The University of Edinburgh — School of Mathematics
2010 — (U01457)

Geometry & Convergence

Problem Sheet 3: Solutions

Assessment 3 due by 12.10 on Friday, 12 February 2010.
Tutorial 3 on Tuesday, 9 February 2010.

Pretutorial questions: 2, and 7.

Tutorial questions: 3, and 5.

Handin questions: 1, 4, and 8.

(1†) (i) In the following question, $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{b} = -5\mathbf{i} + 2\mathbf{k}$ and $\mathbf{c} = -4\mathbf{k} + \mathbf{i}$.

(a) Let $\Pi = \{P \in \mathbb{R}^3 : \mathbf{p} = t\mathbf{a} + s\mathbf{b}, s, t \in \mathbb{R}\}$. Compute a normal vector to Π.

(b) Compute the distance from Π to the point C.

(ii) Compute the volume of the parallelepiped spanned by the vectors $\mathbf{a} = \langle -1, 0, 4\rangle$, $\mathbf{b} = \langle 3, 4, 0\rangle$ and $\mathbf{c} = \langle -5, 0, 2\rangle$.

(iii) Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. Is it true that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

for all $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$? If yes, demonstrate this; if no, give an example where the two differ.

Solution.

(i) (a) A normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b}$. We expand and get $8\mathbf{j} \times \mathbf{k} - 20\mathbf{j} \times \mathbf{i} + 6\mathbf{i} \times \mathbf{k} - 15\mathbf{i} \times \mathbf{i}$, which simplifies to $20\mathbf{k} - 6\mathbf{j} + 8\mathbf{i}$ [mark: 1].

(b) The distance is $d = |\mathbf{n} \cdot \mathbf{c}|/|\mathbf{n}|$, since $\mathbf{0} \in \Pi$. We get $\mathbf{n} = 20\mathbf{k} - 6\mathbf{j} + 8\mathbf{i}$ dotted with $\mathbf{c} = +\mathbf{i} - 4\mathbf{k}$ equals $8 - 4 \times 20 = -72$ and $|\mathbf{n}| = 2 \times 5^{\frac{3}{2}}$ so the distance is $36/5^{\frac{3}{2}}$ [mark: 1].

(ii) 72 [mark: 1].

(iii) No [mark: 1]. Consider [mark: 1]

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}.$$

Total Marks for Question: 5.

(2**) (i) Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ be a fixed vector in $\mathbb{R}^3$. Show that for $\mathbf{x} = \langle x, y, z\rangle$, the map f from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $f(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$ is a linear map.

(ii) Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle, \mathbf{b} = \langle b_1, b_2, b_3\rangle$ be fixed vectors in $\mathbb{R}^3$. Show that for $\mathbf{x} = \langle x, y, z\rangle$, the maps f and g from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $f(\mathbf{x}) = \mathbf{a} \times (\mathbf{b} \times \mathbf{x})$ and $g(\mathbf{x}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{x}$ are both linear maps.

(iii) Find the 3×3 matrices A, B such that $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{x}) = B\mathbf{x}$.
What are these matrices when $\mathbf{a} = \langle 1, 0, 3\rangle$ and $\mathbf{b} = \langle 2, 1, 1\rangle$?

Solution.

(i) From the determinant definition of the cross product we know that for $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$ and for real λ

$$\mathbf{a} \times (\lambda\mathbf{x}) = \lambda(\mathbf{a} \times \mathbf{x})$$
$$\mathbf{a} \times (\mathbf{x} + \mathbf{x}') = \mathbf{a} \times \mathbf{x} + \mathbf{a} \times \mathbf{x}'.$$

Using these facts, we have that

$$f(\lambda\mathbf{x}) = \mathbf{a} \times (\lambda\mathbf{x}) = \lambda(\mathbf{a} \times \mathbf{x}) = \lambda f(\mathbf{x}), \qquad \text{L1}$$
$$f(\mathbf{x} + \mathbf{x}') = \mathbf{a} \times (\mathbf{x} + \mathbf{x}') = \mathbf{a} \times \mathbf{x} + \mathbf{a} \times \mathbf{x}' = f(\mathbf{x}) + f(\mathbf{x}') \qquad \text{L2.}$$

Thus f is linear.

(ii) We can write $g(\mathbf{x}) = \mathbf{c} \times \mathbf{x}$, so it is linear by the previous question. We know that if F and G are linear transformations, then their composition is linear (if the composition is defined). Since $f(\mathbf{x}) = F(G(\mathbf{x}))$ where $G(\mathbf{x}) = \mathbf{b} \times \mathbf{x}$ and $F(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$ are both linear, f is linear.

(iii) To find A such that $f(\mathbf{x}) = A\mathbf{x}$, we have

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{a} \times (\mathbf{b} \times \mathbf{x}) \\ &= (\mathbf{a} \cdot \mathbf{x})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{x} \end{aligned}$$

using the formula of question 4, sheet 2

$$\begin{aligned} &= (a_1x + a_2y + a_3z) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - (a_1b_1 + a_2b_2 + a_3b_3) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} -a_2b_2 - a_3b_3 & a_2b_1 & a_3b_1 \\ a_1b_2 & -a_1b_1 - a_3b_3 & a_3b_2 \\ a_1b_3 & a_2b_3 & -a_1b_1 - a_2b_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \end{aligned}$$

So

$$A = \begin{bmatrix} -a_2b_2 - a_3b_3 & a_2b_1 & a_3b_1 \\ a_1b_2 & -a_1b_1 - a_3b_3 & a_3b_2 \\ a_1b_3 & a_2b_3 & -a_1b_1 - a_2b_2 \end{bmatrix}.$$

To find B such that $g(\mathbf{x}) = B\mathbf{x}$:
we have

$$\begin{aligned} g(\mathbf{x}) &= (\mathbf{a} \times \mathbf{b}) \times \mathbf{x} \\ &= -\mathbf{x} \times (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{x} \times (\mathbf{b} \times \mathbf{a}) \\ &= (\mathbf{x} \cdot \mathbf{a})\mathbf{b} - (\mathbf{x} \cdot \mathbf{b})\mathbf{a} \end{aligned}$$

again using the formula of question 4, sheet 2

$$\begin{aligned} &= \begin{bmatrix} b_1(a_1x + a_2y + a_3z) \\ b_2(a_1x + a_2y + a_3z) \\ b_3(a_1x + a_2y + a_3z) \end{bmatrix} - \begin{bmatrix} a_1(b_1x + b_2y + b_3z) \\ a_2(b_1x + b_2y + b_3z) \\ a_3(b_1x + b_2y + b_3z) \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_2b_1 - a_1b_2 & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \end{aligned}$$

Hence

$$B = \begin{bmatrix} 0 & a_2b_1 - a_1b_2 & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & 0 \end{bmatrix}.$$

When $\mathbf{a} = \langle 1, 0, 3 \rangle, \mathbf{b} = \langle 2, 1, 1 \rangle$ get

$$A = \begin{bmatrix} -3 & 0 & 6 \\ 1 & -5 & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & 3 \\ -5 & -3 & 0 \end{bmatrix}.$$

Remarks:

1. After we have found that $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{x}) = B\mathbf{x}$ the linearity of f and of g is very easy to prove.
2. The fact that $A \neq B$ and so $f(\mathbf{x}) \neq g(\mathbf{x})$ shows that the cross product is non-associative:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{x}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{x},$$

   i.e. the order in which the cross-multiplication of the three vectors is performed will in general make a difference to the result!

(3*) (i) Find a 3×3 orthogonal matrix all of whose entries are $\pm 1/3$ or $\pm 2/3$.
[Recall that rows have length 1, and are orthogonal to each other.]
(ii) Find a 4×4 orthogonal matrix all of whose entries are $\pm 1/2$.

Solution.

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix}.$$

These matrices can have any of their rows or columns multiplied by -1, or any of their rows swapped ('permuted') or columns swapped to obtain other such orthogonal matrices.
[In lectures orthogonal matrices were in fact defined by $A^T A = I$, i.e. by having their *columns* orthogonal.]

(4†) Suppose that A is an $n \times n$ matrix with the property that each row contains exactly one 1 and each column contains exactly one 1, and all other entries are 0. Prove that A is an orthogonal matrix.

Solution.
From the description, each column of A is a vector $\mathbf{e}_i$ for some i **[mark: 1]**; moreover, no two columns of A are equal (otherwise, the row would have two entries with 1) **[mark: 1]**. Therefore, the columns of A are some permutation of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. This means the columns of A are pair-wise orthogonal unit vectors, so A is orthogonal **[mark: 1]**.
Total Marks for Question: 3.

(5*) For the rotation map $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and the reflection map $M_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}$ verify by matrix multiplication that $M_\phi R_\theta = M_{\phi-\theta}$ and that $R_\theta M_\phi = M_{\phi+\theta}$. Use these identities to show, without any further matrix multiplication that $R_{-\theta} M_\phi R_\theta = M_{\phi-2\theta}$.

Solution.
The first two are just matrix multiplication. Then $R_{-\theta}(M_\phi R_\theta) = R_{-\theta} M_{\phi-\theta} = M_{(\phi-\theta)+(-\theta)} = M_{\phi-2\theta}$.

(6) Find the two points of intersection of the ellipse $x^2 + 2y^2 = 3$ with the line $4x - 3y = 1$.

Solution.
On substituting $x = (1+3y)/4$ we get $((1+3y)/4)^2 + 2y^2 = 3$, giving $(y-1)(y+47/41) = 0$. Then using $x = (1+3y)/4$ we see that the two points of intersection are $(1,1)$ and $((1-3.47/41)/4, -47/41) = (-25/41, -47/41)$.

(7**) Find the four points of intersection of the two ellipses

$$E_1 : \frac{y^2}{81} + x^2 = 1, \qquad E_2 : y^2 + \frac{x^2}{16} = 1.$$

Solution.
Since there are no xy terms in either equation, we can solve for x^2 and y^2 directly, to get $x^2 = 256/259, y^2 = 243/259$. Thus, we get the four points of intersection $x = \pm 16/\sqrt{259}, y = \pm\sqrt{243/259}$.

(8†) For which values of $b > 0$ do the two ellipses E_1 and E_2 intersect?

$$E_1 : \frac{y^2}{b^2} + x^2 = 1 \qquad E_2 : y^2 + \frac{x^2}{16} = 1.$$

Solution.
As in the previous question, we can solve for x^2, y^2: **[mark: 1]**

$$x^2 = \frac{16b^2 - 16}{16b^2 - 1} \qquad y^2 = \frac{15b^2}{16b^2 - 1}.$$

Since $x^2 \geq 0$, we get $b \geq 1$ or $b < \frac{1}{4}$, and since $y^2 \leq 0$ (!) for $b < \frac{1}{4}$, this gives the constraint $b \geq 1$ **[mark: 1]**.

[I guess full marks has to be awarded to someone who draws a picture, and says that it's clear from sketch that $b \geq 1$ is a necessary and sufficient condition.]

Total Marks for Question: 2.

(9) For the ellipse $x^2/9 + y^2/4 = 1$, use the standard formulae to calculate its eccentricity e. Find both foci, and both directrixes. Check that for the point $(\frac{3}{2}\sqrt{3}, 1)$ on this ellipse:

- its distance from the left focus is e times its distance from the left directrix;
- its distance from the right focus is e times its distance from the right directrix;
- the sum of its distances to both foci is equal to the length of its major axis (i.e. $2a = 6$).

Solution.

Now $a = 3, b = 2, e = \sqrt{1 - b^2/a^2} = \sqrt{5}/3$. Left focus $F = (-ae, 0) = (-\sqrt{5}, 0)$, left directrix D is the line $x = -a/e = -9/\sqrt{5}$. For $P = (\frac{3}{2}\sqrt{3}, 1)$, $e|DF| = (\sqrt{5}/3)(9/\sqrt{5} + \frac{3}{2}\sqrt{3}) = 3 + \sqrt{15}/2$, so that $(e|DF|)^2 = (3 + \sqrt{15}/2)^2 = 51/4 + 3\sqrt{15}$. Also $|PF|^2 = (\frac{3}{2}\sqrt{3} + \sqrt{5})^2 + 1 = 51/4 + 3\sqrt{15}$, so that $|PF|^2 = (e|DF|)^2$, $|PF| = e|DF| = 3 + \sqrt{15}/2$. Similarly for the right focus $F' = (ae, 0) = (\sqrt{5}, 0)$ and right directrix $D' : x = a/e = 9/\sqrt{5}$ we have $e|D'F'| = (\sqrt{5}/3)(9/\sqrt{5} - \frac{3}{2}\sqrt{3}) = 3 - \sqrt{15}/2$, so that $(e|D'F'|)^2 = (3 - \sqrt{15}/2)^2 = 51/4 - 3\sqrt{15}$, while also $|PF'|^2 = (\frac{3}{2}\sqrt{3} - \sqrt{5})^2 + 1 = 51/4 - 3\sqrt{15}$, so that $|PF'|^2 = (e|D'F'|)^2$, $|PF'| = e|D'F'| = 3 - \sqrt{15}/2$. Then $|PF| + |PF'| = (3 + \sqrt{15}/2) + (3 - \sqrt{15}/2) = 6 = 2a$.

(10) Imagine the parabola $y^2 = 4x$ as a concave mirror. What is its focus? A light ray comes in from the right along the line $y = 3$. Find the slope of the tangent at the point where the light ray hits the mirror. Call this slope $\tan\phi$. Assuming that the light ray is reflected at this point in the usual way (i.e. so that the arriving and reflected ray make the same angle with this tangent), show that the slope of the reflected ray is $\tan(2\phi)$. Hence evaluate this slope, and so find the equation of the reflected ray. Show that this ray passes through the focus of the parabola.

Solution.

Since $a = 1$, its focus is $(1, 0)$. The ray hits the mirror at $(\frac{9}{4}, 3)$. From $2y\frac{dy}{dx} = 4$ we see that the slope of the tangent there is $2/3 = \tan\phi$. Extending the reflected ray back through the mirror we see that it makes an angle of 2ϕ with the horizontal, so its slope is $\tan(2\phi) = \frac{2\tan\phi}{1 - \tan^2\phi} = 12/5$. Hence the equation of the reflected ray is $y - 3 = \frac{12}{5}(x - 9/4)$, or $y = \frac{12}{5}(x - 1)$, which hits the x-axis at $x = 1$.

(11) (Continuation of previous question.) Where does this ray hit the mirror again? Show that the tangent at this second point, of slope $\tan\phi'$ say, is such that $\tan(2\phi') = \tan(2\phi)$. Deduce without further calculation that, after being reflected again, the ray travels out to the right horizontally along the line $y = -4/3$.

Solution.

From $y^2 = 4x = (\frac{12}{5}(x - 1))^2$ we get $(4x - 9)(9x - 4) = 0$, so $x = 4/9$ gives the second point of intersection, at $(4/9, \frac{12}{5}(4/9 - 1)) = (4/9, -4/3)$. From $2y\frac{dy}{dx} = 4$ the slope of the tangent there is $-3/2 = \tan\phi'$. Then $\tan(2\phi') = \frac{2\tan\phi'}{1 - \tan^2\phi'} = 12/5$. Now a ray coming *in* along $y = -4/3$ will, by the previous question, be reflected along the line of slope $\tan(2\phi') = \tan(2\phi)$, i.e. along the line $y = \frac{12}{5}(x - 1)$.

Hence, reversing the ray, we see that our original ray will be reflected out along the line $y = -4/3$.

(12) (Generalising the previous two questions.) Repeat these questions, but now for the general parabola $y^2 = 4ax$, and a horizontal ray $y = 2c$ coming in from the right. Show that the ray always passes through the focus, whatever the values of a and c. Show that after being reflected a second time it comes out along the line $y = -2a^2/c$.

Solution.

Ray hits mirror at $(c^2/a, 2c)$,where tangent has slope $a/c = \tan\phi$, is reflected along the line $y - 2c = \dfrac{2ac}{c^2 - a^2}(x - c^2/a)$, of slope $\dfrac{2ac}{c^2 - a^2} = \tan(2\phi)$, which passes through the focus $(a, 0)$ and hits the mirror again at $(a^3/c^2, -2a^2/c)$. At this point the slope of the tangent is $-c/a = \tan\phi'$ say. Since $\tan(2\phi') = \dfrac{2\tan\phi'}{1 - \tan^2\phi'} = (2(-c/a)/(1 - (-c/a)^2) = \dfrac{2ac}{c^2 - a^2} = \tan(2\phi)$, so that the argument in the previous solution applies, and the ray comes out along the line $y = -2a^2/c$.

(13) Use a suitable rotation transformation $\begin{bmatrix} x \\ y \end{bmatrix} = R_\theta \begin{bmatrix} x' \\ y' \end{bmatrix}$ to put the conic $2x^2 + 4xy + 5y^2 = 1$ into standard form. Give R_θ explicitly.

Solution.

For the associated matrix $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$, one readily finds that its e'values are 1 and 6, with corresponding normalised e'vectors $\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. Hence $R_\theta = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$ and the standard form is ${x'}^2 + 6{y'}^2 = 1$. This is of the standard form ${x'}^2/a^2 + {y'}^2/b^2 = 1$ with $a = 1, b = 1/\sqrt{6}$.

(14) Use the results of the previous question, and a suitable translation, to put the conic $2x^2 + 4xy + 5y^2 + 6x + 4y = 1$ into standard form. Give explicitly the mapping between the old and new coordinate variables x, y and x'', y''.

Solution.

We substitute $x = (2x' + y')/\sqrt{5}, y' = (-x' + 2y')/\sqrt{5}$ into the equation of the conic to obtain

$$ {x'}^2 + 6{y'}^2 + 6(2x' + y')/\sqrt{5} + 4(-x' + 2y')/\sqrt{5} = 1, $$

simplifying to

$$ {x'}^2 + 6{y'}^2 + 8x'/\sqrt{5} + 14y'/\sqrt{5} = 1, $$

which, on completing the squares, can be written

$$ (x' + 4/\sqrt{5})^2 + 6(y' + \frac{7}{6}/\sqrt{5})^2 = 1 + \frac{16}{5} + 6\frac{49}{6^2 5}, $$

or

$$ {x''}^2 + 6{y''}^2 = \frac{35}{6}, $$

where $x'' = x' + 4/\sqrt{5}, y'' = y' + \frac{7}{6}/\sqrt{5}$. On dividing by $\frac{35}{6}$ this can be put in the standard form ${x''}^2/a^2 + {y''}^2/b^2 = 1$ with $a = \sqrt{\frac{35}{6}}, b = \frac{1}{6}\sqrt{35}$.
Finally

$$ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} x'' - 4/\sqrt{5} \\ y'' - \frac{7}{6}/\sqrt{5} \end{bmatrix} $$

gives the transformation between x, y and x'', y'': $x = (2/\sqrt{5})x'' + (1/\sqrt{5})y'' - \dfrac{11}{6}, y = -(1/\sqrt{5})x'' + (2/\sqrt{5})y'' + \dfrac{1}{3}$.

(15) Use a suitable rotation transformation $\begin{bmatrix} x \\ y \end{bmatrix} = R_\theta \begin{bmatrix} x' \\ y' \end{bmatrix}$ to put the conic $4x^2 - 4xy + 7y^2 = 1$ into standard form. Give R_θ explicitly.

Solution.

For the associated matrix $\begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$, its characteristic polynomial is $\lambda^2 - 11\lambda + 24 = (\lambda - 3)(\lambda - 8)$,so that its e'values are 3 and 8, with corresponding normalised e'vectors $\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. Hence $R_\theta = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$ and the standard form is $3{x'}^2 + 8{y'}^2 = 1$. This is of the standard form ${x'}^2/a^2 + {y'}^2/b^2 = 1$ with $a = 1/\sqrt{3}, b = 1/\sqrt{8}$.

Notes:

* The smaller eigenvalue should be taken first, i.e. $\lambda_1 < \lambda_2$ as then the standard form of the ellipse is ${x'}^2/a^2 + {y'}^2/b^2 = 1$ with $a = 1/\sqrt{\lambda_1}, b = 1/\sqrt{\lambda_2}$, and $b < a$.
* Since a rotation matrix R_θ has its two diagonal elements equal (to $\cos\theta$), the signs of the eigenvectors should be chosen so that the matrix of eigenvectors has this property too. Then the eigenvectors should be scaled to have length 1, making the matrix of eigenvectors into an orthogonal matrix, and so a rotation matrix.

(16) Use the results of the previous question, and a suitable translation, to put the conic $4x^2 - 4xy + 7y^2 + 4x + 2y = 1$ into standard form. Give explicitly the mapping between the old and new coordinate variables x, y and x'', y''.

Total Marks for Paper: 10.