The University of Edinburgh
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School of Mathematics
(U01457)

Geometry & Convergence

Problem Sheet 4: Solutions

Assessment 4 due by 12.10 on Friday, 26 February 2010.
Tutorial 4 on Tuesday, 23 February 2010.

Tutorial questions: 1, 3, and 5.

Handin questions: 2, and 4.

Conics

(1*) Put the following conics into standard form.

(i) $\mathcal{X}_0 : 7y^2 + 2xy + 7x^2 = 1$.

(ii) $\mathcal{X}_1 : 7y^2 + 2xy - y + 7x^2 + 11x = 1$.

(iii) What is the length of the semi-minor (resp. semi-majour) axis of $\mathcal{X}_0$?

(iv) What is the centre of $\mathcal{X}_1$?

Solution.

Since the conic $\mathcal{X}_1$ has the same quadratic part as $\mathcal{X}_0$, we will do just $\mathcal{X}_1$. Let

$$S = \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial of S is

$$\det(S - \lambda I) = (7-\lambda)^2 - 1 \qquad \Longrightarrow \quad \text{roots } \lambda = 6, 8.$$

The associated unit eigenvectors are

$$\lambda_1 = 6: \quad S - 6I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \Longrightarrow \mathbf{x}_1 = \frac{1}{\sqrt{2}} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 8: \quad S - 8I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \Longrightarrow \mathbf{x}_2 = \frac{1}{\sqrt{2}} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, we will introduce the coordinate change

$$\mathbf{x} = \frac{1}{\sqrt{2}} \times \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{u}, \qquad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

and simplify $\mathcal{X}_1$ ($\mathcal{X}_0$) to

$$\mathcal{X}_1 : 8v^2 + 6u^2 + 3 \times 2^{\frac{3}{2}} u + 5\sqrt{2} v = 1.$$
$$\mathcal{X}_0 : 8v^2 + 6u^2 = 1.$$

We substitute

$$u = r - \frac{3 \times 2^{\frac{3}{2}}}{2 \times 6} = r - \frac{1}{\sqrt{2}} \qquad v = s - \frac{5\sqrt{2}}{2 \times 8} = s - \frac{5}{2^{\frac{7}{2}}},$$

to get

$$\mathcal{X}_1 : 8s^2 + 6r^2 = 89/16.$$

Remark. We did not need to compute $\mathbf{x}_2$ explicitly, once we know $\mathbf{x}_1$, since $\mathbf{x}_2 \perp \mathbf{x}_1$.

(v) In $\mathcal{X}_0$, we have $|u| \leq 1/\sqrt{6}$, $|v| \leq \frac{1}{2\sqrt{2}}$, so the semi-minor (semi-majour) axis length is $\dfrac{1}{2\sqrt{2}}$ (resp. $\dfrac{1}{\sqrt{6}}$).

(vi) $(u, v) = \left(-\dfrac{1}{\sqrt{2}}, -\dfrac{5}{2^{\frac{7}{2}}} \right)$.

(2†) Put the following conics into standard form.

(i) $\mathcal{X}_0 : 86y^2 - 96xy + 114x^2 = 1$.

(ii) $\mathcal{X}_1 : 86y^2 - 96xy + 45y + 114x^2 + 65x = 1$.

Solution.

Since the conic $\mathcal{X}_1$ has the same quadratic part as $\mathcal{X}_0$, we will do just $\mathcal{X}_1$. Let

$$S = \begin{bmatrix} 114 & -48 \\ -48 & 86 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$ [mark: 1]

The characteristic polynomial of S is

$$\det(S - \lambda I) = (86 - \lambda)(114 - \lambda) - 2304 = \lambda^2 - 200\lambda + 7500$$
$$\implies \text{roots } \lambda = 150, 50. \quad \textbf{[mark: 1]}$$

The associated unit eigenvectors are

$$\lambda_1 = 50: \qquad S - 50I = \begin{bmatrix} 64 & -48 \\ -48 & 36 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} \begin{matrix} \text{row 1 / 16} \\ \text{row 2 / (-12)} \end{matrix}$$
$$\implies \mathbf{x}_1 = \frac{1}{5} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$
$$\lambda_2 = 150: \qquad S - 150I = \begin{bmatrix} -36 & -48 \\ -48 & -64 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{matrix} \text{row 1 / (-12)} \\ \text{row 2 / (-16)} \end{matrix}$$
$$\implies \mathbf{x}_2 = \frac{1}{5} \times \begin{bmatrix} 4 \\ -3 \end{bmatrix}. \quad \textbf{[mark: 1]}$$

Thus, we will introduce the coordinate change

$$\mathbf{x} = \frac{1}{5} \times \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \mathbf{u}, \qquad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

and simplify $\mathcal{X}_1$ ($\mathcal{X}_0$) to

$$\mathcal{X}_1 : 50u^2 + 150v^2 + 75u + 25v = 1,$$
$$\mathcal{X}_0 : 50u^2 + 150v^2 = 1. \quad \textbf{[mark: 1]}$$

We substitute

$$u = r - \frac{75}{2 \times 50} = r - \frac{3}{4} \qquad v = s - \frac{25}{2 \times 150} = s - \frac{1}{12},$$

to get

$$\mathcal{X}_1 : 150s^2 + 50r^2 = \frac{181}{6}. \quad \textbf{[mark: 1]}$$

Total Marks for Question: 5.

(3*) Put the following centred conics into standard form simultaneously.

$$\mathcal{X}_0 : 95y^2 + 216xy + 130x^2 = 1,$$
$$\mathcal{X}_1 : 222y^2 + 480xy + 278x^2 = 1.$$

Do these conics intersect?

Solution.

Let S and R denote the symmetric matrices associated to $\mathcal{X}_0$ and $\mathcal{X}_1$, respectively,

$$S = \begin{bmatrix} 130 & 108 \\ 108 & 95 \end{bmatrix} \qquad R = \begin{bmatrix} 278 & 240 \\ 240 & 222 \end{bmatrix}.$$

We wish to simultaneously diagonalise R and S as quadratic forms. To do this we compute the generalised characteristic polynomial, its roots, and the generalised eigenvectors.

$$\det(R - \lambda S) = (278 - 130\lambda)(222 - 95\lambda) - (240 - 108\lambda)^2$$
$$= 686(\lambda^2 - 5\lambda + 6).$$
$$\implies \text{roots } \lambda = 2, 3.$$

Associated eigenvectors are

$$\lambda_1 = 3: \qquad R - 3S = \begin{bmatrix} -112 & -84 \\ -84 & -63 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \begin{matrix} \text{-row 1 / 28} \\ \text{-row 2 / 21} \end{matrix}$$
$$\implies \mathbf{x}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix},$$
$$\lambda_2 = 2: \qquad R - 2S = \begin{bmatrix} 18 & 24 \\ 24 & 32 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{matrix} \text{row 1 / 6} \\ \text{row 2 / 8} \end{matrix}$$
$$\implies \mathbf{x}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Thus, we will introduce the coordinate change

$$\mathbf{x} = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix} \mathbf{u}, \qquad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

and simplify $\mathcal{X}_0$ ($\mathcal{X}_1$) to

$$\mathcal{X}_0 : 343v^2 + 98u^2 = 1,$$
$$\mathcal{X}_1 : 686v^2 + 294u^2 = 1.$$

The final step is to introduce the substitution

$$u = r/\sqrt{98} \qquad v = s/\sqrt{343} \text{ which implies}$$
$$\mathcal{X}_0 : s^2 + r^2 = 1$$
$$\mathcal{X}_1 : 2s^2 + 3r^2 = 1.$$

Note that $\mathcal{X}_0$ and $\mathcal{X}_1$ do not intersect in $\mathbb{R}^2$, since when solving for r^2, we get $r^2 = -1$.

(4†) Put the following centred conics into standard form simultaneously.

$$\mathcal{X}_0 : 30y^2 + 32xy + 9x^2 = 1,$$
$$\mathcal{X}_1 : 12y^2 + 20xy + 9x^2 = 1.$$

Do these conics intersect?

Solution.

Let S and R denote the symmetric matrices associated to $\mathcal{X}_0$ and $\mathcal{X}_1$, respectively,

$$S = \begin{bmatrix} 9 & 16 \\ 16 & 30 \end{bmatrix} \qquad R = \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}.$$ **[mark: 1]**

We wish to simultaneously diagonalise R and S as quadratic forms. To do this we compute the generalised characteristic polynomial, its roots, and the generalised eigenvectors.

$$\det(R - \lambda S) = (12 - 30\lambda)(9 - 9\lambda) - (10 - 16\lambda)^2$$
$$14\lambda^2 - 58\lambda + 8 \implies \text{roots } \lambda = \frac{1}{7}, 4.$$ **[mark: 1]**

Associated eigenvectors are

$$\lambda_1 = 4 : \qquad R - 4S = \begin{bmatrix} -27 & -54 \\ -54 & -108 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{matrix} \text{-row 1 / 27} \\ \text{-row 2 / 54} \end{matrix}$$
$$\implies \mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$ **[mark: 1]**

$$\lambda_2 = \frac{1}{7} : \qquad R - \frac{1}{7}S = \begin{bmatrix} \frac{54}{7} & \frac{54}{7} \\ \frac{54}{7} & \frac{54}{7} \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\implies \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Thus, we will introduce the coordinate change

$$\mathbf{x} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{u}, \qquad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$ **[mark: 1]**

and simplify $\mathcal{X}_0$ ($\mathcal{X}_1$) to

$$\mathcal{X}_0 : 7v^2 + 2u^2 = 1,$$
$$\mathcal{X}_1 : v^2 + 8u^2 = 1.$$ **[mark: 1]**

The final step is to introduce the substitution

$$u = r/\sqrt{2} \qquad v = s/\sqrt{7} \text{ which implies}$$
$$\mathcal{X}_0 : s^2 + r^2 = 1$$
$$\mathcal{X}_1 : \frac{s^2}{7} + 4r^2 = 1.$$

We see that the conics intersect at $(r, s) = (\pm\sqrt{2}/3, \pm\sqrt{7}/3)$.

Total Marks for Question: 5.

Total Marks for Paper: 10.

Induction

(5*) Prove by induction that $n^2 - n + 2$ is always even for $n = 1, 2, \ldots$.

Solution.

For $n = 1$, we have $n^2 - n + 2 = 2$, which is even. So, let's assume that $n^2 - n + 2$ is even for some $n \geq 1$. Then $(n+1)^2 - (n+1) + 2 = (n^2 - n + 2) + 2n$, which is the sum of even numbers, hence even. Thus, the principal of induction says that $n^2 - n + 2$ is even for all positive integers n.

Remark. It may appear that induction is not needed since $n^2 - n = n(n-1)$ and either n or $n - 1$ is always divisible by n. However, one would want to insist that this "obvious" fact be proven by induction.

(6) Prove by induction that, for $n = 1, 2, \ldots$

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2} \right)^2 .$$

(7) Prove by induction that $3^n > 5n^2$ for $n \geq 4$.

(8) Prove that $\cos(n\pi) = (-1)^n (n = 0, \pm 1, \pm 2, \ldots)$, by induction.

(9) In a computer memory, an arbitrary length vector v_n stores sequentially all the previous vectors $v_1, v_2, \ldots, v_{n-1}$, so the length ℓ_n of v_n satisfies

$$\ell_n = \ell_1 + \ldots + \ell_{n-1} \qquad (n = 2, 3, \ldots .$$

Given that $\ell_1 = 1$, prove by induction that $\ell_n = 2^{n-2}$ for $n = 2, 3, \ldots$.

What about $n = 1$?