The University of Edinburgh — School of Mathematics
2010 — (U01457)

Geometry & Convergence

Problem Sheet 1

Assessment 1 due by 12.10 on Friday, 15 January 2010.

Tutorial questions: 4, 6, and 14.

Handin questions: 1, and 3.

(4*) Find the component of $\mathbf{c} = \langle 2, 1\rangle$ in the direction $\mathbf{v} = \langle 4, 1\rangle$. Hence write $\mathbf{c}$ in the form $\mathbf{c} = \lambda\mathbf{v} + \mathbf{w}$, where $\mathbf{v} \cdot \mathbf{w} = 0$.

Check also that $\mathbf{w}$ itself is the component of $\mathbf{c}$ in the direction $\mathbf{w}$.

Draw a picture to illustrate the question.

(6*) Let $\mathbf{u} = \langle u_1, u_2\rangle, \mathbf{v} = \langle v_1, v_2\rangle$ be two independent vectors in $\mathbb{R}^2$ (i.e. neither is a multiple of the other), and put $\mathbf{u}^\perp = \langle -u_2, u_1\rangle, \mathbf{v}^\perp = \langle -v_2, v_1\rangle$.

Show that $\mathbf{u}^\perp \cdot \mathbf{u} = \mathbf{v}^\perp \cdot \mathbf{v} = 0$.

Writing $\mathbf{x} \in \mathbb{R}^2$ as a linear combination $\lambda\mathbf{u} + \mu\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$, show by taking appropriate dot products that

- $\lambda = (\mathbf{v}^\perp \cdot \mathbf{x})/(\mathbf{v}^\perp \cdot \mathbf{u})$;
- $\mu = (\mathbf{u}^\perp \cdot \mathbf{x})/(\mathbf{u}^\perp \cdot \mathbf{v})$.

Now try this procedure with the example $\mathbf{u} = \langle 2, 3\rangle, \mathbf{v} = \langle 2, -1\rangle$ and $\mathbf{x} = \langle 1, 1\rangle$. Check that indeed $\mathbf{x} = (\mathbf{v}^\perp \cdot \mathbf{x})/(\mathbf{v}^\perp \cdot \mathbf{u})\mathbf{u} + (\mathbf{u}^\perp \cdot \mathbf{x})/(\mathbf{u}^\perp \cdot \mathbf{v})\mathbf{v}$ for this example.

($6\frac{1}{2}$) In Q6, explain why $\mathbf{v}^\perp \cdot \mathbf{u}$ (and, similarly, $\mathbf{u}^\perp \cdot \mathbf{v}$) is nonzero.

(14*) Consider the points $A = (1, 1, 1)$, $B = (1, -1, -1)$, $C = (-1, 1, -1)$, $D = (-1, -1, 1)$ in $\mathbb{R}^3$.

(a) Show that they are all equidistant from the origin $O = (0, 0, 0)$. What is this distance?

(b) Show that they are equidistant from each other, and so form the vertices of a regular tetrahedron. What is this distance?

(c) Find the angle $\angle AOB$.
(This is the so-called *tetrahedral angle*, and is e.g. the angle subtended at the carbon atom by two hydrogen atoms in a methane molecule CH_4.)