# Invariant fibrations of geodesic flows

pi1.tex

Leo T. Butler [1]

*Department of Mathematics and Statistics*
*Queen's University, Kingston, ON, Canada K7L 3N6*

**Abstract**

Let $(\Sigma, \mathbf{g})$ be a compact $C^2$ finslerian 3-manifold. If the geodesic flow of $\mathbf{g}$ is completely integrable, and the singular set is a tamely-embedded polyhedron, then $\pi_1(\Sigma)$ is almost polycyclic. On the other hand, if $\Sigma$ is a compact, irreducible 3-manifold and $\pi_1(\Sigma)$ is infinite polycyclic while $\pi_2(\Sigma)$ is trivial, then $\Sigma$ admits an analytic riemannian metric whose geodesic flow is completely integrable and singular set is a real-analytic variety. Additional results in higher dimensions are proven.

*Key words:* geodesic flows, integrable systems, nonintegrability, momentum map, liouville foliations, 3-manifolds
*1991 MSC:* 37J30, 37K10, 53C60, 53C22, 53D25

## 1 Introduction

A smooth ($C^1$) flow $\phi_t : M \to M$ is *integrable* if there is an open, dense subset $R$ that is covered by angle-action charts $(\theta, I) : U \to \mathbf{T}^k \times \mathbf{R}^l$ which conjugate $\phi_t$ with a translation-type flow $(\theta, I) \mapsto (\theta + t\omega(I), I)$. Evidently, there is an open dense subset $L \subset R$ fibred by $\phi_t$-invariant tori [2]. Let $f : L \to B$ be the $C^1$ fibration which quotients $L$ by these invariant tori and let $\Gamma = M - L$ be the *singular set*. If $\Gamma$ is a tamely-embedded polyhedron, then $\phi_t$ is called *k-semisimple* with respect to $(f, L, B)$. We say $\phi_t$ is semisimple if it is $k$-semisimple with respect to some $(f, L, B)$.

*Email address:* `lbutler@mast.queensu.ca` (Leo T. Butler).
[1] Thanks to Keith Burns, Oleg Bogoyavlenskij and a referee for their comments on this paper. Research partially supported by the Natural Sciences and Engineering Research Council of Canada.

A geodesic flow on a unit-sphere bundle $S\Sigma$ is *completely integrable* if it is integrable with invariant tori of dimension $\dim \Sigma$. It is evident that $\dim \Sigma$-semisimplicity is a definition of topologically-tame complete integrability.

A group $\Delta$ is *polycyclic* of step length $c$ if there is a finite chain of subgroups $1 = \Delta_0 \lhd \cdots \lhd \Delta_c = \Delta$, with $\Delta_{i-1}$ normal in $\Delta_i$ and $\Delta_i/\Delta_{i-1}$ cyclic for all $i$. In the present paper, $\Sigma$ is always a boundaryless manifold.

**Theorem 1** *Let $(\Sigma, \mathbf{g})$ be a compact $C^2$ finslerian 3-manifold. If the geodesic flow of $\mathbf{g}$ is 3-semisimple, then $\pi_1(\Sigma)$ contains a finite-index polycyclic subgroup of step length at most 4.*

Recall that $\Sigma$ is *irreducible* if every tamely-embedded 2-sphere bounds a 3-ball. Evans and Moser classify the solvable groups that appear as the fundamental group of a compact 3-manifold in [16]; in this case they are all polycyclic. Their result, along with [8,9], help to prove

**Theorem 2** *Let $\Sigma$ be a compact, irreducible 3-manifold with $\pi_1(\Sigma)$ infinite polycyclic and $\pi_2(\Sigma) = 0$. Then $\Sigma$ admits an analytic riemannian metric whose geodesic flow is 3-semisimple.*

Four comments on Theorem 2: First, irreducibility is a technical hypothesis that precludes $\Sigma$ from containing fake 3-balls. The Poincaré conjecture would make redundant this hypothesis. Second, Evans and Moser's work, along with that of Hempel and Jaco [20], implies that if $\Sigma$ satisfies the hypotheses of Theorem 2, then $\Sigma$ admits flat geometry, $Nil$-geometry or $Sol$-geometry. These geometries supply the mentioned metrics. Third, if $\pi_1(\Sigma)$ is finite, then the geometrization conjecture implies that $\Sigma$ admits $S^3$-geometry. Fourth, if $\pi_1(\Sigma)$ is infinite polycyclic and $\pi_2(\Sigma) \neq 0$, then Theorem 5.1 of [16] implies that $\pi_1(\Sigma)$ is isomorphic to one of $\pi_1(S^2 \times S^1)$, $\pi_1(P^2 \times S^1)$ or $\pi_1(P^3 \# P^3)$. The universal covering space $\tilde{\Sigma}$ of $\Sigma$ is two-ended and, according to Ian Agol, the geometrization conjecture implies that $\tilde{\Sigma}$ must be homeomorphic to $S^2 \times \mathbf{R}$. Some work establishes that $\Sigma$ itself is a geometric 3-manifold homeomorphic to $P^2 \times S^1$, $P^3 \# P^3$ or one of the two $S^2$-bundles over $S^1$. From the proof of Theorem 2 follows

**Theorem 3** *Assume the geometrization conjecture. Let $\Sigma$ be a compact 3-manifold such that $\pi_1(\Sigma)$ is polycyclic. Then $\Sigma$ admits an analytic riemannian metric whose geodesic flow is 3-semisimple.*

Say $(f', L', B')$ is a *refinement* of $(f, L, B)$ if $B'$ is an open dense subset of $B$, $L' = f^{-1}(B')$, $f' = f|L'$ and $\Gamma' = M - L'$ is a tamely-embedded polyhedron; it is *tractable* if, for each component $L'_i$ of $L'$, either (1) there is an $f$-saturated, codimension-1 submanifold $W_i \subset L'_i$ such that the inclusion map $\iota_{W_i,L'_i}$ is epimorphic on $\pi_1$ or (2) there is a component $L'_j$ satisfying (1) and a map $r : L'_i \to L'_j$ such that $\iota_{L'_i,M}$ is homotopic to $\iota_{L'_j,M} \circ r$. By Lemma 18, $(f, L, B)$

has a tractable refinement, so it can be assumed from the outset that $(f, L, B)$ is tractable.

A group is *small* if it does not contain a free group on two generators.

**Theorem 4** *Let* $(\Sigma, \mathbf{g})$ *be a compact* $C^2$ *finslerian* 4*-manifold. If the geodesic flow of* $\mathbf{g}$ *is* 4*-semisimple and the fundamental group of each component of* $B$ *is small, then* $\pi_1(\Sigma)$ *contains a finite-index polycyclic subgroup of step length at most* 6*.*

Theorems 1 and 4 have similar proofs. Since $(f, L, B)$ is tractable, each component $L_i$ of $L$ "looks like" a $\mathbf{T}^k$-bundle over a codimension-1 submanifold $N \subset B$. $N$ is a 1-manifold in Theorem 1 and a 2-manifold whose fundamental group is small in Theorem 4. In both cases, $\pi_1(N)$ is almost abelian, which implies $\pi_1(L_i)$ is almost polycyclic. Theorems 1 and 4 then follow from Lemma 15, which states that there is a component $L_i$ such that the inclusion map $\iota_{L_i,S\Sigma}$ has a finite-index image in $\pi_1(S\Sigma)$. Similar ideas appear in Taĭmanov's work [32].

Without the condition on $B$ in Theorem 4, it seems difficult to deduce any constraint on $\pi_1(\Sigma)$ with the techniques of this paper. One is led to ask

**Question A**: *is there any obstruction to the existence of a* 4*-semisimple geodesic flow on a* 4*-manifold?*

and more pointedly,

**Question B**: *is there a* 4*-semisimple geodesic flow on* $(S^3 \times S^1)\#(S^3 \times S^1)$*?*

*Algebraic properties of* $\pi_1$ *and recurrence*: A subgroup is *almost normal* if its normalizer is of finite-index.

**Definition 5** *A group is* anabelian *if its only abelian, almost normal subgroup is the trivial group* 1*.*

A Gromov-hyperbolic group is either anabelian or almost cyclic. Also, the fundamental group of a finite-volume riemannian manifold of non-positive curvature is either anabelian or almost abelian.

Recall two notions of recurrence: (1) $x$ is periodic if there is a $T > 0$ such that $\phi_T(x) = x$. $\mathcal{P}(\phi)$ is the closure of the set of periodic points. (2) the $\omega$-limit set of $x$, $\omega(x)$, is the set of points $y$ such that $\phi_{t_k}(x) \to y$ for some sequence $t_k \to +\infty$. The $\alpha$-limit set, $\alpha(x)$, is defined similarly. If $x \in \omega(x) \cap \alpha(x)$, then $x$ is a recurrent point. The limit-point set $\mathcal{L}(\phi)$ is the closure of the recurrent-point set.

Let $\tilde{\Sigma}$ be the universal cover of $\Sigma$. The pullback of $\bullet$ on $S\Sigma$ to $S\tilde{\Sigma}$ is $\bar{\bullet}$.

**Theorem 6** *Let $(\Sigma, \mathbf{g})$ be a compact $C^2$ finsler manifold whose geodesic flow $\phi_t$ is semisimple. If $\pi_1(\Sigma)$ is anabelian, then* $\operatorname{Int} \mathcal{L}(\bar{\phi}) \neq \emptyset$.
*In any $C^2$ neighbourhood of $\mathbf{g}$, there is a finsler metric $\mathbf{k}$ with geodesic flow $\varphi_t$ such that* $\operatorname{Int} \mathcal{P}(\bar{\varphi}) \neq \emptyset$*, and $\phi_t$ & $\varphi_t$ share the same invariant tori.*

Following [24], one can construct a $C^\infty$ riemannian metric $\mathbf{g}$ on $\Sigma$ with $\operatorname{Int} \mathcal{P}(\bar{\varphi}) \neq \emptyset$: Isometrically identify an open disk $D \subset \Sigma$ with $\mathcal{S} = \{(x_0, \ldots, x_n) : x_0^2 + \cdots + x_n^2 = 1, x_n > -1/2\}$. Let $\mathbf{g}$ be a $C^\infty$ metric on $\Sigma$ that equals the round metric on $D \sim \mathcal{S}$. The geodesic flow of $\mathbf{g}$ possesses an open, invariant subset in $S_{\mathcal{S}}\Sigma$ that consists entirely of periodic orbits, and these orbits remain closed on the universal cover. One is led to ask

**Question C**: *if $(\Sigma, \mathbf{g})$ is a compact real-analytic riemannian manifold and $\pi_1(\Sigma)$ is anabelian, is* $\operatorname{Int} \mathcal{L}(\bar{\phi}) = \emptyset$*?*

A connected component $B_i$ of $B$ is *plentiful* if the inclusion $L_i \hookrightarrow S\Sigma$ has a finite-index image on $\pi_1$. Let $(\bar{f}, \bar{L}, \bar{B})$ be the pullback of $(f, L, B)$ to $S\tilde{\Sigma}$.

**Theorem 7** *Let $(\Sigma, \mathbf{g})$ be a compact $C^2$ finsler $n$-manifold whose geodesic flow is $n$-semisimple with respect to $(f, L, B)$. Assume that $\pi_1(\Sigma)$ is anabelian and $\Sigma$ is aspherical. If $B_i$ is plentiful, then $\pi_k(\bar{B}_i)$ is non-trivial for some $k \geq 1$.*

### *1.1 Background and Motivation*

This paper is motivated by a question posed by Kozlov [23]: which compact surfaces admit a riemannian metric with an integrable geodesic flow? Kozlov showed that if the geodesic flow is analytic and has an additional analytic first integral, then the surface's genus is at most one. Bolotin subsequenty generalized Kozlov's argument to non-compact surfaces, with an additional hypothesis on the behaviour of the metric at infinity [3,24]. Further work by Bolotin and Bolotin-Negrini shows that if $(\Sigma, \mathbf{g})$ is a compact real-analytic surface of genus greater than 1, then in a neighbourhood of any non-trivial minimal periodic orbit of the geodesic flow on $S\Sigma$ there is a horseshoe [4,5]. Taĭmanov [32,33] generalized Kozlov's argument to higher dimensions, and obtained three necessary conditions for a compact real-analytic manifold $\Sigma$ to admit a real-analytically integrable geodesic flow: (1) $\pi_1(\Sigma)$ must be almost abelian; (2) $\Sigma$'s first Betti number $b$ is at most $\dim \Sigma$; and (3) there is an injection of algebras $H^*(\mathbf{T}^b; \mathbf{Q}) \hookrightarrow H^*(\Sigma; \mathbf{Q})$. Taĭmanov introduced what he called a *geometrically simple* geodesic flow to prove these results; the methods of the current paper are indebted to Taĭmanov's conceptions.

Subsequently Paternain [27–29] introduced the *entropy approach* to study integrable geodesic flows. He showed that if a $C^\infty$ geodesic flow on a compact manifold $\Sigma$ is integrable with first integrals that either (i) satisfy Ito's non-degeneracy condition [21]; or (ii) generate a $\mathbf{T}^{n-1}$ action; or (iii) admit action-angle variables with singularities, then the topological entropy of the geodesic flow must vanish. By a result of Dinaburg and Bowen, $\pi_1(\Sigma)$ must be of subexponential word growth. These results led Paternain to conjecture that if $\Sigma$ admits a smoothly integrable geodesic flow (not necessarily satisfying any of the hypotheses (i–iii) above), then $\pi_1(\Sigma)$ is of polynomial word growth. Hence, $\pi_1(\Sigma)$ is almost nilpotent [19].

Note that Paternain's conclusions should be improvable. According to Desolneux-Moulis [14], an Ito-nondegenerate first-integral map admits a Whitney stratification. Thus, in cases (i) and (iii), the critical-point set of the first-integral map is a tamely-embedded polyhedron. If the image of the first-integral map also admits a Whitney stratification – and it almost certainly does – then the Kozlov-Taĭmanov theorem could be applied to conclude that $\pi_1(\Sigma)$ is almost abelian.

The Kozlov-Taĭmanov theorem appeared to give a reasonably complete characterization of manifolds with real-analytically integrable geodesic flows, so the examples [9,8] were surprising (see also [11,10,12,7]). In essence, the current author showed integrability, in $C^\infty$ integrals, of the geodesic flows on 3-dimensional manifolds with *Nil*geometry. Bolsinov and Taĭmanov extended this construction to a 3-manifold with *Sol*geometry. The first example showed that the necessary conditions for real-analytic integrablity derived by Kozlov and Taĭmanov are not necessary for smooth integrability (even of a real-analytic geodesic flow); the second examples showed that integrable geodesic flows may have positive topological entropy (even a real-analytic geodesic flow), thereby frustrating any simple generalization of Paternain's work. These examples were of elemental importance in constructing the definitions of semisimplicity and integrability offered in the first paragraph of the present paper.

### *1.2 Outline*

Section 3 proves several technical lemmas based on the technical hypotheses of Definition 9, then presents a proof of the main lemma in [32], suitably generalized to the present setting. In addition to generalizing Taĭmanov's lemma to non-compact manifolds, the proof shows the central importance of condition FI2 of Definition 9. Section 4 proves Theorem 1, Section 5 proves Theorem 4. In Section 6, the Butler and Bolsinov-Taĭmanov examples are shown to be completely integrable and semisimple. This section also shows that the geodesic flows on associated infra-nil- and infra-solv-manifolds are completely integrable and semisimple which suffices to prove Theorem 2.

## 2 Definitions and Notation

Here are several useful conventions

- $\amalg$ denotes disjoint union;
- $\pi : \mathbf{E} \to \Sigma$ is the footpoint projection;
- $\tilde{\Sigma}$ (resp. $\hat{\Sigma}$) is the universal cover (resp. a cover) of $\Sigma$;
- if $L \subset K$ then $\iota_L = \iota_{L,K} : L \to K$ is the inclusion map;
- a *geodesic flow* means the geodesic flow of a complete $C^2$ finsler metric.

Theorems 1,4,6 are proven by applying a more general result about *Hopf-Rinow* flows. As these results have some independent applicability, it seems desirable to expose them.

### *2.1 HR flows*

Let $\pi : \mathbf{E} \to \Sigma$ be a smooth ($C^1$) fibre bundle with compact fibres and let $\phi_t : \mathbf{E} \to \mathbf{E}$ be a flow. $\mathbf{E}$ may have a boundary but $\Sigma$ is boundaryless and $\phi_t$ may only be a local flow.

**Definition 8** *Let $q \in \Sigma$ and assume that for each non-trivial $[c] \in \pi_1(\Sigma; q)$ there is a $p \in \pi^{-1}(q)$ and a $T > 0$ such that $\gamma(t) := \pi\phi_{tT}(p)$, $0 \le t \le 1$, is a closed curve homotopic to c; then $\phi_t$ is* Hopf-Rinow *over q. An HR-flow is a flow that is Hopf-Rinow over q for some $q \in \Sigma$.*

The curve $\gamma(t)$ will be called a *geodesic*. An HR-vector field is one whose flow is HR. If two flows are orbitally equivalent and one is HR, then so is the other. By the Hopf-Rinow theorem, the geodesic flow of a complete $C^2$ finsler metric is an HR-flow on the unit-sphere bundle [18]. A skew product over an HR-flow is also HR. A second class of Hopf-Rinow flows are obtained as follows: let $\Sigma \subset M$ be an open submanifold with a geodesically convex boundary and $\mathbf{E} = S_\Sigma M$. The restriction of the geodesic flow to $\mathbf{E}$ defines a local flow that is HR.

### *2.2 $\mathfrak{F}$-semisimplicity*

Let us turn to a definition that abstracts some essential features of complete integrability. A continuous surjection $f : L \to B$ is a *fibration* if $f$ has the path-lifting property and the fibres of $f$ are path-connected. If $B$ is paracompact and connected, then the fibres of $f$ are of the same homotopy type, and so a "typical" fibre will be denoted by $F$ [34].

**Definition 9** *Let $\mathfrak{F} = \{f_i : L_i \xrightarrow{F_i} B_i\}_{i\ \in A}$ be a collection of fibrations. Let $\phi_t : \mathbf{E} \to \mathbf{E}$ be an HR-flow, $L = \amalg_{i \in A} L_i$ and suppose that:*

*(FI1)* $\Gamma = \mathbf{E} - L$ *is closed, $\phi_t$-invariant and nowhere dense;*
*(FI2) for each $v \in \mathbf{E}$ and open neighbourhood $U \ni v$, there is an open subset $W$, $v \in W \subseteq U$, such that $L \cap W$ has finitely many path-connected components;*
*(FI3) for each $i \in A$, $L_i$ is an open path-connected component of $L$ and either $f_i \circ \phi_t = f_i$ or the inclusion $F_i \hookrightarrow L_i$ induces an isomorphism on $\pi_1$.*

*Then we will say that $\phi_t$ is $\mathfrak{F}$-*semisimple*.*

If $\pi_1(F_i)$ is abelian for each $i \in A$, then $\phi_t$ will be said to be abelian-$\mathfrak{F}$-semisimple. Note that Definition 9 does not require the fibres $F_i$ to be compact.

### *2.3 Related Definitions of Integrability*

Let $(P, \{,\})$ be a Poisson manifold. If $\mathcal{F} \subset C^2(P)$, let $d\mathcal{F}_p = \operatorname{span}\{df_p : f \in \mathcal{F}\}$ and $Z(\mathcal{F}) = \{f \in \mathcal{F} : \{\mathcal{F}, f\} \equiv 0\}$. When $\mathcal{F}$ is a Lie subalgebra of $C^\infty(P)$, $Z(\mathcal{F})$ is the centre of $\mathcal{F}$. Let $l = \sup \dim d\mathcal{F}_p$, $k = \sup \dim dZ(\mathcal{F})_p$. Say that a point $p \in P$ is *strongly regular* for $\mathbf{f}$ if there is a saturated neighbourhood $U \ni p$ and $\mathbf{f}|U$ is a trivial fibre-bundle map.[2] A point $p \in P$ is *$\mathcal{F}$-regular* if there exists $f_1, \ldots, f_l \in \mathcal{F}$ such that $p$ is strongly regular for the map $\mathbf{f} = (f_1, \ldots, f_l)$ and $f_1, \ldots, f_k \in Z(\mathcal{F})$; if $p$ is not $\mathcal{F}$-regular then it is *$\mathcal{F}$-critical*. Let $L(\mathcal{F})$ be the set of $\mathcal{F}$-regular points.

**Definition 10** *$\mathcal{F} \subset C^2(P)$ is* tamely integrable *if*

*I1.* $k + l = \dim P$;
*I2.* $L(\mathcal{F})$ *is an open and dense subset of* $P$;
*I3.* $P - L(\mathcal{F})$ *is a tamely-embedded polyhedron.*

*A hamiltonian flow $\phi_t$ is tamely-$\mathcal{F}$-integrable if it enjoys a tamely integrable set of first integrals.*

See [6] for an analogous definition and further discussion. The conventional definitions of complete and non-commutative integrability fit within the framework of Definition 10.

[2] If $\mathbf{f}$ is proper, then strong regularity is equivalent to regularity.

Let $(\Sigma, \mathbf{g})$ be a complete $C^2$ finslerian manifold. The tangent bundle less its zero section, $\hat{T}\Sigma$, enjoys a Poisson structure and the geodesic flow, $\phi_t$, is a $C^1$ hamiltonian flow on $\hat{T}\Sigma$ with $C^2$ hamiltonian $H$ (see section 3.2 in [18] for further explanation).

**Theorem 11** *Assume $\Sigma$ is compact. Then $\phi_t$ is tamely-$\mathcal{F}$-integrable iff $\phi_t$ is integrable and semisimple.*

**PROOF.** If $\phi_t$ is tamely-$\mathcal{F}$-integrable, let $G$ denote the abelian group of $C^1$ diffeomorphisms of $\hat{T}\Sigma$ generated by the complete flows of $Y_h$, $h \in Z(\mathcal{F})$. By the Sussman-Stefan orbit theorem [22] and I1, the orbits of $G$ in $L(\mathcal{F})$ are embedded $C^1$ submanifolds. I1 and the properness of $H$ imply that for each $G$-orbit in $L(\mathcal{F})$, there is a $G$-invariant open neighbourhood, $U$, and an action of $\mathbf{T}^k$ on $U$, such that the $\mathbf{T}^k$-orbits and $G$-orbits coincide. Thus each connected component of $L(\mathcal{F})$ is fibred by $\mathbf{T}^k$-orbits. Therefore, there is a $C^1$ atlas of $L(\mathcal{F})$, $\mathcal{A} = \{\varphi = (\theta, I) : U \to \mathbf{T}^k \times \mathbf{R}^l\}$ which satisfies the universal property that for all $\theta \in \mathbf{T}^k$, $I \in \mathbf{R}^l$ and 1-parameter subgroups $g^t$ of $G$: $\varphi \circ g^t \circ \varphi^{-1}(\theta, I) = (\theta + t\xi(I), I)$ and $\xi : \mathbf{R}^l \to \mathbf{R}^k$ is $C^1$.

Let $L_o = L(\mathcal{F})$, $B_o = L_o/G$ and $f_o : L_o \to B_o$ be the orbit map. Since $\phi_t$ is a 1-parameter subgroup of $G$, it is integrable with respect to $(f_o, L_o, B_o)$ . By I3 of Definition 10 the complement of $L_o$ is a tamely-embedded polyhedron, so $\phi_t$ is semisimple with respect to $(f_o, L_o, B_o)$.

The opposite implication is straightforward. $\square$

Let $L = L_o \cap S\Sigma$, $B = f(L_o)$, $f = f_o|L$ and $\Gamma = \Gamma_o \cap S\Sigma$. By hypothesis, $\Gamma_o = \hat{T}\Sigma - L_o$ is a tamely-embedded polyhedron. Once can see that there is a triangulation of $T\Sigma$ such that $\Gamma_o$ and $S\Sigma$ are subcomplexes. Therefore $\phi_t|S\Sigma$ is integrable and semisimple with respect to $(f, L, B)$. By defining $A$ to be the set of connected components of $B$, and $\mathfrak{F} = \{f : L \to B\}$ it is immediately apparent that

**Theorem 12** *If $\phi_t$ is tamely-$\mathcal{F}$-integrable, then $\phi_t$ is abelian-$\mathfrak{F}$-semisimple.*

## 3 An Extension of the Kozlov-Taimanov Theorem

This section generalizes the Kozlov-Taĭmanov theorem to $\mathfrak{F}$-semisimple HR flows. We start with a couple elementary lemmas. Let $d_H$ denote the Hausdorff distance between two compacts sets.

**Lemma 13** *Assume that* $\mathbf{E} = \Gamma \amalg L$ *and that* $L$ *is dense and satisfies (FI2). If* $K \subset \mathbf{E}$ *is compact, then given* $\epsilon > 0$*, there is a bounded open set* $U$*,* $K \subseteq U$*,* $d_H(\overline{U}, K) \leq \epsilon$ *such that* $U \cap L$ *has finitely many path-connected components.*

**PROOF.** Let $\epsilon > 0$ be given and let $\mathcal{O}$ denote the set of open subsets of $\mathbf{E}$ whose diameter is less than $\epsilon/2$ and which intersect $L$ in finitely many path-connected components. By (FI2), $\mathcal{O}$ is a covering of $\mathbf{E}$, hence of $K$. By the compactness of $K$ there is a finite subcovering of $K$, which we will denote by $K_1, \ldots, K_k \in \mathcal{O}$. We may assume that $K_i \cap K \neq \emptyset$ for all $i$. Then $U := \cup_{i=1}^{k} K_i$ is a bounded open set containing $K$, and any point in $U$ is at most $\epsilon$ away from a point in $K$. Since $L \cap K_i$ has finitely many path-connected components, $U \cap L$ has finitely many path-connected components. $\square$

**Lemma 14 (Lifting Lemma)** *Assume that* $\phi_t : \mathbf{E} \to \mathbf{E}$ *is a Hopf-Rinow flow that satisfies (FI1 – FI3). Let* $p : \hat{\Sigma} \to \Sigma$ *be a covering of* $\Sigma$*. Then the flow* $\hat{\phi}_t : \hat{\mathbf{E}} \to \hat{\mathbf{E}}$ *is Hopf-Rinow and satisfies (FI1 – FI3).*

**PROOF.** Clearly, the Hopf-Rinow property is satisfied when we pass to a covering, so $\hat{\phi}_t$ is Hopf-Rinow. Let $\hat{\mathbf{E}} = p^*\mathbf{E}$ be the pull-back of $\mathbf{E}$, and let $P : \hat{\mathbf{E}} \to \mathbf{E}$ denote the covering map. We let $\hat{\Gamma} = P^{-1}(\Gamma)$ and $\hat{L} = P^{-1}(L)$. Since $P$ is a local homeomorphism and conditions (FI1) and (FI2) are purely local, they are obviously satisfied. Let $C$ denote the connected components of the set $\hat{L}$, so that $\hat{L} = \amalg_{j \in C} \hat{L}_j$ (we abuse notation and let $\hat{L}_j = j$ for each $j \in C$). Clearly, for each $j \in C$ there is an $i \in A$ such that $P(\hat{L}_j) = L_i$; we define $\check{f}_j : \hat{L}_j \to B_i$ by $\check{f}_j = f_i \circ P$. Since $f_i$ has the path-lifting property and $P$ is a local homeomorphism, $\check{f}_j$ has the path-lifting property [34]; and clearly $\check{f}_j$ is surjective.

Let $\hat{\pi}_j : \hat{B}_j \to B_i$ be a covering space of $B_i$ such that $\operatorname{im} \hat{\pi}_{j,*} = \operatorname{im} \check{f}_{j,*}$. By the usual properties of covering spaces, there is a continuous surjective map $\hat{f}_j$ such that

$$\begin{array}{ccc} \hat{L}_j & \xrightarrow{\hat{f}_j} & \hat{B}_j \\ & \overset{\check{f}_j}{\searrow} & \downarrow \hat{\pi}_j \\ & & B_i \end{array}$$

commutes. Since $\check{f}_j$ has the path-lifting property and $\hat{\pi}_j$ is a local homeomorphism, $\hat{f}_j$ has the path-lifting property. By construction, the fibres of $\hat{f}_j$ are path-connected, and since $\hat{f}_j$ is continuous and surjective, it is a fibration.

Let $\hat{F}_j = \hat{f}_j^{-1}(b)$ for some $b \in \hat{B}_j$. It follows that $\hat{\mathfrak{F}} = \{\hat{f}_j : \hat{L}_j \xrightarrow{\hat{F}_j} \hat{B}_j\}_{j \in C}$ is a collection of fibrations that satisfies (FI1–FI3). $\square$

**Lemma 15 (c.f. Theorem 1 [32])** *Assume that $\phi_t : \mathbf{E} \to \mathbf{E}$ is a Hopf-Rinow flow that satisfies* (FI1) *and* (FI2). *Then there is an $i \in A$ such that the map*

$$\pi_1(L_i) \to \pi_1(\mathbf{E}) \to \pi_1(\Sigma)$$

*has a finite-index image.*

**PROOF.** Assume that the Hopf-Rinow flow $\phi_t : \mathbf{E} \to \mathbf{E}$ satisfies (FI1) and (FI2) of Definition 9. Let $L := \cup_{\alpha \in A} L_\alpha$. Let $q \in \Sigma$ be a point at which $\phi_t$ satisfies the Hopf-Rinow property. Let $\mathbf{E}_q = \pi^{-1}(q)$ and let $Q$ and $P$ be open disks containing $q$ such that $\overline{Q} \subset P$. Let $U = \pi^{-1}(Q)$, $W = \pi^{-1}(P)$. The contractibility of $Q$ (resp. $P$) means that $U$ (resp. $W$) is topologically trivial, so $V \simeq Q \times \mathbf{E}_q$ (resp. $W \simeq P \times \mathbf{E}_q$). Since $\mathbf{E}_q$ is compact, lemma 13 implies that there is a bounded open set $V$ such that $\overline{U} \subset V \subset W$ and $L \cap V$ has a finite number of path-connected components. Let $K_1, \ldots, K_\omega$ be an enumeration of these path-connected components.

Because $K_i \subset W$, $\pi(K_i) \subset P$ so $\pi(K_i)$ can be contracted to the point $q$ in $W$ for all $i$. In addition, for each $i = 1, \ldots, \omega$, $K_i$ is a path-connected subset of $L$. By (FI2) each path-connected subset of $L$ lies in a unique component $L_\alpha$. That is, for each $i \in \{1, \ldots, \omega\}$ there is a unique $\alpha_i \in A$ such that $K_i \subset L_{\alpha_i}$. Finally, by (FI1) $L$ is dense in $\mathbf{E}$ so $\cup_{i=1}^{\omega} K_i$ is a dense subset of $V$.

For each $j = 1, \ldots, \omega$, select a point $v_j \in K_j$ and let $a_j : [0,1] \to P$ be a continuous curve that joins $q$ to $q_j := \pi(v_j)$. Let $a_j^*(t) := a_j(1-t)$ be the curve traversed in the opposite sense. By construction, each $q_j$ lies in the contractible set $P$.

Let us agree to call a curve $\gamma(t) = \pi \circ \phi_t(x)$, for $0 \le t \le T$ a *geodesic*; $\pi(x)$ is the footpoint of the geodesic $\gamma$.

Because $\phi_t$ is a Hopf-Rinow flow, for each non-trivial homotopy class $[\tilde{c}] \in \pi_1(\Sigma; q)$ there is a geodesic in $[\tilde{c}]$ with footpoint $q$. Let $u \in \mathbf{E}_q$ be the initial condition of such a geodesic $\gamma$, and let its length be $T$. The initial condition $u$ may lie in the singular set $\Gamma$, but by the continuity in initial conditions of $\phi_t$, for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $v \in \mathbf{E}$ and $d(u,v) < \delta$, then $d(\phi_t(u), \phi_t(v)) < \epsilon$ for all $t \in [0,T]$. By the openness of $V$, the density of $L \cap V$ in $V$, and the invariance of $L$ (FI2), for all $\epsilon > 0$ sufficiently small there is a $v \in L \cap V$ such that $\phi_T(v) \in L \cap V$ and so $\pi \circ \phi_T(v) \in P$. Because $L \cap V = \cup_{i=1}^{\omega} K_i$, there are $i, j \in \{1, \ldots, \omega\}$ such that $v \in K_i$ and $\phi_T(v) \in K_j$. Indeed, by the $\phi_t$-invariance of each $L_\alpha$ (FI2), there exists an $\alpha$ such that $K_i, K_j \subset L_\alpha$.

Let now $C$ be the arc in $\mathbf{E}$ that consists of an arc $s : [0,1] \to K_i$ joining $v_i$ to $v$, followed by the arc obtained by following the trajectory $\phi_t(v)$ for

$0 \leq t \leq T$, followed by an arc $e : [0,1] \to K_j$ joining $\phi_T(v)$ to $v_j$. Let $c$ be the arc in $\Sigma$ obtained by concatenating $a_i$, $\pi \circ C$ and $a_j^*$. The contractibility of $P$ implies that the curve $c$ is homotopic to the geodesic arc through $u$, namely $\gamma(t) = \pi \circ \phi_t(u)$ for $t \in [0,T]$. Therefore $c \in [\tilde{c}]$.

Take the collection of all such arcs $C$ in $L$ constructed in the previous paragraph. These arcs generate a groupoid: if $C$ ends in $(K_i, v_i)$ and $D$ begins in $(K_i, v_i)$, then their product (concatenation) $D * C$ is defined. Modulo homotopies in $L$ that fix end points, the operation $*$ is associative, and so the equivalence classes of arcs $C$ generates a groupoid $\mathbf{G}$. The collection of homotopy classes of arcs $C$ that begin and end in $(K_i, v_i)$ generates a group $\mathbf{G}_i$ for each $i = 1, \ldots, \omega$. $\mathbf{G}_i$ is a subgroup of $\pi_1(L_{\alpha_i}; v_i)$, where $K_i \subset L_{\alpha_i}$.

Observe that: (i) if $C_{ji}$ is an arc beginning in $(K_i, v_i)$ and ending in $(K_j, v_j)$ then $C_{ji}^{-1} \mathbf{G}_j C_{ji} = \mathbf{G}_i$; (ii) there is a subset $\{C_{ij}\} \subset \mathbf{G}$ of cardinality no greater than $\omega^2$ such that for any element $C \in \mathbf{G}$ there is a $C_{ij}$ such that $C_{ij} C \in \mathbf{G}_i$ for some $i, j$. Claim (i) follows from the observation that if $[C] \in \mathbf{G}_j$, then $[C] * [C_{ji}]$ is a homotopy class (relative to endpoints) of loops that begin at $v_j$, end at $v_i$ and remain in $L$. Therefore, $[C_{ji}]^{-1} * [C] * [C_{ji}]$ is a homotopy class (relative to endpoints) of loops that begin at $v_i$, end at $v_i$ and remain in $L$, i.e. $[C_{ji}]^{-1} * [C] * [C_{ji}] \in \mathbf{G}_i$. Claim (ii) follows from the finiteness of the number of components $K_i$.

The map $C \in \mathbf{G} \to [c] = [a_j^*(\pi \circ C) a_i] \in \pi_1(\Sigma; q)$ is an epimorphism $s$ of groupoids induced by the maps $L \xrightarrow{\iota_L} \mathbf{E} \xrightarrow{\pi} \Sigma$. The restriction $s|\mathbf{G}_i$ is a group homomorphism. By (ii),

$$\pi_1(\Sigma; q) = \cup_{1 \leq i,j \leq \omega} \ c_{ij} H_i$$

where $c_{ij} = s(C_{ij})$ and $H_i = s(\mathbf{G}_i)$. Therefore $\pi_1(\Sigma; q)$ is a finite union of cosets of subgroups. To prove that at least one of the subgroups $H_i$ is of finite index in $\pi_1(\Sigma; q)$ it remains to observe

**Lemma 16 ([32])** *Suppose that a group $G = \cup_{1 \leq i \leq \alpha,\ 1 \leq j \leq \beta} c_{ij} H_i$ where $c_{1,1}, \ldots, c_{\alpha,\beta} \in G$ and $H_1, \ldots, H_\alpha$ are subgroups of $G$. Then there is at least one subgroup $H_i$ of finite index in $G$.*

Since $H_i \leq (\pi \iota_{L_{\alpha_i}})_* \pi_1(L_{\alpha_i}; v_i)$, this completes the proof. $\square$

Combining the lifting lemma with Lemma 15 shows that, possibly after passing to a finite covering, one can assume that for some $i$, $\pi_1(L_i) \to \pi_1(\Sigma)$ is epimorphic.

## 4 3- and 4-manifolds

Recall that if an integrable flow $\phi_t : M \to M$ is semisimple with respect to $\mathfrak{F} = (f, L, B)$, we say that $\mathfrak{F}' = (f', L', B')$ is a *refinement* of $\mathfrak{F}$ if $B'$ is open and dense in $B$, $L' = f^{-1}(B')$, $f' = f|L'$ and $\Gamma' = M - L'$ is a nowhere-dense tamely-embedded polyhedron.

**Definition 17** *$\mathfrak{F} = (f, L, B)$ is* tractable *if, for each connected component $f_i : L_i \to B_i$, either*

1. *there is a compact codimension-1 submanifold $N_i \subset B_i$ such that the inclusion of $W_i = f_i^{-1}(N_i) \hookrightarrow L_i$ is epimorphic on $\pi_1$; or*
2. *there is a component $L_j$ satisfying* 1. *and a map $r : L_i \to L_j$ such that $\iota_{L_i,M}$ is homotopic to $\iota_{L_j,M} \circ r$.*

*Remark*: Condition 2. implies that $\mathrm{im}\, \iota_{L_i,M*} \subset \mathrm{im}\, \iota_{L_j,M*}$. Hence, from the point of view of the fundamental group, one only need concern oneself with those components $L_j$ that satisfy Condition 1.

**Lemma 18** *If $\phi_t : M \to M$ is semisimple with respect to $\mathfrak{F} = (f, L, B)$ and $M$ is compact, then there is a tractable refinement $\mathfrak{F}' = (f', L', B')$ of $\mathfrak{F}$.*

**PROOF.** We break the proof into 3 cases.

*Case 1*: $\dim B = 1$ is trivial.

*Case 2*: $\dim B = 2$.

*Claim*: $B$ is homeomorphic to a compact surface $S$ punctured at a finite number of points.

To prove the claim, it suffices to show that $B$ has finitely-many ends. If $B$ has no ends, it is compact and the claim is trivial. If $B$ is a one-ended surface whose boundary is $\mathbf{T}^1$, then $B$ is homeomorphic to $\mathbf{T}^1 \times [0, 1)$. If $B$ has finitely-many ends, then $B = C \cup E$ where $C$ is a compact surface with boundary and $E$ is a regular neighbourhood of infinity. Hence each component of $E$ is a one-ended surface with a $\mathbf{T}^1$ boundary, i.e. each component is homeomorphic to $\mathbf{T}^1 \times [0, 1)$. This proves the claim if $B$ has finitely-many ends.

Since $f : L \to B$ is proper, it induces a bijection of ends. Let $K$ be a polyhedral neighbourhood of $\Gamma$, and let $K_n$ be a regular neighbourhood of $\Gamma$ in the $n$-th barycentric subdivision of $K$. Let $\Gamma$ have $\kappa$ components. Compactness of $M$ implies that $\kappa$ is finite. Hence $C_n = K_n \cap L$ has $\kappa$ components for all $n$ sufficiently large. Number these components $C_n^j$ such that $C_{n+1}^j \subset C_n^j$ for all

$j$ and all $n$ sufficiently large. This shows that $L$, hence $B$, has $\kappa < \infty$ ends. This establishes the claim.

Let $S$ be as in the claim. Triangulate $S$ so that each puncture point is a barycentre of a 2-simplex. This triangulation induces a decomposition of $B$ into a 1-skeleton $B^{(1)}$ and a finite union of open 2-disks and open cylinders. Let $B' = B - B^{(1)}$, $L' = f^{-1}(B')$ and $f' = f|L'$. For each component $B_i'$ of $B'$, let $N_i$ be an embedded circle such that $\pi_1(N_i) \to \pi_1(B_i')$ is epimorphic. Clearly $W_i = f^{-1}(N_i) \hookrightarrow L_i'$ is surjective on $\pi_1$, so $(f', L', B')$ is tractable.

From [30], the proper fibration $f : L \to B$ is triangulable. Hence, it can be assumed that $f : L \to B$ is a simplicial map of PL manifolds. From this, it is clear that it can be assumed that $f^{-1}(B^{(1)})$ is a compact subcomplex of $L$ and hence a tamely-embedded polyhedral subset of $M$. Since $\Gamma$ and $f^{-1}(B^{(1)})$ are tamely-embedded polyhedra which are separated by an open set, $\Gamma' = \Gamma \cup f^{-1}(B^{(1)})$ is a tamely-embedded polyhedron. Hence $(f', L', B')$ is a tractable refinement of $(f, L, B)$.

*Case 3*: $\dim B \geq 3$.

*Step 1*: Let $\mathbf{G} \subset \mathbf{K}$ be simplicial complexes with polytopes $|\mathbf{G}| \subset |\mathbf{K}|$ such that $\mathbf{G}$ is a full subcomplex in $\mathbf{K}$'s codimension-1 skeleton. Let $\mathbf{U}$ be the subcomplex of $\mathbf{K}$ obtained by deleting all simplices in $\mathbf{K}$ with a vertex in $\mathbf{G}$. From the proof of Lefschetz duality, there is a deformation retraction of $|\mathbf{K}| - |\mathbf{G}|$ onto $|\mathbf{U}|$ [26].

*Step 2*: Let $K$ be a polyhedral neighbourhood of $\Gamma$ in $M$. From step 1, $L$ admits a deformation retraction onto $L_o = L - \operatorname{Int} K$. Let $\Phi : L \times I \to L$ be this deformation retraction.

From [30], the proper fibration $f : L \to B$ is triangulable. Hence, it can be assumed that $f : L \to B$ is a simplicial map of PL manifolds and that $\Phi$ is a simplicial map. Let $B_o = f(L_o)$, $B_1$ be a regular neighbourhood of $B_o$ and $B_+ = \operatorname{Int} B_1$. Let $B_- = B - B_1$, so that $B$ is the disjoint union of $B_+$, $B_-$ and $\partial B_1$. Let $L_\pm = f^{-1}(B_\pm)$.

Let $B' = B_- \cup B_+$, $L' = f^{-1}(B')$ and $f' = f|L'$. Since $\Gamma$ and $f^{-1}(\partial B_1)$ are tamely-embedded polyhedra that are separated by open sets, $\Gamma'$ is a tamely-embedded polyhedron. Hence $(f', L', B')$ is a refinement of $(f, L, B)$.

For the next two paragraphs, assume that $B$ and $B_-$ are connected. Then $B_+$ is connected.

Let $B_+^{(1)} \subset B_+$ be the union of 1-simplices of $B_1$ that lie entirely in $B_+$. Let $N_+ \subset B_+$ be the boundary of a regular neighbourhood of $B_+^{(1)}$ and let $W_+ = f^{-1}(N_+)$. Since $\dim B \geq 3$, $N_+$ is connected and the inclusion $N_+ \hookrightarrow B_+$

is epimorphic on $\pi_1$; hence $W_+ \hookrightarrow L_+$ is epimorphic on $\pi_1$. This proves $L_+$ satisfies 1. in Definition 17.

On the other hand, let $r : L_- \to L_+$ be $r = \iota_{L_o,L_+} \circ \Phi_1 \circ \iota_{L_-,L}$ where $\Phi_t(\bullet) = \Phi(\bullet, t)$. Since $\Phi_0 = id_L$, $\iota_{L_+,L} \circ r$ is homotopic in $L$ to $\iota_{L_-,L}$. Hence $\iota_{L_+,M} \circ r$ is homotopic in $M$ to $\iota_{L_-,M}$. This proves $L_-$ satisfies 2. in Definition 17.

If $B$ or $B_-$ is not connected, the previous two paragraphs can be applied componentwise. This proves the Lemma. $\Box$

**Proof of Theorems 1 and 4.** Let $(\Sigma, \mathbf{g})$ be a compact finslerian $n$-manifold and assume that the geodesic flow $\phi_t : S\Sigma \to S\Sigma$ is integrable and semisimple with respect to $(f, L, B)$. The dimension of $B$ (resp. $L$) is $l$ (resp. $k + l$) and $k + l = 2n - 1$.

By Lemma 18, it can be assumed that $\mathfrak{F} = (f, L, B)$ is tractable. By Lemma 15, there is a connected component $L_i$ of $L$ such that $\mathrm{im}\,(\pi\iota_{L_i})_*$ is of finite index in $\pi_1(\Sigma)$. By the remark following Definition 17, it can be assumed that $L_i$ satisfies Condition 1., hence there is a compact codimension-1 manifold $W_i \subset L_i$ that fibres over a compact codimension-1 manifold $N_i \subset B_i$ such that $\iota_{W_i,L_i*}$ is epimorphic on $\pi_1$. Let us suppress the index $i$ in the following discussion.

The homotopy exact sequence $\pi_2(N) \xrightarrow{\partial_*} \pi_1(\mathbf{T}^k) \to \pi_1(W) \to \pi_1(N) \to 1$, implies there is a short exact sequence

$$1 \to \Omega \to \pi_1(W) \to \pi_1(N) \to 1,$$

where $\Omega$ is abelian of rank at most $k$. Thus $\pi_1(W)$ is polycyclic if $\pi_1(N)$ is polycyclic. The step length of $\pi_1(W)$ is at most $k$ plus the step length of $\pi_1(N)$. Since $\pi_1(L) = \mathrm{im}\,\iota_{W,L*}$, these comments apply to $\pi_1(L)$, too.

If $\phi_t$ is $n$-semisimple, then $k = n$ and $l = n-1$ so $\dim N = n-2$. When $n = 3$, $\pi_1(N) \simeq \mathbf{Z}$, so $\pi_1(L)$ is polycyclic of step length at most 4. When $n = 4$, $N$ is a compact surface. From the hypothesis of Theorem 4, $N$ covers the projective plane or the Klein bottle, so $\pi_1(N)$ contains a finite-index copy of $\mathbf{Z}^s$, $s \leq 2$. Thus, $\pi_1(L)$ contains a finite-index polycyclic subgroup of step length at most 6. $\Box$

## 5 Anabelian fundamental groups and $\mathfrak{F}$-semisimplicity

**Lemma 19** *Let $\pi_1(\Sigma)$ be anabelian, $L \subseteq \mathbf{E}$, and $f : L \xrightarrow{F} B$ be a fibration. If $\pi_1(F)$ is abelian, then either:*

1. *the image of* $L \xrightarrow{\pi\iota_L} \Sigma$ *on* $\pi_1$ *is not of finite index; or*
2. *the composite map* $F \overset{\iota_F}{\hookrightarrow} L \xrightarrow{\pi\iota_L} \Sigma$ *is trivial on* $\pi_1$ *and the homomorphism* $(\pi\iota_L)_* : \pi_1(L) \to \pi_1(\Sigma)$ *factors through a homomorphism* $\xi : \pi_1(B) \to \pi_1(\Sigma)$.

**PROOF.** Because $f : L \to B$ is a fibration, there is a pair of intersecting horizontal and vertical exact sequences

$$\begin{array}{ccccc} & & \ker(\pi\iota_L)_* & & \\ & & \downarrow & & \\ \pi_1(F) & \xrightarrow{\iota_{F,*}} & \pi_1(L) & \xrightarrow{f_*} & \pi_1(B) \\ & & \downarrow (\pi\iota_L)_* & & \downarrow id \\ & & \pi_1(\Sigma) & \xleftarrow{?\xi} & \pi_1(B), \end{array}$$

and $?\xi$ indicates that $\xi$ remains to be defined. If $\mathrm{im}\,(\pi\iota_L)_*$ is of finite index, then $\mathrm{im}\,(\pi\iota_L\iota_F)_*$ is an almost normal abelian subgroup of $\pi_1(\Sigma)$. Hence it is trivial, so $\mathrm{im}\,\iota_{F,*} \subset \ker(\pi\iota_L)_*$ which suffices to define $\xi$ as the diagram suggests. $\square$

**Proof of Theorem 6.** From the definition of integrability, each point on a $\phi_t$-invariant torus is recurrent. For each $v \in L$, let $U$ be an open neighbourhood of $v$ that is contractible in $L$. Because $v \in L$ is recurrent, there is a sequence $t_k \to \infty$ as $k \to \infty$ such that $\phi_{t_k}(v), \to v$ as $k \to \infty$. Without loss of generality, it may be assumed that $\phi_{t_k}(v) \in U$ for all $k$. Let $\gamma_k : \mathbf{T}^1 \to \mathbf{E}$ be the closed loop obtained by concatenating the orbit segment $\phi_t(v)$ for $0 \le t \le t_k$ with an arc $E_k$ from $\phi_{t_k}(v)$ to $v$. It is clear that $E_k$ can be chosen to lie in $U$ and have $\mathrm{length}(E_k) \to 0$ as $k \to \infty$. Since $U$ is contractible in $L$ and $f$ is $\phi_t$-invariant, $f \circ \gamma_k$ is null-homotopic in $B$.

Let $i \in A$ be such that $\mathrm{im}\,(\pi\iota_{L_i})_*$ is of finite index in $\pi_1(\Sigma)$. By Lemma 19, $(\pi\iota_{L_i})_*$ factors through a homomorphism $\xi_i : \pi_1(B_i) \to \pi_1(\Sigma)$. Then, the loop $\gamma_k$ is freely homotopic to a loop in $F_i$, so $\gamma_k$ is null-homotopic in $\mathbf{E}$. Therefore, any lift of $\gamma_k$ to $\bar{\mathbf{E}}$ is also closed. But this implies that any $\bar{v} \in \bar{L}_i$ in the fibre over $L_i$ is recurrent for $\bar{\phi}_t$. Since $\bar{L}_i$ is open, $\mathcal{L}(\bar{\phi})$ has a non-empty interior.

For the second part of the theorem, we use the method of toroidal surgeries [1]. Let $\Phi = (\theta, I) : U \to \mathbf{T}^k \times \mathbf{R}^l$ be a $C^1$ diffeomorphism with proper inverse that conjugates $\phi_t$ with the translation-type flow $T_t(\theta, I) = (\theta + t\xi(I), I)$. Let $D \subset \mathbf{R}^l$ be a small open ball contained in the image of $I$ and let $\Xi : D \to \mathbf{P}^{k-1}$ be the frequency map $\Xi(I) = [\xi_1(I) : \cdots : \xi_k(I)]$. Since $k \le l+1$, in any $C^1$-neighbourhood of $\Xi$, there is an $\Omega = [\omega_1(I) : \cdots : \omega_k(I)]$ such that $\Omega$

is a submersion on $D$ and the support of $\Xi - \Omega$ is the closure of $D$. Let $S_t(\theta, I) = (\theta + t\omega(I), I)$ and define

$$\eta_t(P) = \begin{cases} \phi_t(P) & \text{if } P \notin U, \\ \Phi^{-1} S_t \Phi(P) & \text{if } P \in U. \end{cases}$$

It is clear that $S_t$, hence $\eta_t$, is $C^1$. It is also clear that $\mathcal{P}(\eta)$ contains an open set. From the first part of the theorem, it follows that $\mathcal{P}(\bar{\eta})$ contains an open set.

Finally, $\eta_t$ is orbitally equivalent to the geodesic flow of a finsler metric by the arguments in sections 3.2–3.3 of [12]. □

**Proof of Theorem 7.** Let $n = \dim \Sigma$. Since $\phi_t$ is completely integrable $\dim B = n - 1$.

If $B_i$ is plentiful, the lifting lemma implies that one may assume, possibly after passing to a finite covering of $\Sigma$, that $\operatorname{im}(\pi\iota_{L_i})_* = \pi_1(\Sigma)$ and that $\Sigma$ is orientable. Let us suppress the subscript $i$ in the remainder of the proof.

Assume that $\pi_k(\bar{B}) = 1$ for all $k$. Then $B$ is aspherical and the epimorphism $\xi : \pi_1(B) \to \pi_1(\Sigma)$ of Lemma 19 is an isomorphism. Since $B$ and $\Sigma$ are aspherical and their fundamental groups are isomorphic, they are weakly homotopy equivalent, hence their singular homology groups are isomorphic (Corollary V.4.6 in [34]). But $\Sigma$ is a compact orientable manifold, so its top homology group $H_n(\Sigma)$ is non-zero, while $H_n(B) = 0$. Absurd. □

## 6 Examples

### 6.1 *The Poisson Geometry of $T^*G$*

The proof of Theorem 2 exploits an underlying Lie-theoretic structure. The principal machinery used to construct integrals will be the *momentum map.*

Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$. The dual space, $\mathfrak{g}^*$, has a Poisson bracket defined as follows: The derivative of a smooth function on $\mathfrak{g}^*$ at a point is naturally identified with an element in $\mathfrak{g}$. The Poisson bracket on $\mathfrak{g}^*$ is defined by

$$\{f, h\}_{\mathfrak{g}^*}(\mu) := -\langle \mu, [df_\mu, dh_\mu] \rangle, \tag{1}$$

for all $f, h \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$. A function $f \in C^\infty(\mathfrak{g}^*)$ is a *Casimir* if $E_f = \{., f\}$ is trivial. The set of Casimirs is precisely $Z(C^\infty(\mathfrak{g}^*))$. The vector field $E_f$ is called the *Euler vector field*.

The map

$$\psi(g, \mu) := \mathrm{Ad}_g^* \mu \tag{2}$$

is called the momentum map of $G$'s left-action on $\mathfrak{g}^*$. A right-invariant vector field on $G$ is of the form $\xi_G^R(g) = d_e R_g \, \xi$, for some $\xi \in \mathfrak{g}$, and all $g \in \mathbf{G}$. The cotangent lift of $\xi_G^R$ has the hamiltonian function $h_\xi(g, \mu) = \langle \xi_{\mathbf{G}}^R(g), \mu \rangle$, which equals $\langle \psi(g, \mu), \xi \rangle$. It is known that

*The map $\psi : T^*G \to \mathfrak{g}^*$ is the momentum map of $G$'s left-action on $T^*G$. The map $\omega(g, \mu) = \mu$ is the momentum map of $G$'s right action on $T^*G$. Both maps are submersions.*

The canonical Poisson structure on $T^*G$, $\{,\}_{T^*G}$, is related to that on $\mathfrak{g}$, $\{,\}_{\mathfrak{g}^*}$ as follows: $\{,\}_{T^*\mathbf{G}}$ is right (resp. left) invariant, so the Poisson bracket of right (resp. left) invariant functions is again right (resp. left) invariant. If $\mathcal{R}$ (resp. $\mathcal{L}$) denotes the right (resp. left) invariant functions smooth functions on $T^*G$, then $\mathcal{R} = \psi^* C^\infty(\mathfrak{g}^*)$ (resp. $\mathcal{L} = \omega^* C^\infty(\mathfrak{g}^*)$), and $\psi^*$ (resp. $\omega^*$) is a Lie algebra isomorphism (resp. anti-isomorphism). In addition, because right and left multiplication commute these two subalgebras commute: $\{\mathcal{R}, \mathcal{L}\}_{T^*G} \equiv 0$. Because $\psi$ is a Poisson map, we will abuse notation and use $\{,\}$ to denote either $\{,\}_{T^*G}$ or $\{,\}_{\mathfrak{g}}$, depending on the context.

Therefore, if $H \in \mathcal{L}$, then $H$ Poisson commutes with all hamiltonians $F \in \mathcal{R}$ so $F$ is a first integral of the hamiltonian vector field $Y_H$ on $T^*G$. In addition, the projection map $\omega : T^*\mathbf{G} \to \mathfrak{g}^*$ also satisfies $\omega_* Y_H = -E_H$. Thus, if $k \in C^\infty(\mathfrak{g}^*)$ is a first integral of $E_H$, then $k \circ \omega$ is a first integral of $Y_H$.

Consequently, to prove integrability of $Y_H$ on $T^*G$, it is useful to: (1) find sufficiently many functions in $\mathcal{R}$; and (2) find integrals of $X_H$ on $\mathfrak{g}^*$. With luck, the sum of these two subalgebras of integrals will be sufficient for integrability. In the event that we wish to study $Y_H$ on $T^*(E \backslash G)$, we need to find sufficiently many functions in $\mathcal{R}^E$.

### *6.2 An integrability theorem*

*Standing Hypothesis*: It will be assumed throughout that the Casimirs of $\mathfrak{g}^*$ separate $G$'s coadjoing orbits.

For a left-invariant metric $\mathbf{g}$ on $G$, let $\mathrm{Isom}(\mathbf{g})$ be the isometry group of $\mathbf{g}$ and let $O(\mathbf{g})$ be the group of automorphisms of $\mathfrak{g}$ that are also $\mathbf{g}$-orthogonal. Let $E$ be a discrete, torsion-free subgroup of $\mathrm{Isom}(\mathbf{g})$ that acts freely and uniformly discretely on $G$; $\Sigma = E\backslash G$ is the quotient manifold and $\mathbf{h}$ is the metric on $\Sigma$ induced by $\mathbf{g}$. There is an exact sequence

$$1 \to N \to E \to F \to 1,$$

where $N = E \cap G$ and $F = E/N$ is isomorphic to a finite subgroup of $O(\mathbf{g})$.

For $\alpha \in \mathrm{Aut}(G)$, let $\beta = (d_e\alpha')^{-1}$ be the induced linear isomorphism on $\mathfrak{g}^*$ and let $T^*\alpha$ be the symplectomorphism of $T^*G$ induced by $\alpha$. A calculation shows that $\psi \circ T^*\alpha = \beta \circ \psi$. The natural right-action of $G$ on $T^*G$ extends to an action of $G \star \mathrm{Aut}(G)$ on $T^*G$, and this action factors through to an action on $\mathfrak{g}^*$. Let $\mathcal{E}$ be the subgroup of $\mathrm{GL}(\mathfrak{g}^*)$ induced by $E < G \star \mathrm{Aut}(G)$.

**Theorem 20** *Assume that* $\mathfrak{g}^* = \mathfrak{g}_r^* \amalg \mathfrak{g}_s^*$, $\mathcal{B} \subset C^\omega(\mathfrak{g}^*)$ *and* $\mathcal{A} \subset C^\omega(\mathfrak{g}_r^*)$ *satisfy*

*F1.* $\mathfrak{g}_s^*$ *is a nowhere dense, analytic,* $\mathcal{E}$*-invariant set;*
*F2.* $\mathcal{B}$ *is an integrable subalgebra of* $C^\omega(\mathfrak{g}^*)^F$ *containing the hamiltonian of* $\mathbf{k}$*;*
*F3.* $\mathcal{A}$ *is an integrable subalgebra of* $C^\omega(\mathfrak{g}_r^*)^{\mathcal{E}}$*.*

*Then: if* $\Sigma$ *is compact, then the geodesic flow of* $\mathbf{k}$ *is integrable and semisimple.*

**PROOF.** Let $\tilde{\mathfrak{b}} = \omega^*\mathcal{B}$ and $\tilde{\mathfrak{a}} = \psi^*\mathcal{A}$. Since $\tilde{\mathfrak{b}}$ is left-invariant and $F$-invariant, it induces a subalgebra $\mathfrak{b}$ on $T^*\Sigma$. Similarly, $\tilde{\mathfrak{a}}$ is $\mathcal{E}$-invariant by F3, so it induces a subalgebra $\mathfrak{a}$ on $T^*\Sigma$. Without changing F2 and F3, it can be assumed that both $\mathcal{A}$ and $\mathcal{B}$ contain the Casimirs of $\mathfrak{g}_r^*$. Let $c$ be the index of $G$, and $n$ be the dimension of $G$, so that the dimension of a coadjoint orbit in $\mathfrak{g}_r^*$ is $n-c$ and $G$ enjoys $c$ independent Casimirs on each coadjoint orbit in $\mathfrak{g}_r^*$, by the standing hypothesis. Since $F$ is finite, it can be assumed that each of these Casimirs is also $F$-invariant.

Let $T^*\Sigma_r = \Sigma \times_F \mathfrak{g}_r^*$. Due to analyticity and the hypothesis that both $\mathcal{A}$ and $\mathcal{B}$ are integrable subalgebras, there exists $\lambda : T^*\Sigma_r \to \mathbf{R}^{c+b}$ (resp. $\rho : T^*\Sigma_r \to \mathbf{R}^{c+a}$) whose regular level set $\mathrm{Reg}(\lambda)$ (resp. $\mathrm{Reg}(\rho)$) is an everywhere-dense analytic set such that:

P1. rank $d\lambda_P = b+c$ for $P \in \mathrm{Reg}(\lambda)$ (resp. rank $d\rho_P = a+c$ for $P \in \mathrm{Reg}(\rho)$);
P2. $\lambda_j = \rho_j$ for $j = 1, \ldots, c$;
P3. $\lambda_j$ for $j = 1, \ldots, c$ are induced by $F$-invariant Casimirs of $\mathfrak{g}^*$;
P4. $\{\lambda_i, \lambda_j\} = 0$ for $i = 1, \ldots, b+c$ and $j = 1, \ldots, b_o + c$ (resp. $\{\rho_i, \rho_j\} = 0$ for $i = 1, \ldots, a+c$ and $j = 1, \ldots, a_o + c$);
P5. $a_o + a = b_o + b = n - c$.

Let $\mathbf{f}_j = \lambda_j$ for $j = 1, \ldots, c + b_o$ and $\mathbf{f}_{j+c+b_o} = \rho_{j+c}$ for $j = 1, \ldots, a_o$, $\mathbf{f}_{j+a_o+c} = \lambda_j$ for $j = b_o+1, \ldots, b$ and $\mathbf{f}_{j+b+c} = \rho_{j+c}$ for $j = a_o+1, \ldots, a$. Since $\tilde{\mathfrak{a}} \cap \tilde{\mathfrak{b}}$ is a set of bi-invariant functions on $T^*G$, P1-P3 imply that $\mathrm{Reg}(\mathbf{f}) = \mathrm{Reg}(\lambda) \cap \mathrm{Reg}(\rho)$ is an everywhere-dense analytic subset of $T^*\Sigma_r$ hence of $T^*\Sigma$.

P4-P5 imply that $\{\mathbf{f}_i, \mathbf{f}_j\} = 0$ for $i = 1, \dots, a_o+b_o+c$ and $j = 1, \dots, a+b+c$. Let $\mathcal{F}$ be the subalgebra of $C^\infty(T^*\Sigma)$ generated by the $\mathbf{f}$-pullbacks of compactly-supported smooth functions whose support is contained in the interior of $\mathrm{im}\,\mathbf{f}$. Then $\mathcal{F}$ satisfies:

J1. $k = \dim dZ(\mathcal{F})_P = a_o + b_o + c$ for all $P \in \mathrm{Reg}(\mathbf{f})$;
J2. $l = \dim d\mathcal{F}_P = a + b + c$ for all $P \in \mathrm{Reg}(\mathbf{f})$;
J3. $\Gamma = T^*\Sigma - \mathrm{Reg}(\mathbf{f})$ is a closed, nowhere-dense analytic set.

From P5, $k+l = 2n$ so J1-J2 imply that I1 of Definition 10 is satisfied. Since $\Gamma$ is an analytic subset of $T^*\Sigma$, there is a triangulation of $(T^*\Sigma, \Gamma)$ [25]. Hence $\Gamma$ is a tamely-embedded polyhedron, so $\mathcal{F}$ is a tamely-integrable algebra. Since $\mathcal{F}$ is a set of integrals of the geodesic flow of $\mathbf{k}$, Theorem 11 implies that the geodesic flow is integrable and semisimple. $\square$

#### *6.2.1 Construction of the integrable subalgebra $\mathcal{A}$*

The following two sets of conditions imply the existence of an integrable subalgebra $\mathcal{A} \subset C^\omega(\mathfrak{g}_r^*)^{\mathcal{E}}$. Note that H1-H3 is a special case of G1-G4. The proofs are straightforward and left to the reader. $C_o^\infty$ is the set of smooth functions with compact support.

Assume that $\mathfrak{g}^* = \mathfrak{g}_r^* \amalg \mathfrak{g}_s^*$ is the disjoint union of two sets such that either

H1. $\mathfrak{g}_s^*$ is a closed, nowhere-dense, $\mathcal{E}$-invariant real-analytic set;
H2. there is a real-analytic fibration $\mathbf{C} : \mathfrak{g}_r^* \to B$ such that $\mathrm{Ad}_G^*$ acts transitively on the fibres of $\mathbf{C}$;
H3. for each $b \in B$, $\mathrm{Ad}_N^*$ acts freely and uniformly discretely on $\mathbf{C}^{-1}(b)$;

or

G1. $\mathfrak{g}_s^*$ is a closed, nowhere-dense, $\mathcal{E}$-invariant real-analytic set;
G2. there is a real-analytic $G$-manifold $V$ and $G$-equivariant real-analytic fibrations $\mathfrak{g}_r^* \xrightarrow{\mathbf{p}} V \xrightarrow{\mathbf{C}} B$;
G3. there is a normal subgroup $N_{stab}$ of $N$ such that for each $b \in B$, $N/N_{stab}$ acts freely and uniformly discretely on $\mathbf{C}^{-1}(b)$;
G4. $\dim V + \dim dZ(\mathbf{p}^* C_o^\infty(V)) = n + c$.

## 7 Applications

This section proves Theorem 2. The geometric 3-manifolds $\mathbf{E}^3$, $Nil$ and $Sol$ will play an important role.

### 7.1 $\mathbf{E}^3$

Let $G$ be the Lie group $\mathbf{R}^3$ and let $\mathbf{g}$ be a left-invariant metric on $G$; $\mathbf{E}^3 = (G, \mathbf{g})$. The isometry group of $\mathbf{E}^3$ is naturally isomorphic to $G \star O(3)$. The Bieberbach Theorem says that if $E$ is a uniformly discrete, torsion-free cocompact subgroup of $\mathrm{Isom}(\mathbf{E}^3)$, then there is an exact sequence

$$1 \to N \to E \to F \to 1,$$

where $N = G \cap E$ is the maximal abelian subgroup of $E$ – which is isomorphic to $\mathbf{Z}^3$ – and $F$ is a finite subgroup of $O(3)$. Relative to the obvious trivialization of $T^*G$, the momentum map of $G$'s right action on $T^*G$ is

$$\psi(h, p) = \mathrm{Ad}_h^* p = p.$$

Let $\mathfrak{g}_r^*$ be the subset of $\mathfrak{g}^*$ on which $F$ acts freely, let $B = \mathfrak{g}_r^*$ and let $\mathbf{C} = id$. Conditions H1–H3 are obviously satisfied. Since $G$ is abelian, its coadjoint orbits are all points, so the standing hypothesis is trivially satisfied. In this case, $a_o = b_o = 0$ and $c = 3$. Thus,

**Theorem 21** *Let $\Sigma = E \backslash G$ where $E$ is a uniformly discrete, torsion-free and cocompact subgroup of isometries of $\mathbf{g}$. The geodesic flow of the metric induced by $\mathbf{g}$ on $\Sigma$ is completely integrable and semisimple.*

Note that the orbifold $\mathfrak{g}^*/F$ is naturally coordinatized by the $F$-invariant polynomials on $\mathfrak{g}^*$. The geodesic flows in Theorem 21 are integrable with real-analytic first integrals. The monodromy of the Liouville foliation is also naturally isomorphic to $F$.

The analogous theorem for the geometric 3-manifolds that are modeled on $S^2 \times \mathbf{E}^1$ and $S^3$ is proven in an identical manner and will be omitted.

### 7.2 $Nil$

c.f. [9,11,10]. Let $G = Nil$ denote the set $\mathbf{R}^3$ with the multiplication rule: $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$. $G$ is the 3-dimensional Heisenberg group whose center is $Z(G) = \{(0, 0, z)\}$. The 1-forms $\alpha = dx$,

$\beta = dy$ and $\gamma = dz - \frac{1}{2}(xdy - ydx)$ are left-invariant and form a basis of $\mathfrak{g}^*$. The dual basis $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ of $\mathfrak{g}$ satisfies $\exp(x\mathcal{X} + y\mathcal{Y} + z\mathcal{Z}) = (x, y, z)$ for all $x, y, z \in \mathbf{R}$. Thus, $\mathfrak{g}$ and $G$ can be identified with $\mathbf{R}^3$ in the obvious way, and in this coordinate system the exponential map is just the identity. Let

$$\mathbf{g} = \sum_{\sigma \in \{\alpha,\beta,\gamma\}} \sigma \otimes \sigma. \tag{3}$$

The isometry group of $\mathbf{g}$ is the semi-direct product of $G$ with the subgroup of automorphisms whose derivative at $id$ is orthogonal. Relative to the $(x, y, z)$ coordinate system the automorphism group acts as linear transformations and

$$O(\mathbf{g}) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ s\sin\theta & s\cos\theta & 0 \\ 0 & 0 & s \end{bmatrix} : s = \pm 1, \theta \in \mathbf{R} \right\}.$$

$O(\mathbf{g})$ is a maximal compact subgroup of $\mathrm{Aut}(G)$. It is known that if $E$ is a uniformly discrete, torsion-free subgroup of $G \star \mathrm{Aut}(G)$, then $E$ is conjugate to a subgroup of $G \star O(\mathbf{g})$ and the maximal normal nilpotent subgroup $N$ of $E$ is simultaneously conjugated to a subgroup of the group generated by the elements $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$ [13]. So it may be assumed that $E$ is a subgroup of $G \star O(\mathbf{g})$ with these properties. Then, there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & \rightarrow & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \rightarrow & Z & \rightarrow & Z & \rightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & N & \rightarrow & E & \rightarrow & F & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & L & \rightarrow & Q & \rightarrow & F_o & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1. & &
\end{array}$$

where $Q$ is a crystallographic group of motions of the plane, $L$ is a lattice subgroup of $\mathbf{R}^2$, $F_o$ is a finite group of linear isometries of the plane, and $Z = N \cap Z(G)$ is the center of $N$ and $F$ is isomorphic to a finite subgroup of $O(\mathbf{g})$. All of the maps in the diagram are the obvious ones induced from the exact sequence $1 \to Z(G) \to G \to G/Z(G) \to 1$ where $G/Z(G)$ is identified with $\mathbf{R}^2$.

Write a covector $p \in T_h^*G$ as $p = p_\alpha \alpha + p_\beta \beta + p_\gamma \gamma$; this amounts to trivializing $T^*G$ with respect to the left action of $G$. The momentum map of $G$'s right action on $T^*G$ is then:

$$\psi(h,p) = \mathrm{Ad}_h^* p = (p_\alpha + \frac{1}{2} y p_\gamma)\alpha + (p_\beta - \frac{1}{2} x p_\gamma)\beta + p_\gamma \gamma,$$

where $h = (x,y,z)$. It is clear that the coadjoint orbits of $G$ on $\mathfrak{g}^*$ are planes $\{p_\gamma = c\}$ for non-zero $c$ and single points for $c = 0$. Let $\mathfrak{g}_r^* = \{p \in \mathfrak{g}^* \ : \ p_\gamma \neq 0\}$ and let $\mathfrak{g}_s^*$ be the complement of $\mathfrak{g}_r^*$. It is clear from the explicit description of $O(\mathbf{g})$ and its action on $\mathfrak{g}$, that $\mathfrak{g}_r^*$ is $O(\mathbf{g})$-invariant. Let $B = \mathbf{R}^\times$ and define $\mathbf{C} : \mathfrak{g}_r^* \to B$ by

$$\mathbf{C}(p) := p_\gamma. \tag{4}$$

Then H1 and H2 are satisfied.

Since $N/Z$ is isomorphic to a lattice subgroup $L$ of the plane, it is clear from the explicit description of $\mathrm{Ad}_h^* p$, that $\mathrm{Ad}_N^*$ acts freely and uniformly discretely on each fibre of $\mathbf{C}$. Hence H3 is satisfied.

Let the quadratic form on $\mathfrak{g}^*$ induced by $\mathbf{g}$ be denoted by $g$. It is given by

$$g(p) = \frac{1}{2} \sum_{\sigma \in \{\alpha,\beta,\gamma\}} p_\sigma^2. \tag{5}$$

Let $\mathcal{B} = \mathrm{span}\,\{g, p_\gamma\}$. Since the dimension of $G$ is 3 and its index is 1, $\mathcal{B}$ is an integrable subalgebra of $C^\omega(\mathfrak{g}^*)$. Thus

**Theorem 22** *Let $\Sigma = E\backslash G$ where $E$ is a uniformly discrete, torsion-free cocompact subgroup of isometries of* $\mathbf{g}$. *The geodesic flow of the metric induced by* $\mathbf{g}$ *on $\Sigma$ is integrable and semisimple.*

The following discussion proves the 3-semisimplicity of the geodesic flow of $\mathbf{g}$.

### *7.2.1 Chern classes and monodromy*

The choice of $\mathbf{C}$ in Equation 4 is not unique and it turns out that alternative choices of $\mathbf{C}$ have interesting geometric properties. Let's explain with the simplest case case where $E = N$.

Case 1. $\mathbf{C} = p_\gamma$.

The map

$$\zeta(Nh,p) = (\ g(p),\ p_\gamma,\ -\frac{2p_\beta}{p_\gamma} + x \bmod 1, \frac{2p_\alpha}{p_\gamma} + y \bmod 1\,) \tag{6}$$

is a real-analytic mapping defined of $T^*\Sigma_r \to \mathbf{R}^+ \times (\mathbf{R}^\times) \times \mathbf{T}^1 \times \mathbf{T}^1$. The first two components of $\zeta$ Poisson commute with all components, and $\zeta$ is a proper submersion except on the set $\Sigma \times \{p_\alpha = p_\beta = 0\}$. Thus, the geodesic flow of $\mathbf{g}$ is non-commutatively integrable. It is clear that it is also semisimple. As $\zeta$ is derived from the canonical first-integral map $(Nh,p) \to (g(p), \mathrm{Ad}_N^*\psi(h,p))$ from $T^*\Sigma_r \to \mathbf{R}^+ \times (\mathfrak{g}_r^*/\mathrm{Ad}_N^*)$, this essentially reproves Theorem 22.

Let $L = \Sigma \times \{p_\alpha^2 + p_\beta^2 > 0\}$ be the regular-point set of $\zeta$. $L$ has the structure of a $\mathbf{T}^2$-fibre bundle, so we can compute its monodromy and Chern class. Let $L_+ := \Sigma \times \{p_\alpha^2 + p_\beta^2 > 0, p_\gamma > 0\}$ be one of the two connected components of $L$ and let $\zeta|L_+$ be denoted by $f : L_+ \to B$ where $B = \mathbf{R}^+ \times (\mathbf{R}^\times) \times \mathbf{T}^2$. $L_+$ retracts onto $L_0 := \Sigma \times \{p_\alpha^2 + p_\beta^2 = 1, p_\gamma = 1\} \simeq \Sigma \times \mathbf{T}^1$. Since $\Sigma$ is a principal $\mathbf{T}^1$-bundle over $\mathbf{T}^2$ with projection map $\pi(Nh) = Z(G)Nh$ – i.e. $\pi(N(x,y,z)) = (x \bmod 1, y \bmod 1)$ – $f|L_0$ equals $(Nh, \theta \bmod 1) \to (2, 1, \sigma(\theta) + \pi(Nh))$, where $\sigma(\theta) = (-2\sin\theta, 2\cos\theta)$. Since $\sigma$ is null-homotopic, $f|L_0$ is homotopic to $(Nh, \theta \bmod 2\pi) \to \pi(Nh)$. Thus $f|L_0$ is homotopic to the composition of the canonical projections $\Sigma \times \mathbf{T}^1 \to \Sigma \to \mathbf{T}^2$. Therefore, the monodromy group of $f : L_+ \to B$ is trivial and the Chern class of $f$ is naturally identified with the Chern class of $\pi \times id$, which is non-trivial.

Case 2. $\mathbf{C} = p_\gamma\gamma \oplus \left(\frac{2p_\alpha}{p_\gamma} + y + \mathbf{Z}\right)\alpha$.

In this case,

$$\xi(Nh,p) = (\ g(p),\ p_\gamma,\ -\frac{2p_\beta}{p_\gamma} + x \bmod 1) \tag{7}$$

is real-analytic surjection of $T^*\Sigma_r \to \mathbf{R}^+ \times (\mathbf{R}^\times) \times \mathbf{T}^1$. The components of $\xi$ Poisson commute and $\xi$ is a proper submersion except on the set $\Sigma \times \{p_\alpha = p_\beta = 0\}$. Thus the geodesic flow of $\mathbf{g}$ is completely integrable on $T^*\Sigma$. It is clear that it is also semisimple.

Let $L_+$ be as above, and let $\ell = \xi|L_+$. Clearly, $\ell$ is a proper lagrangian fibration whose image is a manifold with trivial second Čech cohomology group. Hence the Chern class of $\ell$ is trivial. The subgroup $\mathcal{V} = \{(0,y,z)\}$ is normal in $G$, and this normal subgroup endows the manifold $\Sigma$ with the structure of a $\mathbf{T}^2$ bundle over $\mathbf{T}^1$, with projection map defined by $\Pi(Nh) = \mathcal{V}Nh$; that is $\Pi(N(x,y,z)) = x \bmod 1$. Arguments similar to those above show that $\ell|L_0$ is homotopic to the composition of canonical projections $\Sigma \times \mathbf{T}^1 \to \Sigma \to \mathbf{T}^1$. Thus, the monodromy of $\ell|L_0$ is equal to that of $\Pi \times id$, which is non-trivial.

Thus, we have an example of an integrable system that is naturally tangent to a lagrangian foliation which has non-trivial monodromy but a trivial Chern class, and it is also tangent to an isotropic foliation which has trivial monodromy but a non-trivial Chern class.

### *7.3 Sol*

See [8,7]. Let $G$ be $\mathbf{R}^3$ equipped with the multiplication $(x,y,z)*(x',y',z') = (x+x', \exp(x)y'+y, \exp(-x)z'+z)$. Let $\mathcal{V}$ be the normal subgroup $\{(0,y,z)\}$; $\mathcal{V}$ is naturally isomorphic to $\mathbf{R}^2$ and $G = \mathbf{R}^+ \star \mathbf{R}^2$ is the semi-direct product of $\mathbf{R}^+$ with $\mathbf{R}^2$ and is a 3-dimensional solvable Lie group. The 1-forms $\alpha = dx$, $\beta = \exp(-x)\,dy$, and $\gamma = \exp(x)\,dz$ are left-invariant and span $\mathfrak{g}^*$. Let

$$\mathbf{g} = \sum_{\sigma\in\{\alpha,\beta,\gamma\}} \sigma\otimes\sigma \tag{8}$$

be a left-invariant riemannian metric on $G$. The isometry group of $G$ is equal to $G \star O(\mathbf{g})$ where $O(\mathbf{g})$ is the group of orthogonal automorphisms of $G$. The automorphism group of $G$ acts, in the $(x,y,z)$ coordinate system, as a group of linear transformations. The group of orthogonal automorphisms is isomorphic to the dihedral group of order eight with the elements

$$O(\mathbf{g}) = \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{bmatrix} \right\rangle,$$

where the sign of the diagonal elements are independent of each other [31].

Let $d$ be a positive square-free integer and let $K = \mathbf{Q}(\sqrt{d})$ be the totally real quadratic number field obtained by adjoining $\sqrt{d}$ to $\mathbf{Q}$. Let $\sigma$ be the unique non-trivial automorphism of $K$ that fixes $\mathbf{Q}$ and maps $\sqrt{d}$ to $-\sqrt{d}$. Let $\mathcal{O}_K$ be the ring of integers of $K$, $\mathcal{I}$ an ideal of $\mathcal{O}_K$ and let $u > 0$ be a non-trivial unit of $\mathcal{O}_K$. The additive group of $\mathcal{I}$ is a $u$ module, so let $\Delta = \langle u\rangle \star \mathcal{I}$. Define an embedding of $\Delta$ into $G$ by

$$(u,a) \mapsto (\log u, a, \sigma(a)),$$

for all $a \in \mathcal{I}$. It is elementary to verify that this is a group embedding and that the image, $N$, is a discrete cocompact subgroup of $G$.

Write a covector $p \in T^*_h G$ as $p = p_\alpha\alpha + p_\beta\beta + p_\gamma\gamma$; this amounts to trivializing $T^*G$ with respect to the left action of $G$. The momentum map of $G$'s right action on $T^*G$ is

$$\psi(h,p) = \mathrm{Ad}^*_h p = (p_\alpha + y\exp(x)\,p_\beta + z\exp(-x)\,p_\gamma)\alpha + \exp(x)\,p_\beta\beta + \exp(-x)\,p_\gamma\gamma,$$

where $h = (x, y, z)$. It is clear that the regular coadjoint orbits of $G$ on $\mathfrak{g}^*$ are connected components of the level sets of the Casimir $\kappa = p_\beta p_\gamma$. Let $\mathfrak{g}^*_r = \{p \in \mathfrak{g}^* \ : \ \kappa(p) \neq 0\}$ and let $\mathfrak{g}^*_s$ be the complement of $\mathfrak{g}^*_r$. Since $O(\mathbf{g})$ is a group of automorphisms of $G$, $\mathfrak{g}^*_r$ is $O(\mathbf{g})$-invariant. Let $V = \mathbf{R}^\times \beta \oplus \mathbf{R}^\times \gamma$ and let $\mathbf{p}(p) := p_\beta \beta + p_\gamma \gamma$. The action of $\mathrm{Ad}^*_G$ on $\mathfrak{g}^*_r$ factors through the map $\mathbf{p}$. Let $B = (\mathbf{R}^\times) \times \mathbf{Z}_2$ and define $\mathbf{C} : V \to B$ by

$$\mathbf{C}(p_\beta \beta + p_\gamma \gamma) := (p_\beta p_\gamma, \mathrm{sign}(p_\beta)).$$

Thus G1 and G2 are satisfied.

Let $N_{stab} = N \cap \mathcal{V}$. From the explicit description of the coadjoint action, it is clear that $N/N_{stab} \simeq \langle u \rangle$ acts freely and uniformly discretely on the fibres of $\mathbf{C}$. Thus G3 is true.

Note that $V$ is 2-dimensional, $\mathbf{p}^* C^\infty_o(V)$ is an abelian subalgebra of $C^\infty_o(\mathfrak{g}^*)$ and $\dim G = 3$, $\mathrm{ind}\, G = 1$, so G4 is satisfied. Let $\mathcal{B} = \mathrm{span}\{g, \kappa\}$ where $g$ is the quadratic form on $\mathfrak{g}^*$ induced by $\mathbf{g}$. This shows that

**Theorem 23** *Let $\Sigma = E \backslash G$ where $E$ is a uniformly discrete, torsion-free cocompact group of isometries of $\mathbf{g}$. The geodesic flow of $\mathbf{g}$ is completely integrable and semisimple.*

**Proof of Theorem 2.** Let $\Sigma$ be a 3-manifold with $\pi_1(\Sigma)$ infinite polycyclic and $\pi_2(\Sigma) = 0$. From Evans and Moser's theorem [16], $\pi_1(\Sigma)$ is isomorphic to one of the following:

(1) $\mathbf{Z}$, $\mathbf{Z}^2$ or $\mathfrak{K}$, the fundamental group of the Klein bottle;
(2) an extension $1 \to A \to \pi_1(\Sigma) \to \mathbf{Z} \to 1$ where $A = \mathbf{Z}^2$ or $\mathfrak{K}$;
(3) an amalgamation $\langle a, b, x, y \,|\, bab^{-1} = a^{-1}, yxy{-}1 = x^{-1}, a = x^p y^{2q}, b^2 = x^r y^{2s} \rangle$ where $p, q, r, s$ are integers and $|ps - rq| = 1$;
(4) an extension of a group in (2), with $A = \mathbf{Z}^2$, by a finite group of automorphisms.

The groups in (1) are not the $\pi_1$ of a compact, boundaryless 3-manifold with $\pi_2 = 0$. For $\mathbf{Z}^2$ (hence $\mathfrak{K}$), this is Reidemeister's theorem. Similarly, if $\Sigma$ is a compact, boundaryless 3-manifold with $\pi_1(\Sigma) = \mathbf{Z}$ and $\pi_2(\Sigma) = 0$, then $\Sigma$ is a $K(\mathbf{Z}, 1)$-space, hence homotopy equivalent to $\mathbf{T}^1$. But by Poincaré duality the second Betti number of $\Sigma$ is 1. Absurd.

In case (2), irreducibility of $\Sigma$ plus Theorem 3 of [20] imply that $\Sigma$ fibres over $\mathbf{T}^1$ with fibre $\mathbf{T}^2$ or the Klein bottle – hence $\Sigma$ admits flat, *Nil* or *Sol* geometry by Theorem 5.3 of [31].

In cases (3) and (4), $\pi_1(\Sigma)$ contains a finite-index subgroup of type (2). Hence $\Sigma$ is finitely covered by a $\mathbf{T}^2$-bundle over $\mathbf{T}^1$, so $\Sigma$ admits flat, *Nil* or *Sol* geometry by Theorem 5.3 of [31].

Thus, by Theorems 21–23, $\Sigma$ admits a real-analytic riemannian metric whose geodesic flow is 3-semisimple. $\square$

## References


[1] O. I. Bogoyavlenskij. Theory of tensor invariants of integrable Hamiltonian systems. I. Incompatible Poisson structures. *Comm. Math. Phys.* 180:3 (1996) 529–586.

[2] O. I. Bogoyavlenskij. Theory of tensor invariants of integrable Hamiltonian systems. II. Theorem on symmetries and its applications. *Comm. Math. Phys.* 184:2 (1997) 301–365.

[3] S. V. Bolotin. Nonintegrability of the problem of $n$ centers for $n > 2$. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, 3 (1984) 65–68.

[4] S. V. Bolotin. Homoclinic orbits of geodesic flows on surfaces. *Russian J. Math. Phys.*, 1:3 (1993) 275–288.

[5] S. Bolotin and P. Negrini. A variational criterion for nonintegrability. *Russian J. Math. Phys.*, 5:4 (1997) 415–436.

[6] A. V. Bolsinov and B. Ĭovanovich. Integrable geodesic flows on homogeneous spaces. *Mat. Sb.*, 192:7 (2001) 21–40.

[7] A. V. Bolsinov and I. A. Taĭmanov. Integrable geodesic flows on suspensions of automorphisms of tori. *Tr. Mat. Inst. Steklova*, 231 (2001) 46–63.

[8] A. V. Bolsinov and I. A. Taimanov. Integrable geodesic flows with positive topological entropy. *Invent. Math.*, 140:3 (2000) 639–650.

[9] L. T. Butler. A new class of homogeneous manifolds with Liouville-integrable geodesic flows. *C. R. Math. Acad. Sci. Soc. R. Can.*, 21:4 (1999) 127–131.

[10] L. T. Butler. Integrable geodesic flows on $n$-step nilmanifolds. *J. Geom. Phys.*, 36:3-4 (2000) 315–323.

[11] L. T. Butler. New examples of integrable geodesic flows. *Asian J. Math.*, 4:3 (2000) 515–526.

[12] L. T. Butler. Integrable geodesic flows with wild first integrals: the case of two-step nilmanifolds. *Ergodic Theory Dynam. Systems* 23:3 (2003) 771–797.

[13] K. Dekimpe. *Almost-Bieberbach groups: affine and polynomial structures*, volume 1639 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[14] N. Desolneux-Moulis. Dynamique des systèmes hamiltoniens complètement intégrables sur les variétés compactes. In *Géométrie symplectique et mécanique (La Grande Motte, 1988)*, Springer, Berlin, 1990, pp. 75–83.

[15] J. J. Duistermaat. On global action-angle coordinates. *Comm. Pure Appl. Math.*, 33:6 (1980) 687–706.

[16] B. Evans and L. Moser. Solvable fundamental groups of compact 3-manifolds. *Trans. Amer. Math. Soc.*, 168 (1972) 189–210.

[17] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov.* Birkhäuser Boston Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[18] M. Giaquinta and S. Hildebrandt. *Calculus of variations. I–II.* Springer-Verlag, Berlin, 1996.

[19] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes tudes Sci. Publ. Math.* 53 (1981) 53–73.

[20] J. Hempel and W. Jaco. Fundamental groups of 3-manifolds which are extensions. *Ann. of Math.* 95:2 (1972) 86–98.

[21] H. Ito. Action-angle coordinates at singularities for analytic integrable systems. *Math. Z.*, 206:3 (1991) 363–407.

[22] V. Jurdjevic. *Geometric control theory.* Cambridge University Press, Cambridge, 1997.

[23] V. V. Kozlov. Topological obstacles to the integrability of natural mechanical systems. *Dokl. Akad. Nauk SSSR*, 249:6 (1979) 1299–1302.

[24] V. V. Kozlov. *Symmetries, topology and resonances in Hamiltonian mechanics.* Springer-Verlag, Berlin, 1996.

[25] S. Lojasiewicz. Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa (3)*, 18 (1964) 449–474.

[26] J. R. Munkres *Elements of Algebraic Topology*, Addison-Wesley, 1984.

[27] G. P. Paternain. On the topology of manifolds with completely integrable geodesic flows. *Ergodic Theory Dynamical Systems*, 12:1 (1992) 109–121.

[28] G. P. Paternain. Multiplicity two actions and loop space homology. *Ergodic Theory Dynamical Systems*, 13:1 (1993) 143–151.

[29] G. P. Paternain. On the topology of manifolds with completely integrable geodesic flows. II. *J. Geom. Phys.*, 13:3 (1993) 289–298.

[30] H. Putz. Triangulation of fibre bundles. *Canad. J. Math.* 19 (1967) 499–513.

[31] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15:5 (1983) 401–487.

[32] I. A. Taĭmanov. Topological obstructions to the integrability of geodesic flows on nonsimply connected manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51:2 (1987) 429–435, 448.

[33] I. A. Taĭmanov. Topology of Riemannian manifolds with integrable geodesic flows. *Trudy Mat. Inst. Steklov.*, 205 (1994) 150–163.

[34] G. W. Whitehead. *Elements of homotopy theory.* Springer-Verlag, New York, 1978.