The University of Edinburgh — School of Mathematics
2010 — (U01457)

Geometry & Convergence

Problem Sheet 3

Assessment 3 due by 12.10 on Friday, 12 February 2010.
Tutorial 3 on Tuesday, 9 February 2010.

Pretutorial questions: 2, and 7.

Tutorial questions: 3, and 5.

Handin questions: 1, 4, and 8.

(2**) (i) Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ be a fixed vector in $\mathbb{R}^3$. Show that for $\mathbf{x} = \langle x, y, z \rangle$, the map f from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $f(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$ is a linear map.

(ii) Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be fixed vectors in $\mathbb{R}^3$. Show that for $\mathbf{x} = \langle x, y, z \rangle$, the maps f and g from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $f(\mathbf{x}) = \mathbf{a} \times (\mathbf{b} \times \mathbf{x})$ and $g(\mathbf{x}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{x}$ are both linear maps.

(iii) Find the 3×3 matrices A, B such that $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{x}) = B\mathbf{x}$.

What are these matrices when $\mathbf{a} = \langle 1, 0, 3 \rangle$ and $\mathbf{b} = \langle 2, 1, 1 \rangle$?

(3*) (i) Find a 3×3 orthogonal matrix all of whose entries are $\pm 1/3$ or $\pm 2/3$.

[Recall that rows have length 1, and are orthogonal to each other.]

(ii) Find a 4×4 orthogonal matrix all of whose entries are $\pm 1/2$.

(5*) For the rotation map $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and the reflection map $M_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}$ verify by matrix multiplication that $M_\phi R_\theta = M_{\phi-\theta}$ and that $R_\theta M_\phi = M_{\phi+\theta}$. Use these identities to show, without any further matrix multiplication that $R_{-\theta} M_\phi R_\theta = M_{\phi-2\theta}$.

(7**) Find the four points of intersection of the two ellipses

$$E_1 : \frac{y^2}{81} + x^2 = 1, \qquad E_2 : y^2 + \frac{x^2}{16} = 1.$$