AN OPTICAL HAMILTONIAN AND OBSTRUCTIONS TO INTEGRABILITY

LEO T. BUTLER

ABSTRACT. If the geodesic flow of a compact finslerian 3-manifold is completely integrable, and its singular set is a tame polyhedron, then the manifold's π_1 is almost solvable [5]. Is this true for non-commutatively integrable geodesic flows? This note constructs non-commutatively integrable optical hamiltonians on the unit-sphere bundle of homogeneous spaces of $\mathrm{PSL}_2\mathbf{R}$ that have a real-analytic singular set. These flows are not tangent to a lagrangian foliation with a tame singular set.

1. INTRODUCTION

A smooth (C^1) action $\phi : \mathbf{R}^s \times M \to M$ is *integrable* if there is an open, dense subset R that is covered by angle-action charts $(\theta, I) : U \to \mathbf{T}^k \times \mathbf{R}^l$ which conjugate ϕ_t ($t \in \mathbf{R}^s$) with a translation-type map $(\theta, I) \mapsto (\theta + \omega(I)t, I)$ where $\omega : \mathbf{R}^l \to \mathrm{Hom}(\mathbf{R}^s, \mathbf{R}^k)$ is a smooth map. There is an open dense subset $L \subset R$ fibred by ϕ-invariant tori [1]. Let $f : L \to B$ be the C^1 fibration which quotients L by these invariant tori and let $\Gamma = M - L$ be the *singular set*. If Γ is a tamely-embedded polyhedron, then ϕ is called *k-semisimple* with respect to (f, L, B). Semisimplicity is a form of topologically-tame integrability.

Let $\pi : \mathbf{E} \to \Sigma$ be a fibre bundle with compact fibres, and let $\phi : \mathbf{R}^s \times \mathbf{E} \to \mathbf{E}$ be an action. Let $q \in Q \subset \Sigma$ be such that the inclusion $\iota : (\Sigma, q) \subset (\Sigma, Q)$ induces a bijection on π_1. Say that ϕ is *Hopf-Rinow over* Q if for each $\bar{c} \in \pi_1(\Sigma; q)$ there exists $T \in \mathbf{R}^s$ and $v \in \mathbf{E}_Q$ such that $\gamma(t) = \pi \circ \phi_{tT}(v)$ defines a curve $\gamma : ([0,1], \{0,1\}) \to (\Sigma, Q)$ and $\gamma \in \iota_* \bar{c}$. The action is Hopf-Rinow if it is Hopf-Rinow over some Q. A finslerian geodesic flow on the unit-tangent sphere bundle is the standard example of a Hopf-Rinow flow.

Main Result. Let Σ be a finite-volume homogeneous space of $\mathrm{PSL}_2\mathbf{R}$, $S\Sigma$ its unit-tangent sphere bundle.

Theorem 1.1. *There is a Hopf-Rinow action $\varphi : \mathbf{R}^2 \times S\Sigma \to S\Sigma$ that is 2-semisimple. The topological entropy of φ vanishes.*

See Theorem 7.4 below. Since $\pi_1(\Sigma)$ is a lattice subgroup of $\mathrm{PSL}_2\mathbf{R}$, it is not almost solvable, so Theorem 1.1 sharpens

Theorem 1.2 (*c.f.* [5])**.** *If the unit-sphere bundle of a compact 3-manifold Σ admits a 3-semisimple Hopf-Rinow action, then $\pi_1(\Sigma)$ is almost solvable.*

Date: March 28, 2006.

2000 *Mathematics Subject Classification.* 37J30, 37K10 .

Key words and phrases. geodesic flows, integrable systems, nonintegrability, momentum map, Liouville foliations.

This implies that a φ-invariant singular lagrangian fibration has a topologically wild singular set (see the Remark following Theorem 6.3, too).

Theorem 1.2 has a straightforward proof (assume L is connected): L deform retracts onto a $\mathbf{T}^3$-bundle over a circle or point, so $\pi_1(L)$ is solvable. The Hopf-Rinow property implies that $\mathrm{im}(\pi_1(L) \to \pi_1(\Sigma))$ is of finite-index, so $\pi_1(\Sigma)$ is almost solvable.[1] However, in the 2-semisimple case, L deform retracts onto a $\mathbf{T}^2$-bundle over a compact surface, so $\pi_1(\Sigma)$ is unconstrained.

The action φ of Theorem 1.1 has a distinguished 1-parameter subgroup induced by an optical hamiltonian (see Equation 3.3). This optical hamiltonian is the hamiltonian of a charged particle moving in the presence of a magnetic field. The symplectic reduction of this system models the motion of a particle moving on a constant negative curvature surface along curves of constant geodesic curvature. The reduced system has been used to construct examples of magnetic geodesic flows without periodic orbits on an energy surface [6] and as a system which exhibits a transition from complete integrability to chaos [10]. The former example is not Hopf-Rinow, while the latter is Hopf-Rinow only on the energy levels where it is chaotic. The optical-hamiltonian flow of Theorem 1.1 is Hopf-Rinow but, since its entropy vanishes, it is not conjugate to a geodesic flow, so

Question A: *Is there a 2-semisimple* finslerian geodesic flow *on one of the sphere bundles of Theorem 1.1?*

A Second Result. A group is *anabelian* if 1 is the only abelian subgroup whose normalizer is of finite index. The fundamental group of a hyperbolic manifold of finite volume is a basic example of an anabelian group. The π_1's of Theorem 1.1 are not anabelian since they have a non-trivial centre. The following refines Theorem 7 of [5]:

Theorem 1.3. *Let Σ be a compact aspherical n-manifold with π_1 anabelian. If $S\Sigma$ admits a k-semisimple Hopf-Rinow flow, then there is a component $B_i \subset B$ and a natural almost-surjective homomorphism $\pi_1(B_i) \to \pi_1(\Sigma)$ such that either*

(1) $\ker(\pi_1(B_i) \to \pi_1(\Sigma)) \neq 1$*; or*
(2) $\pi_k(B_i) \neq 1$ *for some* $k \geq 2$*; or*
(3) B_i *is homotopy equivalent to a finite covering of* Σ *and* $k < n$*.*

Here is an example that sharpens Theorem 1.3. Let Σ be a finite-volume homogeneous space of $\mathrm{PSL}_2\mathbf{R}$ which is simultaneously a principal S^1 bundle over a compact surface of genus more than one, $S^1 \hookrightarrow \Sigma \xrightarrow{p} \Lambda$. Let S^1 act on S^2 by its standard action. Then S^1 acts freely on $\Sigma \times S^2$ by the anti-diagonal action. One obtains a smooth 4-manifold $\Omega = \Sigma \times_{S^1} S^2$ with $\pi_1(\Omega) = \pi_1(\Lambda)$ –which is anabelian–and $\pi_2(\Omega) = \pi_2(S^2)$. One has (see Theorem 8.6, too):

Theorem 1.4. *There is a Hopf-Rinow action $\varphi : \mathbf{R}^3 \times S\Omega \to S\Omega$ that is 3-semisimple with respect to (f, L, B) and B is homotopy equivalent to Λ. The topological entropy of φ vanishes.*

[1] The definition of a Hopf-Rinow flow in [5] assumes that $Q = \{q\}$. Inspection of the generalized Kozlov-Taimanov Theorem (Lemma 15 of [5]) shows that the present extended definition is precisely the hypothesis necessary for the proof.

This is also the first example of a semisimple Hopf-Rinow action where π_1 is anabelian—hyperbolic, in fact. Theorems 1.3 and 1.4 motivate

Question B: *Is there a semisimple* geodesic flow *on a sphere bundle of Theorem 1.4? on a compact aspherical manifold with* π_1 *anabelian?*

Background: This paper is part of a sequence of papers [8, 9, 3, 4, 5] which try to answer the question: which manifolds admit an integrable geodesic flow? Kozlov showed that if the geodesic flow is analytic and has an additional analytic first integral, then the surface's genus is at most one. Taimanov [9] generalized Kozlov's argument, and obtained three necessary conditions for a compact real-analytic manifold to admit a real-analytically integrable geodesic flow: (1) its fundamental group must be almost abelian; (2) its first Betti number is at most its dimension; and (3) a third condition involving its rational cohomology ring. Taimanov introduced and used the idea of geometric simplicity to prove these results. In [3], Taimanov's results are shown to fail in the smooth category. In [5], the present author generalized geometric simplicity to semisimplicity,[2] showed that the "exotic" smoothly integrable geodesic flows of [3, 2] are 3-semisimple, and that–modulo the geometrization conjecture–these are essentially the only such examples in three dimensions. While these papers are largely concerned with geodesic flows, the obstacles to integrability discovered in [8, 9, 5] are obstacles to the integrabiliy of Hopf-Rinow flows. Given Theorem 1.2, it is natural to explore Seifert 3-manifolds, hence Theorem 1.1.

Outline: Section 2 proves Theorem 1.3. Sections 3 and 4 outline the geometry of $\mathrm{PSL}_2\mathbf{R}$ required to construct the action φ. Section 5 constructs "action-angle" variables on φ's invariant tori and shows how these tori degenerate. Section 6 proves the Hopf-Rinow property of φ by a topological argument. Section 7 proves semisimplicity and completes the proof of Theorems 1.1 & 7.4. Section 8 sketches a proof of Theorems 1.4 & 8.6.

2. Theorem 1.3

Theorem 1.3. By Lemmas 15 and 19 of [5] there is a component $B_i \subset B$ such that the induced map $\pi_1(B_i) \to \pi_1(\Sigma)$ is almost surjective. By passing to a finite covering $\hat{\Sigma}$, it can be assumed the map is onto. If cases 1 & 2 do not hold, then $\pi_1(B_i)$ is isomorphic to $\pi_1(\hat{\Sigma})$ and B_i is aspherical. Thus B_i and $\hat{\Sigma}$ are homotopy equivalent. Hence $\dim B_i \geq n$, so $k \leq n-1$. □

3. The Setting

Preamble. Let Σ be a finite-volume homogeneous space of $\mathrm{PSL}_2\mathbf{R}$. Since Σ is a Seifert manifold, there is a topological surface Λ of genus more than one, and a surjection $p : \Sigma \to \Lambda$ whose fibres are circles. The surface Λ contains a discrete set of cone points which lie under the set $\mathcal{S}$ of "short" fibres of p. The complement of $p(\mathcal{S})$, Λ_*, is a smooth surface. The submersion $p_* : \Sigma_* \to \Lambda_*$ is a principal S^1 bundle, where $\Sigma_* = \Sigma - \mathcal{S}$. Let $S\mathcal{S} = S_{\mathcal{S}}\Sigma$ be the set of unit tangent vectors over $\mathcal{S}$. $S\mathcal{S}$ is a disjoint union of $S^1 \times S^2$ fibred in the obvious way by 2-tori which pinch to circles. $S\mathcal{S}$ may be empty, of course.

[2] Real-analytic integrability implies semisimplicity by Taimanov's work.

Let $\mathbf{G} = \mathrm{PSL}_2\mathbf{R}$, $\mathfrak{g} = psl_2\mathbf{R}$, and let $\mathbf{\Delta} < \mathbf{G}$ be a lattice subgroup. For notational convenience, the following conventions are adopted: the Lie group $\mathbf{H}$ has Lie algebra $\mathfrak{h}$; elements in $\mathbf{G}$ are written as square-bracketed matrices; elements in $\mathfrak{g}$ are round-bracketed, genuine, matrices. From the KAN-decomposition theorem, $\mathbf{G}$ is analytically diffeomorphic to $\mathbf{K}\times\mathbf{A}\times\mathbf{N}$ where $\mathbf{K} = SO_2\mathbf{R}/\{\pm I\}$,

$$\mathbf{A} = \left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} : x \in \mathbf{R}^+ \right\}, \qquad \mathbf{N} = \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} : y \in \mathbf{R} \right\}.$$

The diffeomorphism is $(k,a,n) \mapsto kan$ [7]. Let $\mathbf{B} = \mathbf{AN}$ be the Borel subgroup of upper triangular elements in $\mathbf{G}$. Let $\kappa(\xi) = \det\xi$ be the Cartan-Killing form on $\mathfrak{g}$, and let K be the induced bi-invariant hamiltonian on $T\mathbf{G}$:

$$\mathrm{K}(g,\xi) = \det\xi, \tag{3.1}$$

for all $(g,\xi) \in T\mathbf{G}$. For $\xi \in \mathfrak{g}$, write

$$\xi = \begin{pmatrix} a & b-c \\ b+c & -a \end{pmatrix} = a\alpha + b\beta + c\gamma, \tag{3.2}$$

so that $\kappa(x) = c^2 - a^2 - b^2$. The commutation relations between the basis elements are $[\alpha,\beta] = 2\gamma$, $[\alpha,\gamma] = -2\beta$, $[\beta,\gamma] = 2\alpha$. Define a hamiltonian h by

$$h(\xi) := \frac{1}{2}\left(a^2 + b^2 + (c-\sqrt{2})^2\right), \qquad h : \mathfrak{g} \to \mathbf{R}, \tag{3.3}$$

and let $H : T\mathbf{G} \to \mathbf{R}$ be the induced left-invariant hamiltonian. Relative to the left trivialization of $T\mathbf{G} = \mathbf{G}\times\mathfrak{g}$, Hamilton's equations for H are

$$X_H : \qquad \dot{\xi} = [\xi, \nabla h(\xi)], \quad g^{-1}\dot{g} = \xi, \qquad \text{at } (g,\xi) \in \mathbf{G}\times\mathfrak{g}.$$

Let E_h be the reduction of X_H to $\mathfrak{g}$. Its critical-point set is $\mathcal{C} = \{\xi \in \mathfrak{g} \ : \ \xi \in \mathfrak{k} \text{ or } c = \frac{1}{\sqrt{2}}\}$. One observes that on $\mathbf{G}\times\mathcal{C}$, the flow of X_H is $(g,x) \mapsto (g\,\exp(t\xi),\xi)$. This flow has zero entropy when $\kappa(\xi) \geq 0$.

4. The Momentum Map and Adjoint Orbits

One observes that X_H enjoys the first integrals

$$\Psi_{\mathbf{G}}(g,\xi) = \mathrm{Ad}_g\xi, \quad \Psi_{\mathbf{K}}(g,\xi) = c\gamma, \tag{4.1}$$

the former (resp. latter) is the momentum map of $\mathbf{G}$'s left action (resp. $\mathbf{K}$'s right action) on $T\mathbf{G}$. Note that H and K are both pull-backs of functions on $\mathfrak{g}\oplus\mathfrak{k}$ by $\Psi_{\mathbf{G}}\times\Psi_{\mathbf{K}}$:

$$\begin{aligned} H &= (\Psi_{\mathbf{G}}\times\Psi_{\mathbf{K}})^*\mathrm{h}, \qquad \text{where } \mathrm{h} = -\frac{1}{2}\kappa + c^2 - \sqrt{2}c + 1, \\ \mathrm{K} &= (\Psi_{\mathbf{G}}\times\Psi_{\mathbf{K}})^*\kappa. \end{aligned}$$

To carry through the construction behind Theorem 1.1, one must "push down" $\Psi_{\mathbf{G}}\times\Psi_{\mathbf{K}}$ to a quotient $T(\mathbf{\Delta}\backslash\mathbf{G})$. Since $\mathbf{K}$'s right action, hence $\Psi_{\mathbf{K}}$, naturally descends to $T(\mathbf{\Delta}\backslash\mathbf{G})$, the difficulty lies in pushing down $\Psi_{\mathbf{G}}$. It is only possible to do this on a subspace of $T\mathbf{G}$. To wit, let $\mathfrak{g}_+ = \{\xi \in \mathfrak{g} \ : \ \kappa(\xi) > 0, c > 0\}$ and $\mathfrak{g}_0 = \{\xi \in \mathfrak{g} \ : \ \kappa(\xi) = 0, c > 0\}$. It will be shown that $\Psi_{\mathbf{G}}$ can be pushed down to $\mathbf{\Delta}\backslash\mathbf{G}\times\mathfrak{g}_+ \subset T(\mathbf{\Delta}\backslash\mathbf{G})$. This is the most important step in proving that the hamiltonian flows of H and K generate a 2-semisimple action of $\mathbf{R}^2$ on $H^{-1}(\frac{1}{2}) \subset T(\mathbf{\Delta}\backslash\mathbf{G})$. To demonstrate the action's Hopf-Rinow property, one must ensure that $h^{-1}(\frac{1}{2}) \cap \mathfrak{g}_0 \neq \emptyset$.

Define a map

$$\sigma : \mathfrak{g}_+ \to \mathbf{G}/\mathbf{K} \times \mathbf{R}^+,$$

by $\sigma(\xi) = (u\mathbf{K}, r)$ where $u \in \mathbf{B}$ and r is the unique solution to $\xi = r\,\mathrm{Ad}_u\gamma$.

Lemma 4.1. *Let $\xi \in \mathfrak{g}_+$. Then $\sigma(\xi) = (u, r)$ iff*

$$u = \begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix}, \quad r = \sqrt{\kappa(\xi)}, \quad \text{where} \quad x = \sqrt{\tfrac{r}{b+c}}, \quad y = axr^{-1}.$$

Consequently, σ is an analytic diffeomorphism.

The proof of this lemma is a simple computation.

4.1. The Energy-Momentum Map.

Lemma 4.2. *The image of $\Psi_{\mathbf{G}} \times \Psi_{\mathbf{K}} : \mathbf{G} \times \mathfrak{g}_+ \to \mathfrak{g}_+ \times \mathfrak{k}_+$ is $\{\xi \oplus c\gamma \;:\; r(\xi) \leq c\}$. The critical-value set is $\{\xi \oplus c\gamma \;:\; r(\xi) = c\}$ and the critical-point set is $\mathbf{G} \times \mathfrak{k}_+$.*

The proof is straightforward. It is convenient to define a map

$$\hat{J}(g, \xi) = \mathrm{Ad}_g\xi \oplus c(\xi) - r(\xi), \qquad \hat{J} : \mathbf{G} \times \mathfrak{g}_+ \to \mathfrak{g}_+ \oplus \mathbf{R}^{\geq 0}. \tag{4.2}$$

Lemma 4.3. *The map $\hat{J}$ is continuous and onto $\mathfrak{k}_+ \times \mathbf{R}^{\geq 0}$. Its critical-point set is $\mathbf{G} \times \mathfrak{k}_+$, its critical-value set is $\mathfrak{g}_+ \times 0$ and it is smooth on $\mathbf{G} \times (\mathfrak{g}_+ - \mathfrak{k}_+)$.*

Clearly Lemma 4.2 implies Lemma 4.3. One sees that $H, \mathrm{K}|\mathbf{G} \times \mathfrak{g}_+$ are $\hat{J}$-pullbacks of functions on $\mathfrak{g}_+ \times \mathbf{R}^{\geq 0}$.

5. "Action-Angle" Variables

This section contructs a trivialization of $\hat{J}$ over $\mathfrak{g}_+ \times \mathbf{R}^+$. An "embedding" is a smooth embedding.

Theorem 5.1. *There is a continuous map $\hat{\omega} : \mathbf{K} \times \mathbf{K} \times \mathfrak{g}_+ \times \mathbf{R}^{\geq 0} \to \mathbf{G} \times \mathfrak{g}_+$ such that*

1. $\mathbf{K} \times \mathbf{K} \times \mathfrak{g}_+ \times \mathbf{R}^{\geq 0} \xrightarrow{\hat{\omega}} \mathbf{G} \times \mathfrak{g}_+ \xrightarrow{\hat{J}} \mathfrak{g}_+ \times \mathbf{R}^{\geq 0}$ *is the projection map;*
2. $\hat{\omega}|\mathbf{K} \times \mathbf{K} \times \mathfrak{g}_+ \times \mathbf{R}^+$ *is an embedding;*
3. $\hat{\omega}|\mathbf{K} \times 1 \times \mathfrak{g}_+ \times 0$ *is an embedding.*

Proof. First, a section (g_o, η_o) of $\hat{J}$ will be constructed. Let $\hat{\epsilon} = (\xi, s) \in \mathfrak{g}_+ \times \mathbf{R}^{\geq 0}$ and let $\hat{T}_{\hat{\epsilon}} = \hat{J}^{-1}(\hat{\epsilon})$. Then

$$(g, \eta) \in \hat{T}_{\hat{\epsilon}} \qquad \Longleftrightarrow \qquad \eta = \mathrm{Ad}_{g^{-1}}\xi \;\; \& \;\; c(\eta) - r(\eta) = s. \tag{$*$}$$

Let $\eta_o = b\beta + c\gamma$ where $c = s + r(\xi)$, $b = \sqrt{c^2 - r(\xi)^2}$. These equations define a map $\eta_o : \mathfrak{g}_+ \times \mathbf{R}^{\geq 0} \to \mathfrak{g}_+$. One defines g_o by

$$g_o(\hat{\epsilon}) = u(\xi)\, u(\eta_o(\hat{\epsilon}))^{-1},$$

which, like η_o, is continuous on $\mathfrak{g}_+ \times \mathbf{R}^{\geq 0}$ and smooth on $\mathfrak{g}_+ \times \mathbf{R}^+$. It is clear that $(g_o(\hat{\epsilon}), \eta_o(\hat{\epsilon})) \in \hat{T}_{\hat{\epsilon}}$ for all $\hat{\epsilon}$.

Second, the general solution to (*): Since $\mathbf{K}$ is abelian, (*) is $\mathbf{G}_\xi \times \mathbf{K}$-invariant so $(g = hg_o\phi^{-1}, \eta = \mathrm{Ad}_\phi \eta_o)$ solves (*) for all $h \in \mathbf{G}_\xi, \phi \in \mathbf{K}$. Since $\mathbf{G}_\xi = u(\xi)\mathbf{K}u(\xi)^{-1}$, define

$$g = u(\xi)\theta u(\xi)^{-1}\, g_o(\hat\epsilon)\, \phi^{-1}, \qquad \eta = \mathrm{Ad}_\phi \eta_o(\hat\epsilon),$$

and let $\hat\omega(\theta, \phi, \hat\epsilon) = (g(\theta,\phi,\hat\epsilon), \eta(\phi,\hat\epsilon))$. Property (*) proves (1).

Let us verify the claims (2–3). Assume that $(g = h_i g_o \phi_i^{-1}, \eta = \mathrm{Ad}_{\phi_i}\eta_o)$ is a solution to (*) where $h_i \in \mathbf{G}_\xi$ and $\phi_i \in \mathbf{K}$.

Case 1: If $s = 0$, then $\eta_o \in \mathfrak{k}$ so $1 \times \mathbf{K}$ acts trivially on $\hat T_{\hat\epsilon}$. However, if $\phi_1 = \phi_2 = 1$, then $h_1 = h_2$. Thus $\hat\omega$ is injective when restricted to $\mathbf{K} \times 1 \times \mathfrak{g}_+ \times 0$, which proves (3).

Case 2: If $s > 0$, then $\eta_o \notin \mathfrak{k}$ so $\mathbf{K} \cap \mathbf{G}_{\eta_o}$ is trivial. Thus $\phi_1 = \phi_2$, hence $h_1 = h_2$. Thus $\hat\omega$ is injective, which proves (2).

□

6. The Hopf-Rinow Property

The hamiltonians H and K Poisson commute, so their flows generate an action, $\hat\varphi$, of $\mathbf{R}^2$ on $T\mathbf{G}$, and in particular on "the" unit-sphere bundle $S\mathbf{G} := H^{-1}(\frac{1}{2})$. This action descends to any quotient $S(\mathbf{\Delta}\backslash\mathbf{G})$ by left-invariance. Let $S = h^{-1}(\frac{1}{2})$ be the unit sphere in $\mathfrak{g}_+$, so that $S(\mathbf{\Delta}\backslash\mathbf{G}) = (\mathbf{\Delta}\backslash\mathbf{G}) \times S$. Let φ be the action induced $T(\mathbf{\Delta}\backslash\mathbf{G})$ by $\hat\varphi$.

Lemma 6.1. *Suppose that for each $g \in \mathbf{G}$, there is an $\hat\epsilon = (\xi, s) \in S \times \mathbf{R}^{\geq 0}$ and an $\eta \in \mathfrak{g}_+$ such that $(1,\xi), (g,\eta) \in \hat T_{\hat\epsilon}$. Then the action $\varphi : \mathbf{R}^2 \times S(\mathbf{\Delta}\backslash\mathbf{G}) \to S(\mathbf{\Delta}\backslash\mathbf{G})$ is Hopf-Rinow.*

Proof. Recall that φ is Hopf-Rinow over $\{\mathbf{\Delta}\} \subset \mathbf{\Delta}\backslash\mathbf{G}$, if, for each $\delta \in \mathbf{\Delta}$, there is a $T \in \mathbf{R}^2$ and $\xi \in S$ such that $\hat\varphi_T((1,\xi)) = (\delta, \eta)$. Since the orbit $\hat\varphi_{\mathbf{R}^2}((1,\xi))$ is the level set $\hat T_{\hat\epsilon}$, where $\hat\epsilon = (\xi, s)$ and $s = c(\xi) - r(\xi)$, the lemma is proven. □

Let $\xi = b\beta + c\gamma$, where $b = \cos\chi$ and $c = \sqrt{2} + \sin\chi$ to ensure that $\xi \in S$. Let $s = c(\xi) - r(\xi)$, so that $(1,\xi) \in \hat T_{\hat\epsilon}$. Since $(1,\xi) \in \hat T_{\hat\epsilon}$, this forces $\eta_o = \xi$, $g_o = 1$ while $u(\xi)$ is diagonal with $x^4 = \frac{1+\sin\chi}{1+\cos\chi}$. Thus,

$$(g,\eta) \in \hat T_{\hat\epsilon} \qquad \text{implies} \qquad g = u\theta u^{-1}\phi^{-1},$$

and $\eta = \mathrm{Ad}_\phi \xi$ for some $u \in \mathbf{A}$ and $\phi, \theta \in \mathbf{K}$ (see previous section). Since $\lim_{\chi \to \frac{\pi}{2}} x = \infty$ and $\lim_{\chi \to -\frac{\pi}{2}} x = 0$, it follows by continuity and Lemma 6.1 that

Lemma 6.2. *If the map $\mathrm{f} : \mathbf{A} \times \mathbf{K} \times \mathbf{K} \to \mathbf{G}$, $(u,\theta,\phi) \mapsto u\theta u^{-1}\phi^{-1}$, is surjective, then the action φ is Hopf-Rinow on $S(\mathbf{\Delta}\backslash\mathbf{G})$.*

Theorem 6.3. *The action $\varphi : \mathbf{R}^2 \times S(\mathbf{\Delta}\backslash\mathbf{G}) \to S(\mathbf{\Delta}\backslash\mathbf{G})$ generated by the hamiltonian flows of H and K is Hopf-Rinow.*

Proof. By Lemma 6.2, it suffices to prove that f is onto.

Step 1. f is onto $\mathbf{NK}$.

Since $\mathrm{f}(u,\theta,\phi_1\phi_2) = \mathrm{f}(u,\theta,\phi_2)\phi_1^{-1}$, it suffices to prove that f is onto $\mathbf{N}$. Let $\theta = \exp(\boldsymbol{\theta}\gamma)$, $\phi = \exp(\boldsymbol{\phi}\gamma)$ and $u = \exp(\mathbf{u}\alpha)$. By a direct calculation

$\mathrm{f}(u,\theta,\phi)\in\mathbf{N}$ iff $\tan(\boldsymbol{\phi})=-\exp(-2\mathbf{u})\tan(\boldsymbol{\theta})$. So, $\mathrm{f}\in\mathbf{N}$ implies that the upper right-hand corner entry of f, f_{12}, equals $-2\cosh(2\mathbf{u})\sin(\boldsymbol{\theta})\cos(\boldsymbol{\phi})$.

If $\boldsymbol{\theta}=-\frac{\pi}{4}$, then $\tan(\boldsymbol{\phi})=\exp(-2\mathbf{u})$ and $\mathrm{f}_{12}=\sqrt{2}\cosh(2\mathbf{u})\cos(\boldsymbol{\phi})$. Thus $\lim_{\mathbf{u}\to\infty}\mathrm{f}_{12}=\infty$. If $\boldsymbol{\theta}=\frac{\pi}{4}$, then by symmetry $\lim_{\mathbf{u}\to\infty}\mathrm{f}_{12}=-\infty$. By continuity, f is onto $\mathbf{N}$ and hence $\mathbf{NK}$.

Step 2. f is onto $\mathbf{G}-\mathbf{NK}$.

Let $\mathbf{A}_\pm=\exp(\mathbf{R}^\pm\alpha)$ and $\mathbf{G}_\pm=\mathbf{NA}_\pm\mathbf{K}$. The *kan*-decomposition shows that $\mathbf{G}-\mathbf{NK}$ is the union of its two connected components $\mathbf{G}_+$ and $\mathbf{G}_-$. Let $R_\pm$ be a connected component of $\mathrm{f}^{-1}(\mathbf{G}_\pm)$ contained in $\mathbf{A}_\pm\times\mathbf{K}\times\mathbf{K}$. Let $\mathrm{f}_\pm=\mathrm{f}|R_\pm$. It is claimed that $\mathrm{f}_\pm$ is onto $\mathbf{G}_\pm$. To prove this, it suffices to show that $\mathrm{f}_\pm$ is a proper map of non-zero degree. This will be proven for f_+, as the proof for f_- is symmetric.

f_+ *is proper.* Let $C\subset\mathbf{G}_+$ be a non-empty compact set. If $\mathrm{f}^{-1}(C)$ is empty, then it is compact. Otherwise, assume that $\mathrm{f}^{-1}(C)$ is non-empty. Let $p_n=(u_n,\theta_n,\phi_n)$ be a sequence in $\mathrm{f}^{-1}(C)$. It suffices to prove that p_n has a convergent subsequence.

By compactness of C and $\mathbf{K}$, one may suppose that, possibly after passing to a subsequence, $g_n=\mathrm{f}_+(p_n)$ converges to $g\in C$ and $\theta_n\to\theta,\phi_n\to\phi$ in $\mathbf{K}$. Thus $u_n\theta_n u_n^{-1}\to g\phi$ and $u_n\theta_n u_n^{-1}=\begin{bmatrix}\cos(\boldsymbol{\theta}_n) & -\exp(2\mathbf{u}_n)\sin(\boldsymbol{\theta}_n)\\ \exp(-2\mathbf{u}_n)\sin(\boldsymbol{\theta}_n) & \cos(\boldsymbol{\theta}_n)\end{bmatrix}$ where $u_n=\exp(\mathbf{u}_n\alpha)$, $\theta_n=\exp(\boldsymbol{\theta}_n\gamma)$. If there is a subsequence $\mathbf{u}_{n_k}\to\infty$, then $g\phi\in\mathbf{N}$ so $g\in\mathbf{NK}$ and $g\in\mathbf{G}-\mathbf{NK}$. Absurd. If there is a subsequence $\mathbf{u}_{n_k}\to 0$, then $g\phi\in\mathbf{K}$ so $g\in\mathbf{K}$. Absurd.

Therefore, u_n is bounded away from 1 and ∞. So, after passing to a subsequence, it may be assumed that $u_n\to u\neq 1$. Then $u\theta u^{-1}\phi^{-1}=g\notin\mathbf{K}$, which implies that $\theta\neq 1$. This proves that $p_n=(u_n,\theta_n,\phi_n)$ contains a convergent sequence converging to a limit in R_+. Hence f_+ is proper.

f_+ *has a non-zero degree.* Let $g=\exp(\alpha)\exp(\frac{\pi}{2}\gamma)\exp(-\alpha)$. Since $g=\exp(2\alpha)\times\exp(\frac{\pi}{2}\gamma)$, it is clear that $g\in\mathbf{G}_+$ (and R_+ is non-empty). Then $\mathrm{f}(u,\theta,\phi)=g$ implies that the diagonal elements of $g\phi$ are $\exp(\pm1)\sin(\boldsymbol{\phi})$, while the diagonal elements of $u\theta u^{-1}$ are both $\cos(\boldsymbol{\theta})$. Thus $\sin(\boldsymbol{\phi})=0$, so $\phi=1$. Consequently $\boldsymbol{\theta}=\frac{\pi}{2}\bmod\pi$ and $\mathbf{u}=1$. Thus $\#\mathrm{f}_+^{-1}(g)=1$. A straightforward computation shows that g is a regular value of f_+.

□

Remark. Since the hamiltonian flow of H is not periodic, it is minimal on $\hat{T}_{\hat{\epsilon}}$ for a residual set of $\hat{\epsilon}$. This fact, combined with Lemma 6.2, implies that the hamiltonian flow of H on $S(\mathbf{\Delta}\backslash\mathbf{G})$ is Hopf-Rinow. Theorem 1.4 therefore implies that any singular lagrangian fibration to which the hamiltonian flow of H is tangent has a topologically wild singular set.

7. Quotients

Recall that $\mathbf{\Delta}<\mathbf{G}$ is a lattice subgroup of $\mathbf{G}$; it acts on $\mathbf{G}/\mathbf{K}$ with only a discrete set of "cone points." Let

$$\mathfrak{g}_+^r=\{\xi\in\mathfrak{g}_+\ :\ \mathbf{\Delta}_\xi=1\},$$

which is the set of $\mathbf{\Delta}$-regular elements in $\mathfrak{g}_+$. Under the diffeomorphism σ (Lemma 4.1), the complement of $\mathfrak{g}_+^r$ is homeomorphic to a discrete union of lines. The $\mathbf{\Delta}$-stabilizer of a point in the complement of $\mathfrak{g}_+^r$ is conjugate to a subgroup of $\mathbf{K}$, hence it is cyclic. The set of regular points, $\mathfrak{g}_+^r$, itself is a smooth manifold on which $\mathbf{\Delta}$ acts properly discontinuously and freely.

Let $\mathbf{G}_*$ be the set of elements g in $\mathbf{G}$ such that $\mathbf{\Delta} \cap g\mathbf{K}g^{-1} = \emptyset$. $\mathbf{\Delta}\backslash\mathbf{G}/\mathbf{K}$ is an orbifold, while $\mathbf{\Delta}\backslash\mathbf{G}_*/\mathbf{K}$ is the maximal smooth surface contained in this orbifold. Of course, if $\mathbf{\Delta}$ is torsion free, then $\mathbf{G}_* = \mathbf{G}$ and the above distinctions are trivial.

In terms of the preamble to section 2, one has the following commutative diagram of inclusions ($\rightarrow$), homeo/diffeomorphisms ($\searrow$) and projections ($\downarrow$)

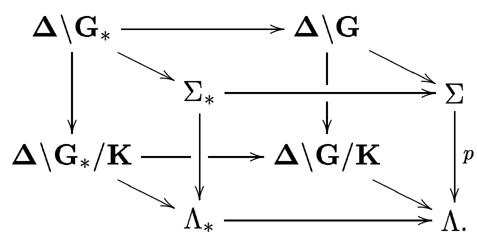

FIGURE 1

7.1. *J* **and its singularities.** The map $\hat{J}$ (Equation 4.2) induces $J : \mathbf{\Delta}\backslash\mathbf{G} \times \mathfrak{g}_+ \to \mathbf{\Delta}\backslash\mathfrak{g}_+ \times \mathbf{R}^{\geq 0}$ by equivariance. J is certainly continuous, and it inherits the singularities of $\hat{J}$, but it may also have additional singularities which we now investigate.

Fix $\hat{\epsilon} = (\xi, s) \in \mathfrak{g}_+ \times \mathbf{R}^{\geq 0}$, let $\epsilon = (\mathbf{\Delta}\xi, s)$ and let $T_\epsilon = J^{-1}(\epsilon)$ be the ϵ-level set of J. Let

$$\hat{\pi}_{\hat{\epsilon}} : \hat{T}_{\hat{\epsilon}} \to T_\epsilon$$

be the restriction of the canonical projection $\pi : T\mathbf{G} \to T(\mathbf{\Delta}\backslash\mathbf{G})$.

Assume that $(g_i, \eta_i) \in \hat{T}_{\hat{\epsilon}}$ and $\hat{\pi}_{\hat{\epsilon}}(g_1, \eta_1) = \hat{\pi}_{\hat{\epsilon}}(g_2, \eta_2)$. This is possible iff

$$\delta = g_2 g_1^{-1} \in \mathbf{\Delta}_\xi \qquad \text{and} \qquad \eta_1 = \eta_2 = \eta \qquad \text{and} \qquad \xi = \mathrm{Ad}_{g_i}\eta.$$

Hence T_ϵ is diffeomorphic to $\mathbf{\Delta}_\xi\backslash\hat{T}_{\hat{\epsilon}}$ which, as a consequence of Theorem 5.1, is diffeomorphic to either $(\mathbf{\Delta}_\xi\backslash\mathbf{G}_\xi) \times \mathbf{K}$ or $\mathbf{\Delta}_\xi\backslash\mathbf{G}_\xi$. This proves that $\hat{\pi}_{\hat{\epsilon}}$ is

(1) a finite covering map for all $\hat{\epsilon}$;
(2) a diffeomorphism for all $\hat{\epsilon} = (\xi, s)$ s.t. $\mathbf{\Delta}_\xi = 1$.

We wish to construct neighbourhoods of the level set T_ϵ. Since $\mathbf{\Delta}_\xi$ is conjugate to a discrete subgroup of $\mathbf{K}$, it is a cyclic group of order $n = |\mathbf{\Delta}_\xi|$. Let $\mathbf{Z}_n$ act on $S^1 \times D^2$ via $(\theta, z) \mapsto (\theta + \frac{p}{n}, e^{2i\pi\frac{q}{n}} z)$ for integers p, q coprime to n. The quotient $(S^1 \times D^2)/\mathbf{Z}_n$ is a manifold with an S^1 action, but there are two types of S^1 orbits: the regular ones through $(0, z)$ for $z \neq 0$; and a short orbit through $(0, 0)$.

Lemma 7.1. *T_ϵ has a neighbourhood in $T(\mathbf{\Delta}\backslash\mathbf{G})$ diffeomorphic to*

(1) *$S^1 \times S^1 \times \mathbf{R}^4$ if $s > 0$ and $\mathbf{\Delta}_\xi = 1$;*
(2) *$S^1 \times D^2 \times \mathbf{R}^3$ if $s = 0$ and $\mathbf{\Delta}_\xi = 1$;*
(3) *$(S^1 \times D^2)/\mathbf{Z}_n \times S^1 \times \mathbf{R}^2$ if $s > 0$ and $\mathbf{\Delta}_\xi \neq 1$;*
(4) *$(S^1 \times D^2)/\mathbf{Z}_n \times \mathbf{R}^3$ if $s = 0$ and $\mathbf{\Delta}_\xi \neq 1$;*

The level set T_ϵ is identified with $S^1 \times S^1 \times 0$ in case 1, $S^1 \times 0 \times 0$ in cases 2 and 4, and $S^1 \times 0 \times S^1 \times 0$ in case 3.

Proof. It is clear that there is a $\mathbf{\Delta}_\xi$-invariant neighbourhood $D_\xi \subset \mathrm{Ad}_{\mathbf{G}}\xi$ such that D_ξ is a disk and $\mu \in D_\xi$ implies that $\mathbf{\Delta}_\mu = 1$ or $\mu = \xi$. The neighbourhood $D = \mathbf{R}^+ D_\xi$ of ξ in $\mathfrak{g}_+$ is then diffeomorphic to $\mathbf{R} \times D^2$. Combine this with $\hat{\omega}$ (Theorem 5.1) and the lemma follows. □

Corollary 7.2. *The map $J : \mathbf{\Delta}\backslash\mathbf{G} \times \mathfrak{g}_+ \to \mathbf{\Delta}\backslash\mathfrak{g}_+ \times \mathbf{R}^{\geq 0}$ is continuous and onto, and it is a smooth submersion on $J^{-1}(\mathbf{\Delta}\backslash\mathfrak{g}_+^r \times \mathbf{R}^+)$.*

7.2. **Semisimplicity.** Define

$$D = h^{-1}([0, \frac{1}{2}]), \qquad S = h^{-1}(\frac{1}{2}),$$

which are subsets of $\mathfrak{g}_+ \cup \mathfrak{g}_0$. The image of $\Psi_{\mathbf{G}}|\mathbf{G} \times D$ is $\mathrm{Ad}_{\mathbf{G}}D$, which is also the image of $\Psi_{\mathbf{G}}|\mathbf{G} \times S$. It is clear that the image of $\hat{J}|\mathbf{G} \times D$ is $\mathrm{Ad}_{\mathbf{G}}D \times \mathbf{R}^{\geq 0}$. The sphere S is the union of two open disks $S_\pm = \{\xi \in S \ : \ \pm(c(\xi) - \frac{1}{\sqrt{2}}) > 0\}$ and the circle $\bar{S}_- \cap \bar{S}_+$ along which r vanishes. Let $\tau_\pm : S_\pm \to \mathbf{R}$ be defined by $\tau_\pm(\xi) = \frac{1}{\sqrt{2}}(1 \pm r) - r$. These functions extend to $\mathrm{Ad}_{\mathbf{G}}S_\pm$ by the adjoint invariance of r. Define

$$\hat{B}_{1,\pm} = \{(\xi, s) \ : \ \xi \in \mathrm{Ad}_{\mathbf{G}}S_\pm \cap \mathfrak{g}_+^r, \ s = \tau_\pm(\xi) > 0\},$$

which is the set of elements in the image of $\hat{J}|\mathbf{G} \times S_\pm$ with trivial $\mathbf{\Delta}$-stabilizer. Let $\hat{B}_1 = \hat{B}_{1,-} \cup \hat{B}_{1,+}$.

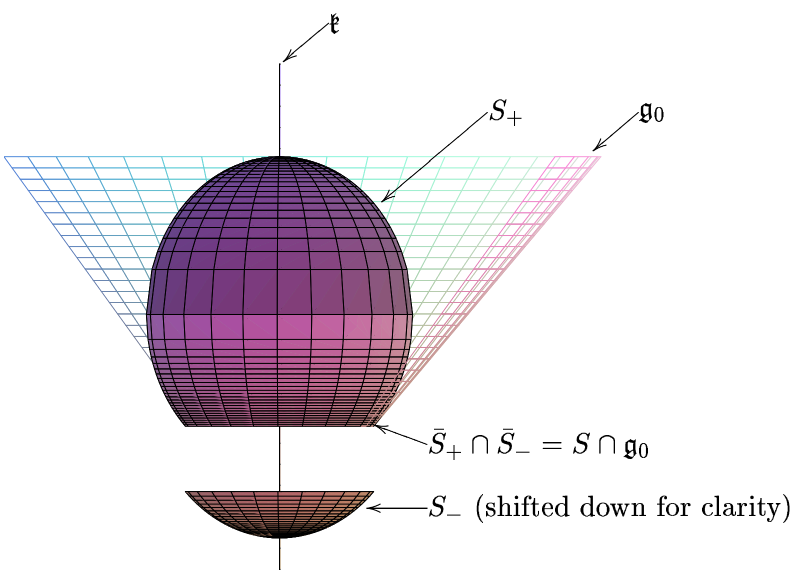

FIGURE 2. A cutaway view of S and $\mathfrak{g}_0$.

Similarly,

$$\hat{B} = (\mathrm{Ad}_{\mathbf{G}}D \cap \mathfrak{g}_+^r) \times \mathbf{R}^+,$$

is the set of regular values of $\hat{J}|\mathbf{G} \times D$ which have trivial $\mathbf{\Delta}$-stabilizer.

Finally, define $\hat{L} = \hat{J}^{-1}(\hat{B})$, $\hat{f} = \hat{J}|\hat{L}$ and $\hat{L}_1 = \hat{J}^{-1}(\hat{B}_1)$, $\hat{f}_1 = \hat{J}|\hat{L}_1$. The set $\hat{L}$ (resp. $\hat{L}_1$) is fibred by regular 2-tori; $\hat{L}$ is connected since $\hat{B}$ is connected, while $\hat{L}_1$ is the union of two components $\hat{L}_{1,\pm}$. Let $\hat{\Gamma}$ (resp. $\hat{\Gamma}_1$) be the complement of $\hat{L}$ in $\mathbf{G} \times D$ (resp. $\hat{L}_1$ in $\mathbf{G} \times S$).

Lemma 7.3. $\hat{\Gamma}$ & $\hat{\Gamma}_1$ *are* $\mathbf{\Delta}$*-invariant, closed, nowhere dense, and real-analytic.*

Proof. It suffices to establish that $\hat{L}$ is an open, real-analytic, $\mathbf{\Delta}$-invariant subset of $\mathbf{G} \times D$. It is clear that $\hat{L}$ is open and $\mathbf{\Delta}$-invariant. From the above definition of $\hat{B}$, it is apparent that

$$(g,\eta) \in \hat{L} \iff h(\eta) \le \frac{1}{2} \ \& \ \eta \notin \mathfrak{k} \ \& \ \mathrm{Ad}_{g^{-1}}\eta \in \mathfrak{g}^r_+.$$

Since $\mathfrak{g}^r_+$ is a real-analytic set, it is clear that $\hat{L}$ is a real-analytic set. □

Recall that $S\Sigma = \mathbf{\Delta}\backslash\mathbf{G} \times S$ and $D\Sigma = \mathbf{\Delta}\backslash\mathbf{G} \times D$.

Theorem 7.4. *The action* φ *of* $\mathbf{R}^2$ *on* $S\Sigma$ *(resp.* $D\Sigma$*) induced by the hamiltonian flows of* H *and* K *is Hopf-Rinow and* 2*-semisimple with respect to* (f_1, L_1, B_1) *(resp.* (f, L, B)*) where*

$$\begin{array}{ccc} L_1 & \xrightarrow{\text{diffeo.}} & \Sigma_* \times S^1 \times \mathbf{R} \times \mathbf{Z}_2 \\ \downarrow f_1 & & \downarrow p_* \times id_{\mathbf{R}} \times id_{\mathbf{Z}_2} \\ B_1 & \xrightarrow{\text{diffeo.}} & \Lambda_* \times \mathbf{R} \times \mathbf{Z}_2 \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\text{diffeo.}} & \Sigma_* \times S^1 \times \mathbf{R} \times \mathbf{R}^{\ge 0} \\ \downarrow f & & \downarrow p_* \times id_{\mathbf{R}} \times id_{\mathbf{R}^{\ge 0}} \\ B & \xrightarrow{\text{diffeo.}} & \Lambda_* \times \mathbf{R} \times \mathbf{R}^{\ge 0}. \end{array}$$

The singular set Γ_1 *is diffeomorphic to* $\big(\Sigma \times (S^1 \cup S^0)\big) \cup SS$*. The action* $\varphi|\Gamma_1$ *is conjugate to*

1. *the action of a maximal compact subgroup of* $\mathrm{PSL}_2\mathbf{R}$ *on* $\Sigma \times S^0$*;*
2. *the action of a unipotent subgroup of* $\mathrm{PSL}_2\mathbf{R}$ *on* $\Sigma \times S^1$*;*
3. *the action of a* 2*-torus on* SS*.*

Thus, all 1*-parameter subgroups of* φ *have zero topological entropy. The fibration* f_1 *(*f*) is non-trivial iff* $\mathbf{\Delta}$ *is cocompact iff the Chern class of* p *is non-zero.*

Proof. The diffeomorphism σ (Lemma 4.1) induces $\mathbf{\Delta}$-equivariant diffeomorphisms

$$\hat{B} \to \mathbf{G}_*/\mathbf{K} \times (0, \sqrt{2}+1] \times \mathbf{R}^+, \qquad \hat{B}_{1,\pm} \to \mathbf{G}_*/\mathbf{K} \times (0, \sqrt{2} \pm 1).$$

The $\mathbf{\Delta}$-equivariance of these diffeomorphisms induces the commutative diagram in Figure 2. $\mathbf{\Delta}$-equivariance implies that Figure 3 is a commutative diagram of $\mathbf{T}^2$-fibre bundles. The commutative diagrams in Figures 2 and 3 show that the horizontal maps in Figure 3 are homotopy equivalences. These facts, along with Lemma 7.3, suffice to prove the diagrams in the present Theorem and the 2-semisimplicity of φ.

Assume that $(\mathbf{\Delta}g, \eta) \in \Gamma_1$. There are three cases to be examined: either $\eta \in S \cap \mathfrak{k}_+$ or $\eta \in S \cap \mathfrak{g}_0$ or $\mathbf{\Delta}_\xi \ne 1$ where $\xi = \mathrm{Ad}_g\eta$. Along $\mathfrak{k}_+$, the hamiltonians h and κ are dependent, hence the action $\hat{\varphi}|\mathbf{G} \times \mathfrak{k}_+$ is a time-change of $\mathbf{K}$. Since $S \cap \mathfrak{k}_+ = S^0$, this proves assertion part 1.

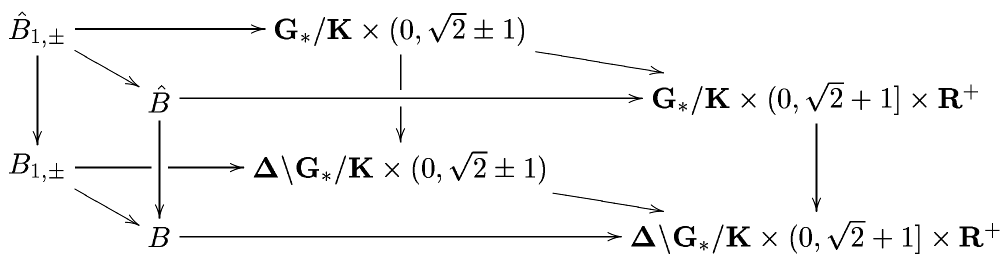

FIGURE 3

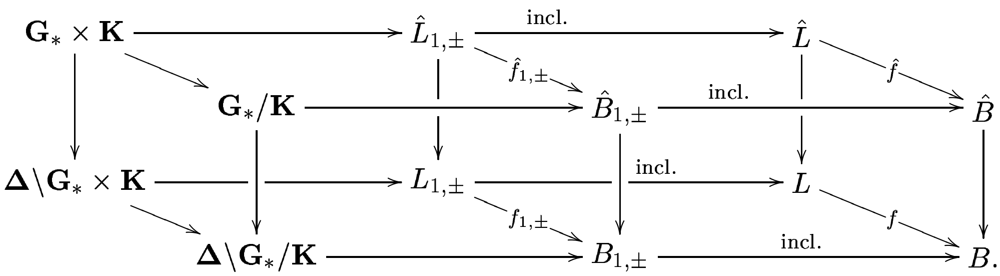

FIGURE 4

Along $S \cap \mathfrak{g}_0 = S^1$, the hamiltonians h and κ are dependent. For each $\eta \in S \cap \mathfrak{g}_0$ the action $\hat{\varphi}|\mathbf{G} \times \eta$ is generated by the unipotent group $\exp(\mathbf{R}\eta)$. This proves part 2.

If $\mathbf{\Delta}_\xi \neq 1$, then Lemma 7.1, cases 3 & 4, provide a J-saturated neighbourhood of the J-level set T_ϵ where $\epsilon = J(\mathbf{\Delta}g, \eta)$. It is clear that the action $\varphi|T_\epsilon$ is transitive on T_ϵ. Since T_ϵ is a torus, this proves part 3.

Since the horizontal maps in Figure 3 are homotopy equivalences, the monodromy of the $\mathbf{T}^2$-fibre bundle f (resp. $f_{1,\pm}$) is trivial while the Chern class of f (resp. $f_{1,\pm}$) pulls back to the Chern class of the principal $\mathbf{T}^2$-bundle

$$\mathbf{K} \times \mathbf{K} \hookrightarrow \mathbf{\Delta}\backslash\mathbf{G}_* \times \mathbf{K} \to \mathbf{\Delta}\backslash\mathbf{G}_*/\mathbf{K}.$$

The Chern class of this fibre bundle generates $H^2(\mathbf{\Delta}\backslash\mathbf{G}_*/\mathbf{K}; \mathbf{Q})$. The latter is non-trivial iff $\mathbf{\Delta}$ is cocompact (whence $\mathbf{G} = \mathbf{G}_*$). □

8. FURTHER CONSTRUCTIONS

8.1. **Generalities.** This section proves a couple results concerning semisimplicity and its behaviour under reduction. More general results are certainly true, but the present ones suffice for this paper. The first observation is elementary:

Lemma 8.1. *The product of semisimple (resp. Hopf-Rinow) actions is semisimple (resp. Hopf-Rinow).*

Let $\varphi : \mathbf{R}^m \times M \to M$ be a k-semisimple action with respect to (f, L, B). Let $\mathbf{T}^n$ be an n-torus that acts freely on M. Assume that $\mathbf{T}^n$ preserves the map f. Let $\pi : M \to M' = M/\mathbf{T}^n$ be the projection map of this principal-fibre bundle. One has the induced structure on M':

$$\begin{array}{ccccc} \mathbf{T}^k & \hookrightarrow & L & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbf{T}^{k'} & \hookrightarrow & L' & \xrightarrow{f'} & B \end{array}$$

where $k' = k - n$, $L' = L/\mathbf{T}^n$ and f' is the map induced by f. Since L is $\mathbf{T}^n$-invariant, its complement Γ is also. Let $\Gamma' = M' - L'$ be the projection of Γ. Finally, let $\varphi' : \mathbf{R}^m \times M' \to M'$ be the induced action.

Lemma 8.2. *If Γ is an analytic set and $\mathbf{T}^n$ acts analytically on M, then φ' is k'-semisimple with respect to (f', L', B').*

It suffices to prove that Γ' is an analytic set. This follows from a simple averaging argument.

Let

$$\begin{array}{ccccc} \mathbf{E}_1 & \hookrightarrow & \mathbf{E} & \hookleftarrow & \mathbf{E}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma_1 & \hookrightarrow & \Sigma_1 \times \Sigma_2 & \hookleftarrow & \Sigma_2 \end{array}$$

be a commutative diagram of natural inclusions. Let $\varphi : \mathbf{R}^m \times \mathbf{E} \to \mathbf{E}$ be an action such that $\mathbf{E}_i$ is an invariant set for $i = 1, 2$. Let φ_i be the action on $\mathbf{E}_i$.

Lemma 8.3. *If φ_i is Hopf-Rinow over q_i, then φ is Hopf-Rinow over $q = (q_1, q_2)$.*

The proof is obtained by including Σ_1 via $q \mapsto (q, q_2)$ and similarly for Σ_2.

Assume that a compact Lie group $\mathbf{T}$ acts freely on Σ and hence on the total space $\mathbf{E}$ of a fibre bundle over Σ. Let

$$\begin{array}{ccc} \mathbf{E} & \twoheadrightarrow & \mathbf{E}/\mathbf{T} \\ \downarrow & & \downarrow \\ \Sigma & \twoheadrightarrow & \Sigma/\mathbf{T} \end{array}$$

be the commutative diagram of projection maps. If φ is a $\mathbf{T}$-equivariant action, then there is a well-defined action φ' on $\mathbf{E}/\mathbf{T}$.

Lemma 8.4. *If $\varphi : \mathbf{R}^m \times \mathbf{E} \to \mathbf{E}$ is Hopf-Rinow, then the induced action $\varphi' : \mathbf{R}^m \times \mathbf{E}/\mathbf{T} \to \mathbf{E}/\mathbf{T}$ is Hopf-Rinow.*

8.2. **A further example.** In this section, $\mathbf{\Delta}$ is a torsion-free discrete group; $\mathbf{K}$ acts freely on $\mathbf{\Delta}\backslash\mathbf{G}$.

Let $TS^2 = \{(v, x) \in \mathbf{R}^3 \times \mathbf{R}^3 \ : \ |x| = 1, \langle v, x\rangle = 0\}$ be the tangent bundle of S^2 and let SO(3) act on TS^2 in its natural manner. The momentum map is

$$\Psi_{\mathrm{SO}(3)}(v, x) = \frac{1}{2}(xv' - vx'), \qquad \Psi_{\mathrm{SO}(3)} : TS^2 \to so(3).$$

Let $\{\mu_i\}$ be an orthonormal basis of $so(3)$ with respect to the trace form. Let $\mu \in so(3)$ and define the functions

$$\ell_1(\mu) = \langle \mu, \mu_1 \rangle, \qquad \ell_2(\mu) = \frac{1}{2}(\langle \mu, \mu_2 \rangle^2 + \langle \mu, \mu_3 \rangle^2)$$

where $\langle \bullet, \bullet \rangle = \mathrm{Tr}(\bullet . \bullet)$ is the trace form. These functions Poisson commute relative to the canonical Poisson structure on $so(3)$, and since $\Psi_{\mathrm{SO}(3)}$ is a Poisson map, the pullbacks

$$F_1 = \langle v, \mu_1 x \rangle = \ell_1 \circ \Psi_{\mathrm{SO}(3)}, \qquad F_2 = \frac{1}{2}(\langle v, \mu_2 x \rangle^2 + \langle v, \mu_3 x \rangle^2) = \ell_2 \circ \Psi_{\mathrm{SO}(3)}$$

also Poisson commute (here $\langle , \rangle$ is the standard inner product).

The function F_1 can be viewed as the momentum map of the action of the group $\exp(\mathbf{R}\mu_1) \subset \mathrm{SO}(3)$. This group can, and will henceforth, be identified with $\mathbf{K} = \mathrm{SO}(2)$. A straightforward computation shows that

Lemma 8.5. *The critical-point set of the map* $F = (F_1, F_2) : TS^2 \to \mathbf{R} \times \mathbf{R}^{\geq 0}$ *is* $F_2^{-1}(0)$. *The critical-value set is* $\mathbf{R} \times 0$. *The regular fibres of* F *are tori.*

Let $\hat{Q}((\mathbf{\Delta} g, \xi) \times (v, x)) = H(\mathbf{\Delta} g, \xi) + F_2(v, x)$, which is a $\mathbf{K}$-invariant fibre-wise positive semi-definite quadratic form. Let $\mathbf{E}$ be the subbundle of $T(\mathbf{\Delta} \backslash \mathbf{G} \times S^2)$ defined by $(\mathbf{\Delta} g, \xi, v, x) \in \mathbf{E}$ iff

$$a^2 + b^2 + (c - \sqrt{2})^2 + \langle v, \mu_2 x \rangle^2 + \langle v, \mu_3 x \rangle^2 = 1, \quad \text{and } c = \langle v, \mu_1 x \rangle.$$

The subbundle $\mathbf{E}$ is the intersection of $\hat{Q}^{-1}(\frac{1}{2})$ with the zero level of the momentum map of $\mathbf{K}$'s anti-diagonal action on $\mathbf{\Delta} \backslash \mathbf{G} \times S^2$. It is clear that $\mathbf{\Delta} \backslash \mathbf{G} \times S$ embeds naturally in $\mathbf{E}$. The functions H, K, F_2 induce a hamiltonian action of $\mathbf{R}^3$ on $T(\mathbf{\Delta} \backslash \mathbf{G} \times S^2)$. This action is tangent to $\mathbf{E}$, it is Hopf-Rinow, 4-semisimple (with an analytic singular set) and it is $\mathbf{K}$-invariant. We can therefore apply Lemmas 8.1–8.4 to conclude that the induced action on $\mathbf{E}/\mathbf{K}$ is Hopf-Rinow and 3-semisimple. Observe that $\mathbf{E}/\mathbf{K}$ is isomorphic to the unit-tangent sphere bundle of $\mathbf{\Delta} \backslash \mathbf{G} \times_{\mathbf{K}} S^2$. The diagram in Theorem 8.6 is a simple consequence of Section 6 and Lemma 8.2. This proves Theorem 8.6 and Theorem 1.4.

Theorem 8.6. *Let* $\varphi : \mathbf{R}^3 \times S\Omega \to S\Omega$ *be the action induced by the hamiltonians* H, K, F_2. *Then* φ *is Hopf-Rinow and* 3*-semisimple with respect to* (f, L, B) *where*

$$\begin{array}{ccc} L & \xrightarrow{\text{diffeo.}} & \Sigma \times \mathbf{T}^2 \times \mathbf{R}^3 \\ \downarrow f & & \downarrow p \times id_{\mathbf{R}^3} \\ B & \xrightarrow{\text{diffeo.}} & \Lambda \times \mathbf{R}^3 \end{array} \qquad \textit{commutes.}$$

Remark 1. This construction works for any action of $\mathbf{K}$ on $\mathbf{\Delta} \backslash \mathbf{G} \times S^2$ that is of the form $\theta.(\mathbf{\Delta} g, x) = (\mathbf{\Delta} g \theta, \theta^n x)$, where $n \in \mathbf{Z} - 0$. The manifold $\Omega = \mathbf{\Delta} \backslash \mathbf{G} \times_{\mathbf{K}} S^2$ contains a canonical $\mathbf{C}^1$-bundle over $\mathbf{\Delta} \backslash \mathbf{G} / \mathbf{K}$. If the Chern class of this $\mathbf{C}^1$-bundle is even, then Ω is trivial as an S^2-bundle; otherwise it is not trivial.

Remark 2. This construction also works for higher-dimensional products $\mathbf{\Delta}\backslash\mathbf{G} \times S^m$. When m is odd (say $m = 3$) there are free actions of $\mathbf{K}$ on this product when $\mathbf{\Delta}$ is discrete but not necessarily torsion-free.

References

[1] O. I. Bogoyavlenskij. Theory of tensor invariants of integrable Hamiltonian systems. II. Theorem on symmetries and its applications. *Comm. Math. Phys.* 184:2 (1997) 301–365.

[2] A.V. Bolsinov and I.A. Taшmanov. Integrable geodesic flows with positive topological entropy, *Invent. Math.* 140:3 (2000) 639–650.

[3] L. T. Butler. New examples of integrable geodesic flows. *Asian J. Math.*, 4:3 (2000) 515–526.

[4] L. T. Butler. Integrable geodesic flows with wild first integrals: the case of two-step nilmanifolds. *Ergodic Theory Dynam. Systems* 23:3 (2003) 771–797.

[5] L. T. Butler. Invariant fibrations of geodesic flows. *Topology* 44:4 (2005) 769–789.

[6] V. L. Ginzburg. On the existence and non-existence of closed trajectories for some Hamiltonian flows. *Math. Z.* 223:3 (1996) 397–409.

[7] S. Helgason. Differential geometry, Lie groups, and symmetric spaces. Graduate Studies in Mathematics, 34. AMS, Providence, RI, 2001 (orig. 1978).

[8] V.V. Kozlov. Topological obstacles to the integrability of natural mechanical systems, *Dokl. Akad. Nauk SSSR* 249:6 (1979) 1299–1302.

[9] I. A. Taшmanov. Topological obstructions to the integrability of geodesic flows on nonsimply connected manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51:2 (1987) 429–435, 448.

[10] I. A. Taшmanov. An example of the transition from chaos to integrability in magnetic geodesic flows. (Russian) *Mat. Zametki* 76:4 (2004) 632–634; translation in *Math. Notes* 76:3-4 (2004) 587–589

School of Mathematics, The University of Edinburgh, 6214 James Clerk Maxwell Building, Edinburgh, UK, EH9 3JZ

E-mail address: `l.butler@ed.ac.uk`