largest family of pairwise disjoint subsets of $\mathbf{B}_F$. Therefore, $\boldsymbol{\pi}$ must be map this family to itself. Since $\boldsymbol{\pi}(\emptyset) = \emptyset$, $\boldsymbol{\pi}$ maps the singleton sets to singletons. □

Let the permutation of $\mathbf{B}_F$ induced by $\boldsymbol{\pi}$ be denoted by $\boldsymbol{\pi}$, too. Equation (6.13) is thereby simplified to

$$\forall \tau \in \mathbf{B}_F : \quad n_\sigma \, f^* \hat{\sigma} = n_\tau \, \hat{\tau} \qquad \iff \qquad \boldsymbol{\pi}(\tau) = \sigma. \tag{6.15}$$

Intuitively, one wants to say that $\boldsymbol{\pi}$ should not mix up the real and non-real embeddings, so the coefficients on both sides of (6.15) ought to be equal. To prove this, observe that (6.15) implies that

$$\forall u \in \mathcal{U}_F^+ : \quad f \circ \ell(u) = \sum_{\tau \in \mathbf{G}_F} \frac{n_\tau}{n_{\boldsymbol{\pi}(\tau)}} \ln |\tau(u)| \cdot \tau. \tag{6.16}$$

Since $f \in \mathrm{Aut}(\mathfrak{L}_F)$, the right-hand side lies in $\mathfrak{L}_F \subset \mathrm{V}_{o,F}$ for all u. Let $\xi = \sum_{\tau \in \mathbf{G}_F} \frac{n_\tau}{n_{\boldsymbol{\pi}(\tau)}} \hat{\tau} \in \mathrm{V}_F^*$; one sees that $\langle \xi, \ell(u) \rangle = 0$ since $f \circ \ell(u) \in \mathrm{V}_{o,F}$. Since $\mathfrak{L}_F$ spans $\mathrm{V}_{o,F}$, this shows that $\xi \in \mathrm{V}_{o,F}^\perp$. Since

$$\mathrm{V}_{o,F}^\perp = \mathrm{span} \left\{ \tau - \hat{\tau}, \epsilon \; : \; \tau \in \mathbf{G}_F^c, \epsilon = \sum_{\tau \in \mathbf{G}_F} \hat{\tau} \right\}, \tag{6.17}$$

and the coefficients $n_\tau / n_{\boldsymbol{\pi}(\tau)}$ are constant under the involution $\tau \mapsto \bar{\tau}$, one sees that ξ must be a multiple of ϵ. Therefore, $n_\tau / n_{\boldsymbol{\pi}(\tau)}$ must be independent of τ. Since $\boldsymbol{\pi}$ is a permutation, this forces $n_\tau / n_{\boldsymbol{\pi}(\tau)}$ to be identically equal to unity. This proves

Lemma 6.9. *The permutation $\boldsymbol{\pi}$ of $\mathbf{G}_F$ preserves the type of each embedding. In particular,*

$$\forall \tau \in \mathbf{B}_F : \quad f^* \hat{\sigma} = \hat{\tau} \qquad \iff \qquad \boldsymbol{\pi}(\tau) = \sigma. \tag{6.18}$$

Lemma 6.10. *For each $\tau \in \mathbf{B}_F$, there exists a homomorphism $\zeta_\tau : \mathcal{U}_F^+ \to S^1$ such that*

1. *for all $u \in \mathcal{U}_F^+$, $\tau(\alpha(u)) = \zeta_\tau(u) \cdot \sigma(u)$ where $\boldsymbol{\pi}(\sigma) = \tau$;*
2. *ζ_τ maps $\mathcal{U}_F^+$ into $S^1 \cap \mathcal{U}_K$ where K is the normal closure of F.*

Proof. The equation $f(\ell(u)) = \ell(\alpha(u))$ implies, via equation (6.18), that $|\tau(\alpha(u))| = |\sigma(u)|$ when $\sigma = \boldsymbol{\pi}^{-1}(\tau)$. Therefore, there is a unit modulus number $\zeta = \zeta_\tau(u)$ such that $\tau(\alpha(u)) = \zeta \cdot \sigma(u)$. The number ζ is a ratio of numbers in conjugates of F, hence it lies in the smallest field containing all conjugates of F, K. Moreover, one sees that ζ_τ is a ratio of two homomorphisms, hence it is a homomorphism. Finally, since ζ is a ratio of units of K, it is a unit of K. □

6.4. **Strictly Hyperbolic Number Fields.** Lemma 6.10 shows that, if one can force ζ_τ to be trivial, then α is an automorphism of F and $\boldsymbol{\pi}$ is induced by right composition by α^{-1}. One expects that this is always the case: the symmetries of the number field $F/\mathbb{Q}$ ought to appear as symmetries (=topological conjugacies) of the Hamiltonian system, and vice versa. However, when K contains infinite order elements in S^1, it is difficult to say anything meaningful about ζ_τ. This is quite likely related to the fact that if $u \in \mathcal{U}_K \cap S^1$ has infinite order, then the induced automorphism of the torus $\mathrm{V}_K/\mathcal{O}_K$ is *partially hyperbolic*.

Definition 7. *A unit $u \in \mathcal{U}_F$ is* hyperbolic *if none of its conjugates have unit modulus. F is* hyperbolic *if its only non-hyperbolic units are roots of unity. F is* strictly hyperbolic *if its normal closure, K, is hyperbolic.*

In other words, F is hyperbolic iff

$$\#\mathcal{U}_F \cap S^1 < \infty. \tag{6.19}$$

If F is hyperbolic, then $\mathcal{U}_F^+$ acts on the torus $\mathrm{V}_F/\mathrm{N}_F$ as a group of Anosov automorphisms; if F is *strictly hyperbolic*, then the 'closure' of $\mathcal{U}_F^+$, $\mathcal{U}_K^+$, acts on the torus $\mathrm{V}_K/\mathcal{O}_K$ as a group of Anosov automorphisms.

Strict hyperbolicity is a property of the normal closure K: K itself is strictly hyperbolic and so, therefore, are all its subfields. Examples of strictly hyperbolic number fields are legion; there also appear to be many hyperbolic but not strictly hyperbolic number fields.

Examples.
(1) F is *totally real* if all its conjugates are real. In this case, its normal closure is also totally real and so $\mathcal{U}_K \cap S^1 = \{\pm 1\}$. Thus, all totally real number fields are strictly hyperbolic.
(2) Let ζ be a p-th root of unity for some odd prime p. The field $K = \mathbb{Q}(\zeta)$ has the totally real subfield $F = \mathbb{Q}(\zeta + \zeta^{-1})$ of index 2. The Dirichlet theorem on the group of units implies that $\mathcal{U}_F$ is of finite index in $\mathcal{U}_K$. Since F is totally real, K is strictly hyperbolic.
(3) More generally, let $K/\mathbb{Q}$ be a non-real, normal extension of $\mathbb{Q}$. If K has a totally real subfield F of index 2, then, as above, $\mathcal{U}_F$ is a finite-index subgroup of $\mathcal{U}_K$, hence K is strictly hyperbolic.
(4) A penultimate, concrete example: let $F = \mathbb{Q}(a)$ where a is the unique real root of $p(x) = x^3 + 3x - 1$. The discriminant of p is $d = -27 \times 5$, so $\sqrt{d} \notin \mathbb{Q}$, which implies that F is not a normal extension of $\mathbb{Q}$ (p's roots are approximately $0.3222, -0.1611 \pm 1.7544\sqrt{-1}$, which also implies F cannot be normal). Therefore, the normal closure of F is a degree 6 extension K. The group $\mathcal{U}_K^+$ has rank 2 since K has no real embeddings, while a and one of its conjugates are multiplicatively independent units in $\mathcal{U}_K^+$, neither of which lies on S^1. This means that $\mathcal{U}_K \cap S^1$ must be finite, so K and F are strictly hyperbolic.
(5) Let us end with an example of a hyperbolic number field that is not strictly hyperbolic. Let a, b, c be the roots of $p(x) = x^3 + 3x - 1$ where a is the real root as in the previous example. It is clear that $|b| = 1/\sqrt{a}$. Let $E = \mathbb{Q}(\sqrt{a})$, which is a real, degree 6 extension of $\mathbb{Q}$ and let $E' = \mathbb{Q}(\sqrt{b})$ and $E'' = \mathbb{Q}(\sqrt{c})$ be the conjugates of E. It is claimed that E is hyperbolic, that is, if $u \in \mathcal{U}_E$ has a conjugate v of unit modulus, then $u = \pm 1$. To verify this claim, let $u \in E$ have a conjugate of unit modulus. Without loss of generality, this conjugate can be assumed to be some $v \in E'$. Since $\bar{b} = c$, one sees that $\bar{v} \in E''$ and that $v\bar{v} = 1$ implies that $v, \bar{v} \in E' \cap E''$. The field $E' \cap E''$ is of degree $1, 2, 3$ or 6. It cannot be 6, since $E' \neq E''$, so its degree is $1, 2$ or 3. The degree of $E' \cap E''$ cannot be 3 so it must be 1 or 2. If the degree is 1, then the claim is proved; if the degree is 2, then v is a

unit in a complex quadratic number field, hence v is a root of unity. This implies that u is a root of unity in the real field E, hence $u = \pm 1$ as claimed.

On the other hand, the normal closure L of E contains $\sqrt{a}$ and b and therefore the unit modulus number $\eta = b\sqrt{a}$. If η were an n-th root of unity, then $1 = \eta^{4n} = b^{4n}a^{2n}$; but a and b are multiplicatively independent in $K = \mathbb{Q}(a, b, c)$, so $n = 0$. This shows that $\eta \in \mathcal{U}_L \cap S^1$ has infinite order and completes the proof that E is hyperbolic but not strictly hyperbolic.

Let us turn to a theorem which demonstrates the importance of strictly hyperbolic number fields. The choice of the set $\mathbf{B}_F$ involves an arbitrariness which it has been possible to avoid up to this point. To work around this arbitrariness, let the map $\boldsymbol{\pi}$ be extended to a map of $\mathbf{G}_F$ by

$$\boldsymbol{\pi}(\tau) = \overline{\boldsymbol{\pi}(\bar{\tau})} \qquad \forall \tau \notin \mathbf{B}_F. \tag{6.20}$$

Theorem 6.11. *If F is strictly hyperbolic, then there is a $\beta \in \mathrm{Aut}(F/\mathbb{Q})$ such that*

1. *the induced maps* $\mathcal{U}_F/\mathcal{R}_F \underset{\beta}{\overset{\alpha}{\longrightarrow}} \mathcal{U}_F/\mathcal{R}_F$ *coincide;*
2. $\boldsymbol{\pi}(\tau) = \tau \circ \beta^{-1} \qquad \forall \tau \in \mathbf{G}_F$;
3. $f = R_{\beta^{-1}}\big|_{\mathrm{V}_{o,F}}$ *where* $R_\beta : \mathrm{V}_F \to \mathrm{V}_F$ *is the linear transformation induced by precomposition with* $\beta \in \mathrm{Aut}(F/\mathbb{Q})$.

Recall that $\mathcal{R}_F$ is the set of units in $\mathcal{U}_F$ all of whose conjugate lie on S^1. If F is strictly hyperbolic, then $\mathcal{R}_F = \mathcal{U}_F \cap S^1$.

Proof. For the purposes of this proof, it is convenient to extend $\alpha \in \mathrm{Aut}(\mathcal{U}_F^+)$ to an automorphism of $\mathcal{U}_F = \mathcal{U}_F^+ \oplus \mathcal{R}_F$ by extending α as the identity on $\mathcal{R}_F$. The choice of extension of α is immaterial. The extension of α permits the extension of the homomorphism ζ_τ (Lemma 6.10), too. Since F is strictly hyperbolic, all conjugates of $\mathcal{U}_F \cap S^1$ lie in S^1. Since α maps $\mathcal{U}_F \cap S^1$ to itself, this implies that the extended homomorphism ζ_τ maps $\mathcal{U}_F$ into S^1.

Let $\mathcal{U}_F^1 = \cap_{\tau \in \mathbf{B}_F} \ker \zeta_\tau$. Since $\mathcal{U}_K \cap S^1$ is finite, $\ker \zeta_\tau$ is a finite-index subgroup of $\mathcal{U}_F^+$ for all τ; thus $\mathcal{U}_F^1$ is a finite-index subgroup. Lemma 6.10.1 implies that

$$\forall u \in \mathcal{U}_F^1,\ \tau \in \mathbf{B}_F : \qquad \sigma(\alpha(u)) = \tau(u) \qquad \text{where } \boldsymbol{\pi}(\tau) = \sigma. \tag{6.21}$$

This implies that $\sigma(\mathcal{U}_F^1) \subset \tau(\mathcal{U}_F)$; and since σ, τ are injective, the group $\sigma(\mathcal{U}_F^1)$ is a finite-index subgroup of $\tau(\mathcal{U}_F)$. Therefore, $\tau(F) \cap \sigma(F)$ contains elements that are of degree $\deg F$. Thus, the two fields coincide:

$$\forall \tau \in \mathbf{B}_F : \qquad \tau(F) = \sigma(F) \qquad \text{where } \boldsymbol{\pi}(\tau) = \sigma. \tag{6.22}$$

Fix $\sigma, \tau \in \mathbf{G}_F$ with $\boldsymbol{\pi}(\tau) = \sigma$ and define

$$\beta_\sigma := \sigma^{-1} \circ \tau. \tag{6.23}$$

Then, $\beta_\sigma|_{\mathcal{U}_F^1} = \alpha|_{\mathcal{U}_F^1}$ and $\beta_\sigma \in \mathrm{Aut}(F/\mathbb{Q})$. Because $\mathcal{U}_F^1$ is a finite-index subgroup of $\mathcal{U}_F$ it contains elements of degree $\deg F$. It is clear that two automorphisms of $F/\mathbb{Q}$ which coincide on an element of degree $\deg F$, coincide on F. Therefore, there is a single $\beta \in \mathrm{Aut}(F/\mathbb{Q})$ such that $\beta_\sigma = \beta$ for all σ.

Moreover, from (6.22) and the remarks in the first paragraph, one knows that ζ_σ maps $\mathcal{U}_F$ into $\mathcal{U}_F \cap S^1$. Consequently, $\sigma^{-1} \circ \zeta_\sigma$ maps $\mathcal{U}_F$ into $\mathcal{U}_F \cap S^1$. Since

$$\alpha(u) = \sigma^{-1}(\zeta_\sigma(u)) \cdot \beta(u) \qquad \forall u \in \mathcal{U}_F, \tag{6.24}$$

one sees that the invariance of S^1 under embeddings of F implies that the induced maps $\mathcal{U}_F/\mathcal{U}_F \cap S^1 \underset{\beta}{\overset{\alpha}{\longrightarrow}} \mathcal{U}_F/\mathcal{U}_F \cap S^1$ are equal. Since $\mathcal{U}_F^+$ is a non-canonical lifting of $\mathcal{U}_F/\mathcal{U}_F \cap S^1$ to $\mathcal{U}_F^+$, one may declare that $\alpha = \beta|_{\mathcal{U}_F^+}$.

Finally, equation (6.23) implies that

$$\forall \tau \in \mathbf{B}_F : \qquad \boldsymbol{\pi}(\tau) = \sigma \qquad \Longleftrightarrow \qquad \sigma = \tau \circ \beta^{-1}. \tag{6.25}$$

Therefore, the way in which $\boldsymbol{\pi}$ is extended to $\mathbf{G}_F$ shows that $\boldsymbol{\pi}(\tau) = \tau\circ\beta^{-1}$ for all $\tau \in \mathbf{G}_F$. This proves the theorem. $\square$

6.5. **Topological conjugacy classes.** The results of the previous section afford the opportunity to classify the Hamiltonian flows of the Bogoyavlenskij-Toda-type Hamiltonians (equation 3.16) up to topological conjugacy — at least in some situations.

Standing Hypothesis: For the remainder of section 6, unless explicitly stated otherwise, it is assumed that F is a strictly hyperbolic number field.

6.5.1. *Root bases and Dynkin Diagrams.* Recall that for each root basis Ψ there is a labelled graph $\Gamma(\Psi)$, called the Dynkin diagram, whose vertices are the points of Ψ. A pair of distinct vertices r, s have $4\langle\langle r, s\rangle\rangle^2/|r|^2|s|^2$ edges connecting them, and if $|r| > |s|$ then there is an arrow pointing from r to s. The vertex r has the label ω_r. The Coxeter diagram is obtained from the Dynkin diagram by erasing the labels and arrows. If Ψ is a root system other than $A_{2n}^{(2)}$, then one says that a permutation $\rho \in S(\Psi)$ is an *automorphism* of the Dynkin diagram $\Gamma(\Psi)$ iff the permutation leaves the Dynkin diagram unchanged with the exception of the numbering of the roots. $\mathrm{Aut}(\Gamma(\Psi))$ is the automorphism group of $\Gamma(\Psi)$. Note that $\rho \in \mathrm{Aut}(\Gamma(\Psi))$ iff $\omega_r = \omega_{\rho(r)}$ and $\langle\langle r, s\rangle\rangle = \langle\langle \rho(r), \rho(s)\rangle\rangle$ for all $r, s \in \Psi$. For the root system $A_{2n}^{(2)}$, one defines the automorphism group, $\mathrm{Aut}(\Gamma(A_{2n}^{(2)}))$, to be the group generated by the permutation that maps $r_j \to r_{n+2-j}$ for all j (see figures 8–9).

In the above discussion, one sees that the Cartan-Killing form must be normalised. We adopt the following normalisation: the shortest roots of $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$ have length $1/\sqrt{2}$; all other root systems' shortest roots have unit length. This normalisation implies that the longest root(s) of $G_2^{(1)}$ and $D_4^{(3)}$ have length $\sqrt{3}$, while all other root systems' longest roots have length $\sqrt{2}$.

Proposition 6.1. *Assume that $F/\mathbb{Q}$ is strictly hyperbolic and $\#\mathbf{B}_F > 2$. Let $\rho_i \in \mathfrak{B}_i = \mathfrak{B}(\mathbf{B}_F, \boldsymbol{\Psi}_i)$ be bijections and let $\mathbf{H}_i$ be defined by Equation 3.16 with Hamiltonian flow φ^i. If there is an energy-preserving conjugacy of φ^1 with φ^2, then μ (equation 6.7) induces $\nu : \boldsymbol{\Psi}_2 \to \boldsymbol{\Psi}_1$ which is an isomorphism of Coxeter diagrams. Thus,*

Case A: *if* $\boldsymbol{\Psi}_1 \notin \left\{C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}\right\}$, *then*
- (a) $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2$*;*
- (b) *the constants $c_1 = c_2$ in the definition of Φ_i (theorem 3.2);*
- (c) $\nu \in \mathrm{Aut}(\Gamma(\Psi))$.

Case B: *if* $\boldsymbol{\Psi}_1 \in \left\{C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}\right\}$, *then*
- (a) $\boldsymbol{\Psi}_2 \in \left\{C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}\right\}$*;*
- (b) *the constants c_1 and c_2 in the definition of Φ_i (theorem 3.2) are related by the following diagram:*

$$\begin{tikzcd} & C_n^{(1)} \arrow[loop above, "\times 1"] \arrow[dl, "\times 1"'] & \\ A_{2n}^{(2)} \arrow[loop left, "1\times"] \arrow[rr, "\times \frac{1}{2}"] & & D_{n+1}^{(2)} \arrow[loop right, "\times 1"] \arrow[ul, "\times 2"'] \end{tikzcd}$$

where the factor $\star$ yields $c_2 = c_1 \times \star$.

(c) *if* $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$*, then* $\nu \in \mathrm{Aut}(\Gamma(\Psi))$.

Proof. From equation (6.15) and corollary 6.4, one knows that μ in equation (6.7) maps a root in $\mathfrak{h}_2^*$ to a non-zero multiple of a root in $\mathfrak{h}_1^*$. Let ν denote the induced bijection $\mathbf{\Psi}_2 \to \mathbf{\Psi}_1$. Since μ maps $r \in \Psi_2$ to a scalar multiple of the root $\nu(r) \in \Psi_1$, one can write $\mu(r) = a_r \nu(r)$ for some coefficients a_r.

To determine the coefficients a_r, note that $\phi_i(\tau) = n_\tau^{-1} \omega_{r_i} r_i$ where $\rho_i(\tau) = r_i$. One computes that

$$\begin{aligned}
\mu(r) &= \frac{c_2}{c_1} \times \phi_1 \circ f^* \circ \phi_2^{-1}(r) && \text{by definition of } \mu,\\
&= \frac{c_2}{c_1} \times \frac{n_\tau}{\omega_r} \times \phi_1 \circ f^*(\hat{\tau}) && \text{where } \rho_2(\tau) = r,\\
&= \frac{c_2}{c_1} \times \frac{n_\sigma}{\omega_r} \times \phi_1(\hat{\sigma}) && \text{where } n_\sigma\, \hat{\sigma} = n_\tau\, f^* \hat{\tau},\\
&= \frac{c_2}{c_1} \times \frac{\omega_{\nu(r)}}{\omega_r} \times \nu(r) && \text{where } \rho_1(\sigma) = s, \phi_1(\hat{\sigma}) = \frac{\omega_s}{n_\sigma} s \text{ and } \nu(r) = s.
\end{aligned}$$

Therefore, since μ is an isometry

$$\langle\langle r, s \rangle\rangle_2 = \left(\frac{c_2}{c_1}\right)^2 \times \frac{\omega_{\nu(r)}\, \omega_{\nu(s)}}{\omega_r\, \omega_s} \times \langle\langle \nu(r), \nu(s) \rangle\rangle_1 \qquad \forall r, s \in \Psi_2. \tag{6.26}$$

This implies that the Coxeter diagrams of $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ are isomorphic. Inspection of figures 8–9 shows that $\{\mathbf{\Psi}_1, \mathbf{\Psi}_2\}$ is contained in one of the following sets:

$$\begin{array}{lll}
\left\{A_n^{(1)}\right\} & \left\{B_n^{(1)}, A_{2n-1}^{(2)}\right\} & \left\{C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}\right\}\\
\left\{D_n^{(1)}\right\} & \left\{G_2^{(1)}, D_4^{(3)}\right\} & \\
\left\{E_n^{(1)}\right\}_{n=6,7,8} & \left\{F_4^{(1)}, E_6^{(1)}\right\} &
\end{array} \tag{6.27}$$

Case A. Suppose that we are in one of the cases covered by the first two columns of (6.27). Note that since $c_i \in \mathbb{N}$, one obtains that

$$\frac{|r|_2}{|\nu(r)|_1} = \frac{c_2}{c_1} \times \frac{\omega_{\nu(r)}}{\omega_r} \in \mathbb{Q} \qquad \forall r \in \Psi_2. \tag{6.28}$$

The possible ratios of root lengths is $1, \sqrt{2}$ and $\sqrt{3}$ or ratios of the these three numbers. Therefore, the ratios are always 1. This proves that ν is itself an isometry. Therefore $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$.

To prove that $c_1 = c_2$, note that since ν is a permutation of $\mathbf{\Psi}$, it has finite order. If $r \in \Psi$ is a fixed point of ν^k for some $k \geq 1$, then equation 6.28 implies that

$$1 = \left(\frac{c_2}{c_1}\right)^k \times \frac{\omega_{\nu(r)}}{\omega_r} \times \frac{\omega_{\nu^2(r)}}{\omega_{\nu(r)}} \times \cdots \times \frac{\omega_{\nu^k(r)}}{\omega_{\nu^{k-1}(r)}}, \tag{6.29}$$

so $1 = \left(\frac{c_2}{c_1}\right)^k$.

Case B. In this case, following equation (6.28), the rational ratios that are possible are $1, 2$ or $1/2$. A simple check shows that the natural Coxeter isomorphisms $C_n^{(1)} \to A_{2n}^{(2)}$, $A_{2n}^{(2)} \to D_{n+1}^{(2)}$ and $D_{n+1}^{(2)} \to C_n^{(1)}$ satisfy this constraint with c_2/c_1 equal to 1, $1/2$ and 2 respectively (see figure 3). These Coxeter isomorphisms are unique up to the action of the automorphism groups. If $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$, then these considerations imply that ν is a Dynkin diagram automorphism and $c_2 = c_1$.

$\square$

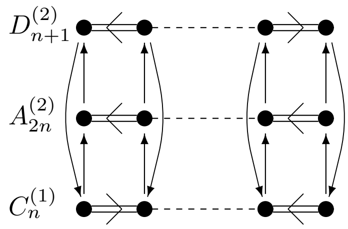


FIGURE 3. The natural Coxeter isomorphisms $\nu : \mathbf{\Psi}_2 \to \mathbf{\Psi}_1$.

Remark 6.3. In [11, Lemma 24] there is a simpler version of theorem 6.1. It is assumed there that $\mathbf{\Psi}_i \neq A^{(2)}_{2n}$ for both i and $c_2 = c_1$. In this case, ν must be an automorphism of the Dynkin diagram.

7. Topological Entropy

In the proof of the complete integrability, Theorem 4.1, one sees that the singular set of the algebra $\mathsf{L} + \mathsf{R}$ is the union of $\mathbf{L}^{-1}(\mathfrak{R}^c)$ and $\mathfrak{Z} = \mathrm{k}^{-1}(0)$. Theorem 5.1 shows that the non-wandering set of the Hamiltonian flow φ of $\mathbf{H}$, restricted to the invariant set $\mathfrak{Z} = \mathrm{W}^{\pm}(\mathbf{V}^{\perp})$, is $\mathbf{V}^{\perp}$. What happens on the other part of the singular set, $\mathbf{L}^{-1}(\mathfrak{R}^c)$?

7.1. **The $A^{(1)}_n$ lattice.** Thanks to the work of Foxman and Robbins [15, 16], this question is answerable for the $A^{(1)}_n$ lattice.

Theorem 7.1. *Let $\mathbf{\Psi} = A^{(1)}_n$ and $\mathbf{H}$ be a Bogoyavlenskij-Toda-like Hamiltonian defined in equation 3.16. Then $\mathbf{L}^{-1}(\mathfrak{R}^c)$ is stratified by symplectic manifolds that are invariant under the Hamiltonian flow of $\mathbf{H}$. Moreover, $\mathbf{H}$ restricted to each stratum is completely integrable.*

Proof. Let $h = 2\kappa \in C^{\infty}(\mathrm{e}+\mathcal{L}^*_0+\mathcal{L}^*_{+1})$ be the Cartan-Killing form, so that $\mathbf{H} = \mathbf{L}^* h$ (equation 3.16). Foxman and Robbins [15, 16] proved that h admits action-angle variables with singularities, which means that for each point $p \in \mathrm{e}+\mathcal{L}^*_0+\mathcal{L}^*_{+1}$, there are coordinates (x, y, u, v) on a neighbourhood of p in $\mathcal{O}^R_p$ such that the canonical symplectic form on $\mathcal{O}_p$ and h take the form

$$\sum_{i=1}^{k} \mathrm{d}x_i \wedge \mathrm{d}y_i + \sum_{i=k+1}^{n} \mathrm{d}u_i \wedge \mathrm{d}v_i, \qquad h = h(x, \rho)$$

$$\rho_i = u_i^2 + v_i^2,\ i = k+1, \dots n,$$

where $k = 0, \dots, n$ is the co-rank of the singularity (and if $k = n$, then there is no singularity). The set $\rho = 0$ is an invariant symplectic submanifold and h is completely integrable on this submanifold.

Let $X_k \subset \mathrm{e} + \mathcal{L}^*_0 + \mathcal{L}^*_{+1}$ be one of these symplectic sub-manifolds of dimension $2k$ and co-dimension $2l$ where $n = k + l$. Let $Y_k = \mathbf{L}^{-1}(X_k)$.

Because $\mathbf{L}|\mathrm{k}^{-1}(\mathbb{R} - 0)$ is a submersion onto $\mathrm{e} + \mathcal{L}^*_0 + \mathcal{L}^*_{+1}$, Y_k is a submanifold of $T^*\Sigma$ of co-dimension $2l$. Moreover, since $\mathbf{L}$ is a Poisson submersion, Y_k is also a symplectic submanifold. Since X_k is invariant under the hamiltonian flow of h, Y_k is similarly invariant.

From the above description of the singular action-angle variables, the algebra $\mathsf{L}|Y_k$ is equal to $\mathbf{L}^* Z^{\infty}(\mathcal{L}^*)|X_k$, which contains k functionally independent elements. On the other hand, the algebra $\mathsf{R}|Y_k$ contains $\dim \mathrm{V}_E$ functionally independent elements. Therefore, in total, there are $k + \dim \mathrm{V}_E$ functionally independent integrals of $\mathbf{H}$ at each point of Y_k. Since $\dim Y_k = 2(k + \dim \mathrm{V}_E)$, this proves the complete integrability of $\mathbf{H}|Y_k$, which proves the theorem. □

Remark 7.1. It is clear from the proof that $\mathbf{H}|\mathfrak{U}$ is completely integrable with singular action-angle variables. This is the mildest kind of singularity that a completely integrable may have. It is a stark contrast with the sort of singularity that develops along $\mathfrak{Z} = \mathrm{k}^{-1}(0)$.

It is natural to conjecture that the Foxman-Robbins theorem is true for all Bogoyavlenskij-Toda lattices.

Corollary 7.2. *Let $\mathbf{\Psi} = A_n^{(1)}$ and $\mathbf{H}$ be a Toda-like hamiltonian defined in equation 3.16. The topological entropy of $\varphi_1|\mathbf{H}^{-1}(\frac{1}{2})$, the time-1 map of the hamiltonian flow of $\mathbf{H}$, equals*

$$h_{top} = \frac{[E:F]}{c} \times \sqrt{\text{floor}\left(\frac{n+1}{2}\right)} \tag{7.1}$$

where $n = dim\,\mathrm{V}_{o,F}$ and $c \in \frac{1}{2}\mathbb{Z}^+$ as in theorem 3.2.

Proof. Since φ admits singular action-angle variables on $\mathrm{k}^{-1}(\mathbb{R}-0)$, we see that the topological entropy of φ is generated entirely in $\mathrm{k}^{-1}(0) = \mathrm{W}^{\pm}(\mathbf{V}^{\perp})$. The non-wandering set of $\varphi|\mathrm{W}^{\pm}(\mathbf{V}^{\perp})$ is $\mathbf{V}^{\perp}$ by theorem 5.1. Thus

$$h_{top}(\varphi|\mathbf{H}^{-1}(\frac{1}{2})) = h_{top}(\varphi|\mathbf{V}_1^{\perp}) = \frac{[E:F]}{c} \times \sqrt{\text{floor}\left(\frac{n+1}{2}\right)} \quad \text{by table 5.} \tag{7.2}$$

□

7.2. **The remaining Bogoyavlenskij-Toda lattices.** As in [11, Section 3], the universal covering space $\tilde{\Sigma} = \mathrm{V}_E \times \mathrm{V}_{o,F}$ admits the structure of a solvable Lie group. The element $v \in \mathrm{V}_{o,F}$ acts by right translation by the one-parameter subgroup

$$\tilde{\phi}_t^v(\mathsf{y},\mathsf{x}) = (\mathsf{y} + t\cdot v, \mathsf{x}). \tag{7.3}$$

This flow descends to a flow ϕ^v on Σ. As in [11, Lemma 12],

$$h_{top}(\phi^v) = [E:F] \times \sum_{\tau \in \mathbf{B}_F} n_\tau \, \langle \hat{\tau}, v\rangle^+ \tag{7.4}$$

where $u^+ = \max\{u, 0\}$.

Let $\mathbf{H}$ be defined by equation (3.16) and let

$$\mathbf{V}_1^{\perp} = \mathbf{V}^{\perp} \cap \mathbf{H}^{-1}(\frac{1}{2}) \tag{7.5}$$

where $\mathbf{V}^{\perp}$ is defined in section 5. If $v = \mathcal{Q}\cdot\mathsf{X}$ with $\mathsf{X} \in \mathrm{V}_{o,F}^*$, and $\langle \mathcal{Q}\cdot\mathsf{X}, \mathsf{X}\rangle = 1$, then $\Delta\cdot(\mathsf{y},\mathsf{x},0,\mathsf{X}) \in \mathbf{V}_1^{\perp}$. The topological entropy of the Hamiltonian flow φ of $\mathbf{H}$ is therefore equal to

$$\begin{aligned}
&\frac{1}{[E:F]} \times h_{top}(\varphi|\mathbf{V}_1^{\perp}) \\
&= \max_{\mathsf{X}:\langle \mathcal{Q}\cdot\mathsf{X},\mathsf{X}\rangle = 1} \sum_{\tau\in\mathbf{B}_F} n_\tau \, \langle \hat{\tau}, \mathcal{Q}\cdot\mathsf{X}\rangle^+ \\
&= \max_{\mathsf{X}:\langle \mathcal{Q}\cdot\mathsf{X},\mathsf{X}\rangle = 1} \sum_{\tau\in\mathbf{B}_F} n_\tau \, \langle\langle \phi_\rho(\hat{\tau}), \phi_\rho(\mathsf{X})\rangle\rangle \\
&= \max_{s\in\mathfrak{h}:\langle\langle s,s\rangle\rangle = 1} \sum_{\mathbf{r}\in\mathbf{\Psi}} \frac{\omega_r}{c} \times \langle r, s\rangle^+ \qquad \text{where } \phi_\rho(\hat{\tau}) = \frac{\omega_r}{n_\tau c} r, s = \phi_\rho(\mathsf{X}) \\
&= c^{-1} \times \max_{I\subset\mathbf{\Psi}} \left| \sum_{\mathbf{r}\in I} \omega_r \, r \right|
\end{aligned} \tag{7.6}$$

The right-hand side of 7.6 is computed in [11, Lemma 13 and Theorem 3]. These results are summarised in table 5.

$$\boxed{h_{top}(\varphi|\mathbf{V}_1^{\perp}) = h \times \tfrac{[E:F]}{c}}$$

$\mathbf{\Psi}$	h	$\mathbf{\Psi}$	h	$\mathbf{\Psi}$	h
$B_n^{(1)},\ n \geq 3$	$2\sqrt{n-1}$	$A_{2n-1}^{(2)},\ n \geq 3$	$\sqrt{2(n-1)}$		
$G_2^{(1)},\ (n=2)$	$2\sqrt{3}$	$D_4^{(3)},\ (n=2)$	2		
$F_4^{(1)},\ (n=4)$	$2\sqrt{6}$	$E_6^{(2)},\ (n=4)$	$2\sqrt{3}$		
$C_n^{(1)},\ n \geq 2$	$\sqrt{2n}$	$A_{2n}^{(2)},\ n \geq 2$	$2\sqrt{n}$	$D_{n+1}^{(2)},\ n \geq 2$	$\sqrt{n}$
$E_6^{(1)},\ (n=6)$	$2\sqrt{3}$	$E_7^{(1)},\ (n=7)$	$2\sqrt{6}$	$E_8^{(1)},\ (n=8)$	$2\sqrt{15}$
$A_n^{(1)},\ n \geq 2$	$\sqrt{\text{floor}\left(\frac{n+1}{2}\right)}$	$D_n^{(1)},\ n \geq 4$	$\sqrt{2(n-2)}$	$A_2^{(1)},\ (n=1)$	$\sqrt{2}$

TABLE 5. Entropies of the Bogoyavlenskij-Toda-like systems. The root systems in the first 4 rows have isomorphic Coxeter graphs; the root systems in the last 2 rows have unique Coxeter graphs. $n = \dim \mathrm{V}_{o,F}$.

Table 5 permits one to give lower bounds on the number of Bogoyavlenskij-Toda-like systems which are not energy-preserving topologically conjugate.

Proposition 7.1. *For each $n \geq 2$, table 6 displays Bogoyavlenskij-Toda-like systems, defined in (3.16), that are not topologically conjugate via an energy-preserving conjugacy.*

n	RootSystems	Total
2	$A_2^{(1)}, C_2^{(1)}, G_2^{(1)}, A_{2\cdot 2}^{(2)}$	4
3	$A_3^{(1)}, C_3^{(1)}, A_{2\cdot 3}^{(2)}, A_{2\cdot 3-1}^{(2)}$	4
4	$A_4^{(1)}, B_4^{(1)}, A_{2\cdot 4}^{(2)}, A_{2\cdot 4-1}^{(2)}$	4
5	$A_5^{(1)}, B_5^{(1)}, C_5^{(1)}, D_5^{(1)}, A_{2\cdot 5}^{(2)}, A_{2\cdot 5-1}^{(2)}$	6
6	$A_6^{(1)}, B_6^{(1)}, D_6^{(1)}, A_{2\cdot 6}^{(2)}, A_{2\cdot 6-1}^{(2)}$	5
7	$A_7^{(1)}, B_7^{(1)}, C_7^{(1)}, D_7^{(1)}, A_{2\cdot 7}^{(2)}, A_{2\cdot 7-1}^{(2)}$	6
8	$A_8^{(1)}, B_8^{(1)}, C_8^{(1)}, D_8^{(1)}, E_8^{(1)}, A_{2\cdot 8-1}^{(2)}$	6
≥ 9 even	$A_n^{(1)}, B_n^{(1)}, D_n^{(1)}, A_{2\cdot n}^{(2)}, A_{2\cdot n-1}^{(2)}$	5
≥ 9 odd	$A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2\cdot n}^{(2)}, A_{2\cdot n-1}^{(2)}$	6

TABLE 6. Minimal number of Bogoyavlenskij-Toda-like systems that are not iso-energetically topologically conjugate.

Proof. Use table 5 to determine a list of root systems the ratio of whose entropies do not lie in $\frac{1}{2}\mathbb{Z}$. Note that this list is not unique. □

7.3. **Summary.** If the results from table 5 are combined with proposition 6.1, one obtains the much stronger result:

Theorem 7.3. *Let $F/\mathbb{Q}$ be a strictly hyperbolic number field with $n+1 = \#\mathbf{B}_F > 2$. The number of iso-energetic topological conjugacy classes of Hamiltonian flows constructed from Equation (3.16) is* at least

$$\sum_{\operatorname{rank}\mathbf{\Psi}=n} \#\left(\operatorname{Aut}(\Gamma(\mathbf{\Psi}))\backslash\mathfrak{B}(\mathbf{\Psi})/\operatorname{Aut}(F/\mathbb{Q})\right). \tag{7.7}$$

where we sum over all rank n root systems except $D^{(2)}_{n+1}$.

Proof. By Proposition 6.1, we know that if the Bogoyavlenskij-Toda-like Hamiltonian flows φ^i are conjugate by an energy-preserving conjugacy, then there are two possibilities

Case A. The root systems coincide, $c_1 = c_2$ and the map $\nu = \mu$ is an automorphism of $\Gamma(\mathbf{\Psi})$. The definition of μ (equation 6.7 and *supra* 6.26) implies that the maps ϕ_1, ϕ_2 are related by

$$\phi_1 = \mu \cdot \phi_2 \cdot R^*_\beta \qquad\qquad \beta \in \operatorname{Aut}(F/\mathbb{Q}) \tag{7.8}$$

where theorem 6.11 is used. Conversely, given any ϕ_2, a ϕ_1 defined as in equation (7.8) is induced by a bijection $\rho_1 \in \mathfrak{B}$.

Case B. The two root systems differ, as in Case B of Proposition 6.1. The topological entropy of $\varphi^i|\mathbf{V}_1^\perp \cap \mathbf{H}_i(\frac{1}{2})$ is an invariant of energy-preserving conjugacy by Lemma 6.2. Table 5 implies that the root systems must therefore be $\{\mathbf{\Psi}_1, \mathbf{\Psi}_2\} = \left\{A^{(2)}_{2n}, D^{(2)}_{n+1}\right\}$ or $\left\{C^{(1)}_n, D^{(2)}_{n+1}\right\}$. Since the sum (7.7) counts the conjugacy classes from only one of these two root systems, there is no double counting. This proves the theorem.

□

Remark 7.2. In [11, Example 3, p. 541], the case where $F = E = \mathbb{Q}(\alpha)$, with α a root of the cubic $x^3 - 4x + 2$, was considered (*c.f.* example 4.1.1 *supra*). F is a cubic, totally-real, non-normal extension of $\mathbb{Q}$. Thus, $\operatorname{Aut}(F/\mathbb{Q})$ is trivial and F is strictly hyperbolic. If one sums over the rank 2 root systems and divides out by the order of their automorphism groups, then Theorem 7.3 implies that there are at least

$$1 + 3 + 6 + 3 + 6 = 19 \qquad \text{(summing over } A^{(1)}_2, C^{(1)}_2, G^{(1)}_2, A^{(2)}_{2\cdot 2}, D^{(3)}_4) \tag{7.9}$$

iso-energetic topological conjugacy classes. In [11, theorem 8], the lower bound of 10 was conjectured.[5] This lower bound depended on Gel'fond's conjecture concerning the algebraic independence of rationally-independent sets of logarithms of algebraic numbers. The results of the present paper, using dynamical systems theory, has proven this lower bound.

In a similar vein, if $F = E$ is a totally real quartic field with $\operatorname{Aut}(F/\mathbb{Q}) = 1$, then one has at least

$$3 + 4 \times 12 = 51 \qquad \text{(summing over } A^{(1)}_3, B^{(1)}_3, C^{(1)}_3, A^{(2)}_{2\cdot 3}, A^{(2)}_{2\cdot 3-1}) \tag{7.10}$$

iso-energetic topological conjugacy classes.

Remark 7.3. Theorem 7.3 provides a means to compute a lower bound on the number of iso-energetic topological conjugacy classes when $\operatorname{Aut}(F/\mathbb{Q})$ is non-trivial, too. Both $\mathbf{\Psi}$ and $\mathbf{B}_F$ are unnaturally isomorphic to the set $\{1, \ldots, n+1\}$. Theorem 6.11, part 2, shows that the representation of $\operatorname{Aut}(F/\mathbb{Q})$ in the group of permutations of $\mathbf{B}_F$, $S(\mathbf{B}_F)$, is the natural right regular representation (one should view $\mathbf{B}_F = \mathbf{G}_F/(\cdot \sim \bar{\cdot})$). By definition, the automorphism group of the Dynkin diagram is a subgroup of the group of permutations of the roots, $S(\mathbf{\Psi})$. Therefore, the

[5] Inexplicably, only the first three root systems are included in that sum, so the conjectural lower bound ought to be 19.

unnatural isomorphisms of $\mathbf{\Psi}$ and $\mathbf{B}_F$ with $\{1,\ldots,n+1\}$ identify the set of bijections $\mathfrak{B}(\mathbf{\Psi})$ with the symmetric group of $\{1,\ldots,n+1\}$, S_{n+1}, with the resulting equivariant diagram (where left/right arrows denote the standard left (resp. right) actions)

$$\begin{tikzcd}
\mathrm{Aut}(\Gamma(\mathbf{\Psi})) \arrow[r, hook] \arrow[d, "\cong"'] \arrow[dr, hook] & S(\mathbf{\Psi}) \arrow[r] \arrow[d, "\cong"'] & \mathfrak{B}(\mathbf{\Psi}) \arrow[d, "\cong"] & S(\mathbf{B}_F) \arrow[l] \arrow[d, "\cong"] & \mathrm{Aut}(F/\mathbb{Q}) \arrow[l, hook'] \arrow[d, "\cong"] \arrow[dl, hook'] \\
G \arrow[r, hook] & S_{n+1} \arrow[r, "\mathrm{id.}"] & S_{n+1} & S_{n+1} \arrow[l, "\mathrm{id.}"'] & H. \arrow[l, hook']
\end{tikzcd} \tag{7.11}$$

This implies that $\#(G\backslash S_{n+1}/H)$ equals $\#(\mathrm{Aut}(\Gamma(\mathbf{\Psi}))\backslash\mathfrak{B}(\mathbf{\Psi})/\mathrm{Aut}(F/\mathbb{Q}))$. Table 7 shows the cardinality of each of these sets for $n \leq 9$. The table is computed by a C++ program written by the author; the computations were checked using the GAP software package [18]. The source code and instructions are freely available from the author's web-page.

Table 7. The minimum number of iso-energetic topological conjugacy classes of Bogoyavlenskij-Toda-like systems. The *Total* column is based on Theorem 7.3 and Tables 8–9 of Coxeter graph automorphism groups.
$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, $D_n =$ the dihedral group of order $2n$, $Q =$ the quaternion group of order 8.

rank Galois grp	$\#\left(\mathrm{Aut}(\Gamma(\mathbf{\Psi}))\backslash\mathfrak{B}(\mathbf{\Psi})/\mathrm{Aut}(F/\mathbb{Q})\right).$ Root systems (grouped with isomorphic Coxeter diagrams)					Total
rank = 2 Aut$(F/\mathbb{Q})$	$A_2^{(1)}$	$C_2^{(1)}/A_{2\cdot 2}^{(2)}/D_{2+1}^{(2)}$	$G_2^{(1)}/D_4^{(3)}$			Total
1	1	3×2	6×2			19
$\mathbb{Z}_3$	1	1×2	2×2			7
rank = 3 Aut$(F/\mathbb{Q})$	$A_3^{(1)}$	$C_3^{(1)}/A_{2\cdot 3}^{(2)}/D_{3+1}^{(2)}$	$B_3^{(1)}/A_{2\cdot 3-1}^{(2)}$			Total
1	3	12×2	12×2			51
$\mathbb{Z}_2\oplus\mathbb{Z}_2$	3	6×2	3×2			21
$\mathbb{Z}_4$	2	4×2	3×2			16
rank = 4 Aut$(F/\mathbb{Q})$	$A_4^{(1)}$	$C_4^{(1)}/A_{2\cdot 4}^{(2)}/D_{4+1}^{(2)}$	$B_4^{(1)}/A_{2\cdot 4-1}^{(2)}$	$D_4^{(1)}$	$F_4^{(1)}/E_6^{(2)}$	Total
1	12	60×2	60×2	5	120×2	497
$\mathbb{Z}_5$	4	12×2	12×2	1	24×2	101
rank = 5 Aut$(F/\mathbb{Q})$	$A_6^{(1)}$	$C_6^{(1)}/A_{2\cdot 6}^{(2)}/D_{6+1}^{(2)}$	$B_6^{(1)}/A_{2\cdot 6-1}^{(2)}$	$D_6^{(1)}$		Total
1	60	360×2	360×2	90		1 590
$\mathbb{Z}_6$	14	64×2	60×2	17		279
S_3	19	72×2	60×2	21		304
rank = 6 Aut$(F/\mathbb{Q})$	$A_6^{(1)}$	$C_6^{(1)}/A_{2\cdot 6}^{(2)}/D_{6+1}^{(2)}$	$B_6^{(1)}/A_{2\cdot 6-1}^{(2)}$	$D_6^{(1)}$	$E_6^{(1)}$	Total
1	360	$2\,520\times 2$	$2\,520\times 2$	630	840	11 910
$\mathbb{Z}_7$	54	360×2	360×2	90	120	1 704
rank = 7 Aut$(F/\mathbb{Q})$	$A_7^{(1)}$	$C_7^{(1)}/A_{2\cdot 7}^{(2)}/D_{7+1}^{(2)}$	$B_7^{(1)}/A_{2\cdot 7-1}^{(2)}$	$D_7^{(1)}$	$E_7^{(1)}$	Total

continued next page

Table 7, continued from previous page						
1	2 520	20 160 × 2	20 160 × 2	5 040	20 160	108 360
$\mathbb{Z}_8$	332	2 544 × 2	2 520 × 2	642	2 520	13 622
$\mathbb{Z}_2^3$	420	2 688 × 2	2 520 × 2	714	2 520	14 070
$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	362	2 592 × 2	2 520 × 2	666	2 520	13 772
Q	333	2 544 × 2	2 520 × 2	642	2 520	13 623
D_4	391	2 640 × 2	2 520 × 2	690	2 520	13 921
rank = 8 Aut$(F/\mathbb{Q})$	$A_8^{(1)}$	$C_8^{(1)}/A_{2\cdot 8}^{(2)}/D_{8+1}^{(2)}$	$B_8^{(1)}/A_{2\cdot 8-1}^{(2)}$	$D_8^{(1)}$	$E_8^{(1)}$	Total
1	20 160	181 440 × 2	181 440 × 2	45 360	362 880	1 154 160
$\mathbb{Z}_9$	2 246	20 160 × 2	20 160 × 2	5 040	40 320	128 246
$\mathbb{Z}_3^2$	2 256	20 160 × 2	20 160 × 2	5 040	40 320	128 256
rank = 9 Aut$(F/\mathbb{Q})$	$A_9^{(1)}$	$C_9^{(1)}/A_{2\cdot 9}^{(2)}/D_{9+1}^{(2)}$	$B_9^{(1)}/A_{2\cdot 9-1}^{(2)}$	$D_9^{(1)}$		Total
1	181 440	1 814 400 × 2	1 814 400 × 2	453 600		7 892 640
$\mathbb{Z}_{10}$	18 264	181 632 × 2	181 440 × 2	45 456		789 864
D_5	18 724	182 400 × 2	181 440 × 2	45 840		792 244

8. Conclusion

The current paper shows that there is a rich family of completely integrable Hamiltonian systems to be found on the cotangent bundles of compact 2-step *Sol*-manifolds. In addition to the questions in the introduction, let us mention the following question which arises from lemma 6.10 and theorem 6.11.

Question F. *Let F be a number field that is not strictly hyperbolic. Assume that there is an automorphism α of $\mathcal{U}_F$ and a permutation $\boldsymbol{\pi}$ of $\mathbf{G}_F$ such that*

$$\begin{aligned} \forall u \in \mathcal{U}_F,\ \forall \tau \in \mathbf{G}_F: &\qquad |\tau(\alpha(u))| = |\sigma(u)| \qquad \text{where } \boldsymbol{\pi}(\tau) = \sigma, \text{ and} \\ \forall \tau \in \mathbf{G}_F: &\qquad \boldsymbol{\pi}(\bar{\tau}) = \overline{\boldsymbol{\pi}(\tau)} \end{aligned} \tag{8.1}$$

Is it true that there is an automorphism β of $F/\mathbb{Q}$ such that $\alpha = \beta|_{\mathcal{U}_F}$? In other words, is it true that $\cap_{\tau\in\mathbf{G}_F} \ker \zeta_\tau$ is always a finite-index subgroup of $\mathcal{U}_F$?

It appears the likely that the answer is *yes*. To explain: If u_i is a basis of $\mathcal{U}_F^+$ and $\alpha \in \mathrm{Aut}(\mathcal{U}_F)$, then $\alpha(u_i) = \epsilon_i \times \prod_j u_j^{a_{ji}}$ for some integer matrix $A = [a_{ji}]$ that is invertible over the integers, and some root of unity $\epsilon_i \in \mathcal{U}_F$. From the condition (8.1), one knows that the system of linear equations

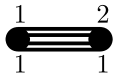
$$\sum_j a_{ji} \ln |\boldsymbol{\pi}\sigma(u_i)| = \ln |\sigma(u_i)| \tag{8.2}$$

is satisfied for all $j = 1, \ldots, \#\mathbf{B}_F - 1$ and embeddings $\sigma \in \mathbf{B}_F$. For a fixed permutation $\boldsymbol{\pi}$, one can treat (8.2) as a linear system that determines A. If there is an integer solution, then this determines an automorphism α; if not, then there is no such automorphism.

Salem number fields are good candidates to investigate question F because these number fields have many infinite order units of modulus one. By means of `Maxima` [24], it has been numerically verified that the answer to the refined question is *yes* for the 13 lowest degree number fields generated by the 'small' Salem numbers listed by Mossinghoff, based on [8, Table 1] and [25, Table 1].

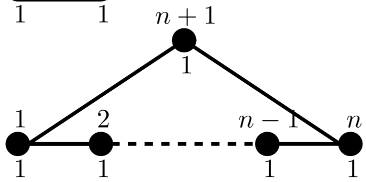
Root System | Dynkin Diagram | root number / weight | Automorphism Group

$A_1^{(1)}$



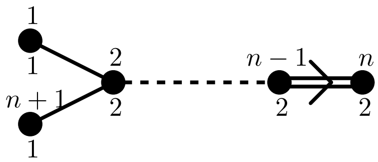
$\mathbf{Z}_2$

$A_n^{(1)}$



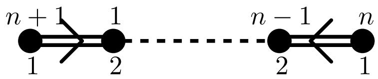
$D_{n+1}\,(n \geq 2)$

$B_n^{(1)}$



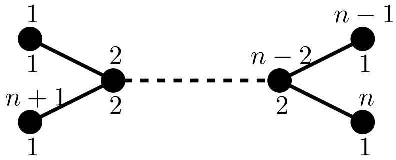
$\mathbf{Z}_2 \quad (n \geq 3)$

$C_n^{(1)}$



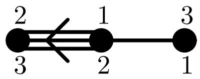
$\mathbf{Z}_2 \quad (n \geq 2)$

$D_n^{(1)}$



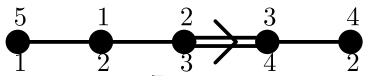
$S_4 \quad (n = 4)$
$\mathbf{Z}_2^3 \quad (n > 4)$

$G_2^{(1)}$



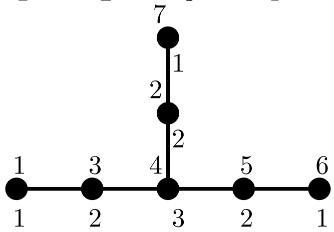
1

$F_4^{(1)}$



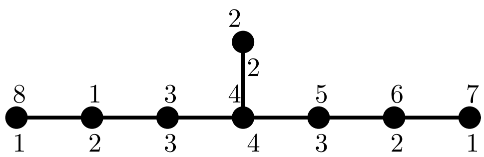
1

$E_6^{(1)}$



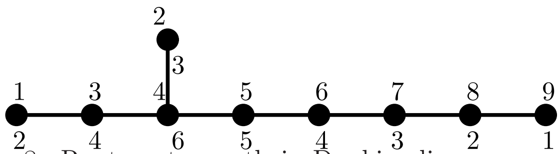
D_3

$E_7^{(1)}$



$\mathbf{Z}_2$

$E_8^{(1)}$



1

TABLE 8. Root systems, their Dynkin diagrams and automorphism groups. Symmetries are indicated by arrows. D_n is the symmetry group of a regular n-gon.

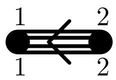
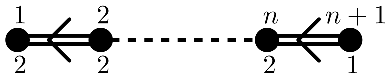
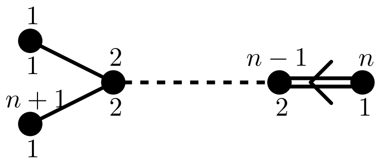
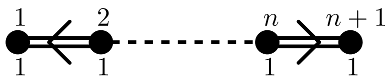
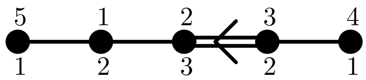
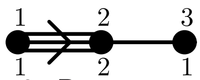
Root System	Dynkin Diagram	root number / weight	Automorphism Group
$A_2^{(2)}$			1
$A_{2n}^{(2)}$			$\mathbf{Z}_2$ (see text, $n \geq 2$)
$A_{2n-1}^{(2)}$			$\mathbf{Z}_2$ $(n \geq 3)$
$D_{n+1}^{(2)}$			$\mathbf{Z}_2$ $(n \geq 2)$
$E_6^{(2)}$			1
$D_4^{(3)}$			1

TABLE 9. Root systems, their Dynkin diagrams and automorphism groups. The shortest roots of $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$ have length $1/\sqrt{2}$; all other root systems' shortest roots have unit length. The longest root(s) of $G_2^{(1)}$ and $D_4^{(3)}$ have length $\sqrt{3}$; all other root systems' longest roots have length $\sqrt{2}$.

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