The University of Edinburgh
2010

School of Mathematics
(U01457)

Geometry & Convergence

Problem Sheet 5

Assessment 5 due by 12.10 on Friday, 12 March 2010.
Tutorial 5 on Tuesday, 9 March 2010.

Pretutorial questions: 3, and 12.

Tutorial questions: 4, 5, 6, and 11.

Handin questions: 1, 2, 7, and 10.

(3**) A certain algorithm takes time $T(n)$ to sort a set of 2^n elements, and time $T(n+1) = T(n) \times n^2$ to sort a set of 2^{n+1} elements. Show by induction that

$$T(n) = ((n-1)!)^2 T(1) \qquad (n = 1, 2, \ldots).$$

(4*) Prove by induction that $3^n - 2n^2 - 1$ is divisible by 8, for $n = 1, 2, \ldots$.

(5*) The Fibonacci numbers f_n are defined by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Prove by strong induction that

$$\phi^{n-2} \leq f_n \leq \phi^n \qquad (n = 1, 2, \ldots),$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$, the so-called *Golden Ratio*.

[Use the fact that $1 + \phi = \phi^2$.]

(6*) Let $\lfloor x \rfloor$ be the *floor* of x, i.e. the largest integer $\leq x$. Prove by induction that

$$n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor \qquad (n = 1, 2, \ldots).$$

(11*) Define a sequence $(a_n)_{n \in \mathbb{N}}$ by $a_n = \dfrac{2n^2 - 1}{n^3 - 2}$. Prove that this sequence tends to 0 as $n \to \infty$.

(12**) Prove that the sequence $(t_n)_{n \in \mathbb{N}}$ defined by $t_n = \dfrac{2n + \sin n}{3n}$ tends to a limit as $n \to \infty$.