The University of Edinburgh
2010

School of Mathematics
(U01457)

Geometry & Convergence

Problem Sheet 2: Solutions

Assessment 2 due by 12.10 on Friday, 29 January 2010.
Tutorial 2 on Tuesday, 26 January 2010.

Pretutorial questions: 2, and 4.

Tutorial questions: 7, 8, and 16.

Handin question: 3.

(1) (a) For the line $7x - 3y = 1$, write down a normal to the line, and a point on the line. Find the distance of the point $(3, 4)$ from the line.

(b) More generally, for the line $ax + by = c$, write down a normal, and show that the point $(ac/(a^2 + b^2), bc/(a^2 + b^2))$ lies on the line. Deduce that the distance of the point (x_0, y_0) from the line is $|ax_0 + by_0 - c|/\sqrt{a^2 + b^2})$.

Solution.

(a) Normal is $\mathbf{n} = (7, -3)$ and $(1, 2)$ is on line. Distance of the point $(3, 4)$ from the line is the component of $(1, 2)-(3, 4) = (-2, -2)$ in the direction of $\mathbf{n}$, which is $|\mathbf{n} \cdot (-2, -2)|/|\mathbf{n}| = 8/\sqrt{58}$.

(b) Normal is $\mathbf{n} = (a, b)$, and the point $(ac/(a^2 + b^2), bc/(a^2 + b^2))$ lies on the line as $a \times ac/(a^2 + b^2)a + b \times bc/(a^2 + b^2) = c(a^2 + b^2)/(a^2 + b^2) = c$. Distance of the point (x_0, y_0) from the line is the component of $(ac/(a^2 + b^2), bc/(a^2 + b^2)) - (x_0, y_0)$ in the direction of $\mathbf{n} = (a, b)$, which is $|(a, b) \cdot (a \times ac/(a^2 + b^2)a + b \times bc/(a^2 + b^2) - (ax_0 + by_0))|/|\mathbf{n}| = |c - ax_0 - by_0|/\sqrt{a^2 + b^2}$.

(2**) (a) Which points in the plane $\mathbb{R}^2$ are equidistant from the x- and y-axes?

(b) Find the point that is equidistant from the x-axis, the y-axis and the line $3x + 4y = 36$.

(c) What is the largest radius of a circle that will fit inside the triangle specified by the three lines in (b)? (This circle is called the *incircle* of the triangle, and its centre is the *incentre* of the triangle).

Solution.

(a) Since the distance from the x-axis (resp. y-axis) to the point (x, y) is $|x|$ (resp. $|y|$), it is the set $|x| = |y|$.

(b) The line $3x + 4y = 36$ has normal $\langle 3, 4\rangle$, and the point $(12, 0)$ lies on it. Hence the distance of $P = (x, y)$ from the line is the length of the component of $\langle 12, 0\rangle - \langle x, y\rangle$ in the direction $\langle 3, 4\rangle$, i.e. $|\langle 12 - x, -y\rangle \cdot \langle 3, 4\rangle| / |\langle 3, 4\rangle| = |36 - 3x - 4y| /5$. Therefore, we get the equations for the point P:

$$x^2 = y^2, \qquad 25x^2 = (36 - 3x - 4y)^2$$

so we get the equations

$$(y = x): \qquad 49x^2 - 504x + 1296 = 25x^2$$

or

$$(y = -x): \qquad x^2 + 72x + 1296 = 25x^2$$

with solutions

$$x = y = 3, x = y = 18 \quad \text{or } x = -y = -6, x = -y = 9. \quad (1)$$

(c) Clearly 3.

(3†) (a) Given a line in $\mathbb{R}^3$ in parametric form $\mathbf{a} + t\mathbf{v}$, and a point $\mathbf{b}$ in $\mathbb{R}^3$, sketch the plane containing $\mathbf{b}$ and the line.

(b) Deduce that the distance of $\mathbf{b}$ from the line is $|\mathbf{a}-\mathbf{b}-\lambda\mathbf{v}|$, where $\lambda\mathbf{v}$ is the component of $\mathbf{a}-\mathbf{b}$ in the direction $\mathbf{v}$.

(c) How far is the point $(1,2,3)$ from the line $(1,0,-1)+t(-1,3,2)$?

(d) This method works for a point and a line in $\mathbb{R}^n$ for any n. In $\mathbb{R}^4$, how far is the point $(1,-1,1,0)$ from the line $(0,1,1,0)+t(1,1,0,-1)$?

Solution.

(b) From sketch (not included!), we see that there is a right-angled triangle with hypotenuse $\mathbf{a}-\mathbf{b}$, and one of the other sides $\lambda\mathbf{v}$. The length of the third side $|\mathbf{a}-\mathbf{b}-\lambda\mathbf{v}|$ is therefore the distance of $\mathbf{b}$ from the line, as this side is orthogonal to $\mathbf{v}$ with $\mathbf{b}$ one of its endpoints **[marks: 3]**.

(c) Now from lectures, the component of $\mathbf{a}-\mathbf{b}$ in the direction $\mathbf{v}$ is **[mark: 1]**

$$\lambda\mathbf{v}=\frac{(\mathbf{a}-\mathbf{b})\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v},$$

which is $\dfrac{-14}{14}\mathbf{v}=-\mathbf{v}$ **[mark: 1]**. Hence the distance of $\mathbf{b}$ from the line is $|\mathbf{a}-\mathbf{b}-(-\mathbf{v})|=|(-1,1,-2)|=\sqrt{6}$ **[mark: 1]**.

(d) Using the same formula we get $\lambda=1/3$ **[mark: 1]**, $|\mathbf{a}-\mathbf{b}-\mathbf{v}/3|=(-4/3,5/3,0,1/3)|=\sqrt{42}/3$ **[mark: 1]**.

Presentation: **[mark: 1]**. **Total Marks for Question: 9.**

(4**) If $\mathbf{a}\neq 0$ and $\mathbf{a}\times\mathbf{b}=0$, prove that $\mathbf{b}$ is a scalar multiple of $\mathbf{a}$.

Solution.

We have $0=|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Hence $\sin\theta=0$, so that $\mathbf{b}$ is a scalar multiple of $\mathbf{a}$ (or $\mathbf{b}=0$, whence $\mathbf{b}=0\mathbf{a}$).

(5) Verify by expanding both sides that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

(6) Show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{0}$.

Solution.

We use the previous question. So

$$\begin{aligned}&\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})\\&\qquad=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\\&\qquad=\mathbf{0}\end{aligned}$$

Cancelling the first and last terms, the 2nd and 3rd terms and the 4th and 5th terms using the symmetry of the scalar product.

(7*) Solving the equation $\mathbf{a}\times\mathbf{x}=\mathbf{b}$, for given vectors $\mathbf{a},\mathbf{b}$ in $\mathbb{R}^3$.

(a) Show that if $\mathbf{a}\times\mathbf{x}=\mathbf{b}$ then $\mathbf{a}\cdot\mathbf{b}=0$. We assume that this condition holds for the rest of the question.

(b) Use Q4 above to show that $-(\mathbf{a}\times\mathbf{b})/|\mathbf{a}|^2$ is a solution of $\mathbf{a}\times\mathbf{x}=\mathbf{b}$.

(c) If $\mathbf{x}=\mathbf{u}-(\mathbf{a}\times\mathbf{b})/|\mathbf{a}|^2$ is another solution, show that $\mathbf{u}\times\mathbf{a}=0$. Hence use Q3 above to write down the general solution $\mathbf{x}$ to $\mathbf{a}\times\mathbf{x}=\mathbf{b}$.

[In this question we've seen that the solution set $\{\mathbf{x}:\mathbf{a}\times\mathbf{x}=\mathbf{b}\}$ is either empty (when $\mathbf{a}\cdot\mathbf{b}\neq 0$) or a line. Note the contrast with the equation $\mathbf{a}\cdot\mathbf{x}=b$, whose solution set is a plane.]

Solution.

(a) We have on dotting the equation with $\mathbf{a}$, $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{x})=0=\mathbf{a}\cdot\mathbf{b}$, as $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{x})$ is a determinant with two rows equal to $\mathbf{a}$.

(b) From Q4 we have $\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}$, which is $-|\mathbf{a}|^2\mathbf{b}$ using part (a), giving the result.

(c) Easily get $\mathbf{u} \times \mathbf{a} = 0$ by substitution. Hence by Q3 $\mathbf{u}$ is a multiple of $\mathbf{a}$, giving the result:

$$\mathbf{x} = -(\mathbf{a} \times \mathbf{b})/|\mathbf{a}|^2 + t\mathbf{a},$$

where t is any scalar.

(8*) (Converse to question 6). Show that a given line $\mathbf{w} + t\mathbf{a}$ in $\mathbb{R}^3$ is the solution set of the equation $\mathbf{a} \times \mathbf{x} = \mathbf{a} \times \mathbf{w}$.

Solution.

Follows by substitution, using $\mathbf{a} \times \mathbf{a} = 0$, that the given line is at least part of the solution set. But from Q6 we know that such equations (here $\mathbf{b} = \mathbf{a} \times \mathbf{w}$) have a line as the solution set, so this line must be the whole solution set.

(9) Consider the points $P = (7, 0, -1)$, $Q = (2, 5, 4)$, $R = (2, -4, -2)$. Find

(a) the angle between $\overrightarrow{QP}$ and $\overrightarrow{QR}$;

(b) the parametric form for the line joining the points P and Q;

(c) the equation of the plane containing P, Q and R.

(10) If $\mathbf{a}$ and $\mathbf{b}$ lie in the xy-plane show that $|[\mathbf{a}, \mathbf{b}, \mathbf{k}]|$ equals the area of the parallelogram generated by $\mathbf{a}$ and $\mathbf{b}$.

Solution.

Now $|[\mathbf{a}, \mathbf{b}, \mathbf{k}]| = |\mathbf{k} \cdot (\mathbf{a} \times \mathbf{b})| = |\mathbf{a} \times \mathbf{b}|$, as $\mathbf{a} \times \mathbf{b}$ is a scalar multiple of $\mathbf{k}$. Then the fact that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|.|\mathbf{b}| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, gives the result.

(11) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of three points P, Q and R. Show that P, Q and R lie on a line (then we say that they are *collinear*) if and only if $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{0}$.

Solution.

Use previous question, for parallelogram with sides $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$. The three vectors are collinear iff the area of this parallelogram is 0.

(12) Suppose $\mathbf{n} \cdot \mathbf{x} = d$ and $\mathbf{n}' \cdot \mathbf{x} = d'$ are two non-parallel planes.

(a) Show that $[\mathbf{n}, \mathbf{n}', \mathbf{n} \times \mathbf{n}'] \neq 0$. *answer: As planes not parallel,* $\mathbf{n}$ *and* $\mathbf{n}'$ *are not scalar multiples of each other, so, by Q3,* $\mathbf{n} \times \mathbf{n}' \neq 0$ *and so* $[\mathbf{n}, \mathbf{n}', \mathbf{n} \times \mathbf{n}'] = [\mathbf{n} \times \mathbf{n}', \mathbf{n}, \mathbf{n}'] = (\mathbf{n} \times \mathbf{n}') \cdot (\mathbf{n} \times \mathbf{n}') = |\mathbf{n} \times \mathbf{n}'|^2 \neq 0$.

(b) Let A be the matrix with rows given by $\mathbf{n}$, $\mathbf{n}'$ and $\mathbf{n} \times \mathbf{n}'$ respectively. Explain why A is invertible. *answer:* $\det A = [\mathbf{n}, \mathbf{n}', \mathbf{n} \times \mathbf{n}'] \neq 0$, *As* $\mathbf{n}$ *and* $\mathbf{n}'$ *span a plane,so* $\mathbf{n}, \mathbf{n}'$ *and* $\mathbf{n} \times \mathbf{n}'$ *span* 3*-space, so their determinant* $\det A$ *is nonzero. Hence* A *is invertible.*

(c) Consider the vector $\mathbf{b} = (d, d', 0)$ and let $\mathbf{a} = A^{-1}\mathbf{b}$. Show that $\mathbf{a} + t\mathbf{n} \times \mathbf{n}'$ gives the intersection line of the two planes. *answer: Consider the solution* $\mathbf{a}$ *of* $A\mathbf{a} = \mathbf{b}$. *Note that it lies on both planes, so on the line of intersection. The direction of this line is orthogonal to both* $\mathbf{n}$ *and* $\mathbf{n}'$, *so is* $\mathbf{n} \times \mathbf{n}'$. *Hence the line is* $\mathbf{a} + t\mathbf{n} \times \mathbf{n}'$.

(d) Can we choose other vectors for $\mathbf{b}$? Explain. *answer: We can choose any* $\mathbf{b} = (d, d', s)$, *for any real number* s.

(13) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be two linear maps. Show that the function h defined by $h(\mathbf{a}) = f(\mathbf{a}) + g(\mathbf{a})$ is also linear.

(14) For the following maps (a) identify a suitable domain and range, and (b) state with reasons if the map is linear or not ($\mathbf{b}$ and $\mathbf{c}$ are fixed non-zero vectors and λ is a scalar):

(a) $f(\mathbf{x}) = \mathbf{0}$

(b) $f(\mathbf{x}) = \mathbf{x} + \mathbf{b}$

(c) $f(\mathbf{x}) = \lambda\mathbf{x}$

(d) $f(\mathbf{x}) = (\mathbf{b} \cdot \mathbf{x})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{x}$

(e) $f(\mathbf{x}) = (\mathbf{b} \cdot \mathbf{x})\mathbf{c} + (\mathbf{b} \cdot \mathbf{x})\mathbf{x}$

(f) $f(\mathbf{x}) = [\mathbf{x}, \mathbf{b}, \mathbf{c}]$

(g) $f(\mathbf{x}) = |\mathbf{x}|$

answer: (a) Any m and n (b), (c), (d) and (e) $m = n$, (f) $m = 1$ and $n = 3$, (g) $m = 1$ answer: (a), (c), (d), (f) are linear.

Solution.

(a) Any domain and range. This is linear since $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\lambda\mathbf{0} = \mathbf{0}$.

(b) Same domain and range. This is not linear, for example $f(2\mathbf{x}) = 2\mathbf{x} + \mathbf{b} \neq 2(\mathbf{x} + \mathbf{b})$.

(c) Same domain and range. This is linear since $\lambda(\mathbf{x}+\mathbf{b}) = \lambda\mathbf{x}+\lambda\mathbf{b}$ and $\lambda(\mu\mathbf{x}) = \mu(\lambda\mathbf{x})$

(d) Same domain and range. This is linear since each term is linear (see Q12).

(e) Same domain and range. This is not linear because the second term is not linear: $(\mathbf{b} \cdot (\lambda\mathbf{x}))(\lambda\mathbf{x}) = \lambda^2(\mathbf{b} \cdot \mathbf{x})\mathbf{x}$.

(f) $\mathbb{R}^3 \to \mathbb{R}$. This is linear because the scalar product is linear.

(g) $\mathbb{R}^n \to \mathbb{R}$. This is not linear because $f(-\mathbf{x}) = f(\mathbf{x})$.

NB:For the non-linear cases a specific counterexample is really needed, guided by the above.

(15) For those functions in the previous question which are linear maps find their corresponding matrices and decide whether they are invertible or not.

Solution.

We know that the matrix of a linear map f is the matrix with columns $f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)$.

(a) Zero matrix. This is clearly not invertible.

(c) λ times the identity matrix. This is invertible so long as $\lambda \neq 0$.

(d) $f(\mathbf{x}_i) = b_i\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{x}_i$ are the columns of the matrix. We can describe this by saying it is the matrix whose columns are $\mathbf{c}$ times b_i minus $\mathbf{b} \cdot \mathbf{c}$ times the identity matrix. This is not invertible because all vectors of the form $\mathbf{a} = \lambda\mathbf{c}$ are mapped to zero under f.

(f) The matrix is the row vector $\mathbf{b} \times \mathbf{c}$ and cannot invert because it is not square.

(16*) For the plane given parametrically by $\imath(t, u) = (3, 0, 0) + t(-3, 4, 0) + u(-3, 0, 6)$ write down three points on the plane that don't lie on one line. Use these to find a normal to the plane. Find the distance of the point $(1, 2, 3)$ from the plane.

Solution.

For instance $\mathbf{a} = (3, 0, 0)$ (from $t = u = 0$),$\mathbf{b} = (0, 4, 0)$ (from t=1,u=0), $\mathbf{c} = (0, 0, 6)$ (from $t = 0, u = 1$) all lie on plane. Then a normal is $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (-3, 4, 0) \times (-3, 0, 6) = 24\mathbf{i} + 18\mathbf{j} + 12\mathbf{k} = 6(4, 3, 2)$, so we can take a normal to be $\mathbf{n} = (4, 3, 2)$. Then the distance of $(1, 2, 3)$ from the plane is $|((3, 0, 0) - (1, 2, 3)) \cdot (4, 3, 2)|/|\mathbf{n}| = 4/\sqrt{29}$.

(17) Show directly that for the projection matrix $P_\theta = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

(a) $P_\theta^2 = P_\theta$ and

(b) $(P_\theta P_\phi)^2 = \lambda P_\theta P_\phi$, where $|\lambda| \leq 1$ is some real number depending on θ and ϕ.

Explain these facts geometrically.

Solution.

Parts (a) and (b) are just a matter of computation.

Since P_θ projects a vector $\mathbf{x}$ onto the vector $\mathbf{n}_\theta = (\cos\theta, \sin\theta)$, one can see geometrically, on drawing the vectors $\mathbf{n}_\phi$ and $\mathbf{n}_\theta$, that $P_\theta\mathbf{x}$ is in the direction $\mathbf{n}_\theta$, and then projecting it on to $\mathbf{n}_\phi$ a second time and then onto $\mathbf{n}_\theta$ a second time shortens it further.

(18) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map which is not the identity and satisfies $f \circ f = f$. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the 2×2 matrix corresponding to f. Show that

(a) $\det P = 0$,

(b) the eigenvalues of P are either 0 or 1,

(c) $0 \le a \le 1$ and $0 \le d \le 1$.

(d) Either

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } P = \begin{bmatrix} \cos^2\theta & \alpha \\ \dfrac{\cos^2\theta\sin^2\theta}{\alpha} & \sin^2\theta \end{bmatrix}$$

for some angle $0 \le \theta \le \pi/2$ and non-zero real number α.

(e) Find the eigenvectors of P and show that the eigenvectors are orthogonal to each other if and only if P is the projection to a line through the origin.

Solution.

If $\mathbf{a}$ is an eigenvector with eigenvalue λ then $f(\mathbf{a}) = \lambda\mathbf{a}$. But $f(f(\mathbf{a})) = f(\lambda\mathbf{a}) = \lambda f(\mathbf{a})$ and $f(f(\mathbf{a})) = f(\mathbf{a}) = \lambda\mathbf{a}$. So $\lambda^2 = \lambda$. hence $\lambda = 0$ or 1 and $f(\mathbf{a}) = \mathbf{0}$ or $\mathbf{a}$. In the former case we have eigenvector $(-\alpha, \cos^2\theta)$ and the latter case $(-\alpha, \cos^2\theta - 1)$. These are orthogonal if $\alpha^2 + \cos^2\theta(1-\cos^2\theta) = 0$. So $\alpha = \pm\cos\theta\sin\theta$. Then P is projection to the line making an angle $\theta + \pi/2$ with the x-axis.

Total Marks for Paper: 9.