The University of Edinburgh School of Mathematics
2010 (U01457)

Geometry & Convergence

Problem Sheet 5: Solutions

Assessment 5 due by 12.10 on Friday, 12 March 2010.
Tutorial 5 on Tuesday, 9 March 2010.

Pretutorial questions: 3, and 12.

Tutorial questions: 4, 5, 6, and 11.

Handin questions: 1, 2, 7, and 10.

($1^\dagger$) Prove by induction that, for fixed $a \neq 1$ and $n = 1, 2, \ldots$

$$1 + a + a^2 + \ldots + a^{n-1} = \frac{a^n - 1}{a - 1}.$$

Solution.

Base Case: For $n = 0$, we have $LHS_n = 1$ and $RHS_n = (a-1)/(a-1) = 1$ **[mark: 1]**. Induction case: Assume that, for a fixed $n \geq 1$, the $LHS_n = RHS_n$ **[mark: 1]**. Then

$$\begin{aligned} LHS_{n+1} &= (1 + a + \cdots + a^{n-1}) + a^n \\ &= \frac{a^n - 1}{a - 1} + a^n, && \text{by induction hypothesis} \\ &= \frac{a^{n+1} - 1 + a^{n+1} - a^n}{a - 1} \\ &= \frac{a^{n+1} - 1}{a - 1}. \\ &= RHS_{n+1} && \textbf{[mark: 1]} \end{aligned}$$

This proves, by the principle of mathematical induction, that the formula is true for all $n \in \mathbb{N}$. **Total Marks for Question: 3.**

($2^\dagger$) Define a_n $(n = 0, 1, 2, \ldots)$ by $a_0 = 1$ and $a_{n+1} = a_n + 2^n + 1$. Show by induction that

$$a_n = 2^n + n \qquad (n = 0, 1, 2, \ldots).$$

Solution.

Base Case: For the base case of $n = 0$, we have $LHS_n = a_0 = 1$ and $RHS_n = 2^0 + 0 = 1$ **[mark: 1]**.

Induction Case: Assume that, for a fixed $n \geq 0$, the $LHS_n = RHS_n$ **[mark: 1]**. Then

$$\begin{aligned} LHS_{n+1} &= a_{n+1} = a_n + 2^n + 1 \\ &= (2^n + n) + 2^n + 1 && \text{by induction hypothesis} \\ &= 2^{n+1} + (n+1) \\ &= RHS_{n+1}. && \textbf{[mark: 1]} \end{aligned}$$

This proves, by the principle of mathematical induction, that the formula is true for all $n \in \mathbb{N}$. **Total Marks for Question: 3.**

(3^{**}) A certain algorithm takes time $T(n)$ to sort a set of 2^n elements, and time $T(n+1) = T(n) \times n^2$ to sort a set of 2^{n+1} elements. Show by induction that

$$T(n) = ((n-1)!)^2 T(1) \qquad (n = 1, 2, \ldots).$$

Solution.

Base Case: for $n = 1$, since $0! = 1$, we have $LHS_n = RHS_n$.

Induction Step: Assume that for a fixed $n \geq 1$, the above equation is true. Then

$$\begin{aligned} T(n+1) &= T(n) \times n^2 && \text{by definition of } T(n) \\ &= n^2 \times ((n-1)!)^2 \times T(1) && \text{by induction hypothesis} \\ &= (((n+1)-1)!)^2 T(1), \end{aligned}$$

which proves that $LHS_{n+1} = RHS_{n+1}$. The principle of mathematical induction implies that the identity holds for all $n \in \mathbb{N}$.

(4*) Prove by induction that $3^n - 2n^2 - 1$ is divisible by 8, for $n = 1, 2, \ldots$.

Solution.

Base Case: When $n = 1$, we have $3^n - 2n^2 - 1 = 0 \equiv 0 \bmod 8$, hence is divisible by 8.

Induction step: Assume that $3^n - 2n^2 - 1 \equiv 0 \bmod 8$. Then

$$\begin{aligned} &3^{n+1} - 2(n+1)^2 - 1 \\ &= 3(3^n - 2n^2 - 1) + 4n^2 + 4n \\ &\equiv 4n(n+1) \bmod 8 && \text{by the induction hypothesis} \\ &\equiv 0 \bmod 8, && \text{one of } n \text{ or } n+1 \text{ is even.} \end{aligned}$$

This proves, by the principle of mathematical induction, that $3^n - 2n^2 - 1 \equiv 0 \bmod 8$ for all $n \in \mathbb{N}$.

(5*) The Fibonacci numbers f_n are defined by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Prove by strong induction that

$$\phi^{n-2} \leq f_n \leq \phi^n \qquad (n = 1, 2, \ldots),$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$, the so-called *Golden Ratio.*

[Use the fact that $1 + \phi = \phi^2$.]

Solution.

Base Case: Let $n = 1$. Then,

$$\begin{aligned} &4 \leq 5 \leq 9 && \implies 2 \leq \sqrt{5} \leq 3 \\ &3 \leq \sqrt{5} + 1 \leq 4 && \implies \frac{3}{2} \leq \frac{\sqrt{5}+1}{2} \leq 2 \end{aligned}$$

Thus, with $n = 1$,

$$\phi^{-1} = \frac{2}{\sqrt{5}+1} \leq \frac{1}{2} < 1 = f_1 \leq \frac{3}{2} \leq \frac{\sqrt{5}+1}{2} = \phi^1$$

Induction step: Assume the inequality is true for all integers between 1 and some fixed $n \geq 1$. Then,

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} && \text{by definition of } f_{n+1} \\ &\geq \phi^{n-2} + \phi^{n-3} && \text{by induction hypothesis} \\ &\geq \phi^{n-3}(1 + \phi) \\ &= \phi^{(n+1)-2} \end{aligned}$$

which proves the LHS of the inequality for $n + 1$. While

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} && \text{by definition of } f_{n+1} \\ &\leq \phi^n + \phi^{n-1} && \text{by induction hypothesis} \\ &\leq \phi^{n-1}(1 + \phi) \\ &= \phi^{(n+1)} \end{aligned}$$

which proves the RHS of the inequality for $n + 1$.

Therefore, by the principle of (strong) mathematical induction, the inequalities are true for all $n \in \mathbb{N}$.

(6*) Let $\lfloor x \rfloor$ be the *floor* of x, i.e. the largest integer $\leq x$. Prove by induction that

$$n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor \qquad (n = 1, 2, \ldots).$$

Solution.

Base Case: For $n = 1$, we have $\lfloor n/2 \rfloor = 0$ and $\lfloor (n+1)/2 \rfloor = 1$ which shows that $LHS_n = RHS_n$ for $n = 1$.

Induction Step: Let us observe that $\lfloor x + 1 \rfloor = \lfloor x \rfloor + 1$ for all real numbers x, so $\lfloor (n+2)/2 \rfloor = \lfloor n/2 \rfloor + 1$ for all $n \in \mathbb{N}$. Assume that the equation is true for some fixed $n \geq 1$. Then

$$\begin{aligned} n + 1 &= \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + 1 && \text{by induction hypothesis} \\ &= \left\lfloor \frac{n+2}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor, && \text{by the observation.} \end{aligned}$$

By the principle of mathematical induction, the proves the identity for all $n \in \mathbb{N}$.

(7†) What is wrong with the following inductive argument:

" **Theorem**. Let $x > 0$ be fixed. Then $x^{n-1} = 1$ for $n = 1, 2, \ldots$.

Proof. If $n = 1$ then $x^{1-1} = x^0 = 1$, so result true for $n = 1$. Assuming the result true for $1, 2, \ldots$, we have

$$x^{(n+1)-1} = x^n = x^{n-1} \times x^{n-1}/x^{n-2} = 1 \times 1/1 = 1,$$

so that the result holds for $n+1$ as well." (Knuth)

Solution.

This is a subtle flaw. Let $P(n)$ be the statement "$x^{n-1} = 1$". The proof has verified $P(1)$ is true. The inductive step then proves that if $P(n)$ and $P(n-1)$ are true, then $P(n+1)$ is true. However, in mathematical induction, we are only allowed to assume $P(n)$ in order to prove $P(n+1)$.

In strong mathematical induction, we are allowed to assume $P(1), \cdots, P(n)$ in order to prove $P(n+1)$. In particular, we are only allowed to assume $P(1)$ to prove $P(2)$. However, above we assume $P(0)$ and $P(1)$ to prove $P(2)$ **[marks: 2]**. **Total Marks for Question: 2.**

(8) Show that if, for some proposition $P(n)$,

(a) $P(1)$ is true

(b) $P(n)$ true $\implies P(2n)$ and $P(2n+1)$ both true $(n = 1, 2, \ldots)$

then $P(n)$ is true for $n = 1, 2, \ldots$.

[Use induction on the length of the binary representation of n].

(9) *(Esoteric variant of induction.)* Show that if, for some statement $P_2(n)$,

(a) $P_2(1)$ is true

(b) $P_2(n)$ true $\implies P_2(2n)$ true $(n = 1, 2, \ldots)$

(c) $P_2(n+1)$ true $\implies P_2(n)$ true (!) $(n = 1, 2, \ldots)$

then $P_2(n)$ is true for $n = 1, 2, \ldots$.

Convergence of sequences and series

(10†) Define a sequence $(t_n)_{n \in \mathbb{N}}$ by $t_n = \dfrac{2n+1}{n^3}$. Prove that this sequence tends to 0 as $n \to \infty$.

Solution.

`Rough Work.` Let us obtain a simple upper bound on $|t_n - 0|$:

$$\begin{aligned} |t_n - 0| &\leq \frac{1}{n^2} \times \frac{2 + 1/n}{1} && \text{dividing top/bottom by } n/n^3 \\ &\leq \frac{3}{n^2} && \text{since } n \geq 1,\ 1/n \leq 1 \\ &\leq \frac{1}{n} && \text{for } n \geq 3. \end{aligned}$$

[mark: 1]

`End of Rough Work`

Proof. Let $\epsilon > 0$ be given. Choose N to be the largest of 3 and $\epsilon^{-1} + 1$. If $n \geq N$, then

$$\begin{aligned} |t_n - 0| &\leq \frac{1}{n} && \text{by rough work, since } n \geq 3 \\ &\leq \frac{1}{N} && \text{since } n \geq N \\ &< \epsilon. \end{aligned}$$

By the definition of convergence, t_n converges to 0 **[mark: 1]**.

Total Marks for Question: 2.

(11*) Define a sequence $(a_n)_{n\in\mathbb{N}}$ by $a_n = \dfrac{2n^2-1}{n^3-2}$. Prove that this sequence tends to 0 as $n \to \infty$.

Solution.

Since $a_n \sim 2/n$ for large n, it should converge to 0.

`Rough Work.` Let us obtain a simple upper bound on $|a_n - 0|$:

$$\begin{aligned} |a_n - 0| &= \left|\frac{2n^2-1}{n^3} - 2\right| \\ &\le \left|\frac{2n^2}{\frac{1}{2}n^3}\right| && \text{since } |n^3-2| = \frac{1}{2}n^3 + (\frac{1}{2}n^3 - 2) \\ & && > \frac{1}{2}n^3 \text{ for } n \ge 2 \\ &\le \frac{2n^2}{\frac{1}{2}n^3} && \text{since } n \ge 1 \\ &\le \frac{4}{n} && \text{for } n \ge 2. \end{aligned}$$

So we want $4/n < \epsilon$, i.e. $n > 4/\epsilon$.
`End of Rough Work`

Proof. Let $\epsilon > 0$ be given. Choose N to be the largest of 2 and $4\epsilon^{-1} + 1$. If $n \ge N$, then

$$\begin{aligned} |a_n - 0| &\le \frac{4}{n} && \text{by rough work, since } n \ge 2 \\ &\le \frac{4}{N} && \text{since } n \ge N \\ &< \epsilon && \text{since } 4/N < 1/(\epsilon^{-1} + 1/4) < \epsilon. \end{aligned}$$

By the definition of convergence, a_n converges to 0.

(12**) Prove that the sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_n = \dfrac{2n + \sin n}{3n}$ tends to a limit as $n \to \infty$.

Solution.

Since $t_n = \dfrac{\sin n}{3n} + \dfrac{2}{3}$, and $|\sin n|$ is bounded, t_n should converge to 2/3.

`Rough Work.`

$$\begin{aligned} \left|t_n - \frac{2}{3}\right| &= \frac{|\sin n|}{3n} \\ &\le \frac{1}{3n} && \text{since } |\sin x| \le 1,\ x \in \mathbb{R} \\ &\le \frac{1}{n}. \end{aligned}$$

`End of Rough Work`

Proof. Let $\epsilon > 0$ be given. Choose N to equal $\epsilon^{-1} + 1$ so that $1/N < \epsilon$. Then, if $n \ge N$,

$$\begin{aligned} \left|t_n - \frac{2}{3}\right| &\le \frac{1}{n} && \text{by rough work} \\ &\le \frac{1}{N} && \text{since } n \ge N \\ &< \epsilon. \end{aligned}$$

This proves that t_n converges to 2/3.

(13) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ tends to the limit A, while the sequence $(b_n)_{n\in\mathbb{N}}$ tends to B. Prove that the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ tends to $A + B$.

(14) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ tends to the limit A, while the sequence $(b_n)_{n\in\mathbb{N}}$ tends to B. Prove that the sequence $(a_n \cdot b_n)_{n\in\mathbb{N}}$ tends to AB.

(15) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ tends to a limit A, and the sequence $(b_n)_{n\in\mathbb{N}}$ tends to a limit B. Does the sequence $a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$ tend to a limit?

(16) Prove that the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2+1}$ converges.

(17) Prove that the series $\sum_{n=1}^{\infty} \dfrac{1}{n+100000}$ diverges.

(18) (Harder) You know (e.g. from the Group Theory course) that the rationals are countable. This means that there is a sequence $(t_n)_{n\in\mathbb{N}}$ that contains each rational number exactly once. (In fact there are many such sequences, obtained by re-ordering $(t_n)_{n\in\mathbb{N}}$ in any way you want to.)

- Prove that $(t_n)_{n\in\mathbb{N}}$ does not tend to a limit.
- On the other hand, prove that for every real number q there is a subsequence of $(t_n)_{n\in\mathbb{N}}$ that tends to q.

(19) Evaluate the recurring decimal $0.142857142857\ldots$ exactly as a rational.

Total Marks for Paper: 10.