The University of Edinburgh — School of Mathematics
2010 — (U01457)

Geometry & Convergence

Problem Sheet 1: Solutions

Assessment 1 due by 12.10 on Friday, 15 January 2010.

Tutorial questions: 4, 6, and 14.

Handin questions: 1, and 3.

($1^{\dagger}$) Sine and cosine rules. Consider the standard triangle with angles A, B, C opposite sides a, b, c. Drop a perpendicular from the angle A onto the side a. (Draw this.)

(a) Show that $b \sin C = c \sin B$. Deduce the sine rule $a/\sin A = b/\sin B = c/\sin C$.

(b) Show that the right-angled triangle with hypotenuse c has other sides $b \sin C$ and $a - b \cos C$. Deduce the cosine rule $c^2 = a^2 + b^2 - 2ab \cos C$.

Solution.

(a) Length of perpindicular is $b \sin C$ and also $c \sin B$ **[mark: 1]**. Hence $b/\sin B = c/\sin C$ and so, applying this result to sides a, b and angles A, B also $a/\sin A = b/\sin B$ **[mark: 1]**.

(b) Apply Pythagoras' Theorem to this right-angled triangle, and use $\sin^2 C + \cos^2 C = 1$ **[marks: 2]**.

Correctly labeled diagram and presentation: **[mark: 1]**.

Total Marks for Question: 5.

(2) *"Angle at centre is twice that at circumference."*

(a) Draw a circle with centre O, and a triangle ABC whose vertices A, B, C lie on the circle. Join A and O, and let the angle $\angle OAB = \beta$, $\angle OAC = \gamma$. Find $\angle AOB$ in terms of β and $\angle AOC$ in terms of γ.

Deduce that $\angle BOC = 2(\beta + \gamma) = 2\angle BAC$.

What happens when AC is a diagonal of the circle?

(b) Let BC be a fixed line segment. A point A moves on the plane so that the angle $\angle BAC$ stays constant. It starts at B and ends at C. Show that A traces out an arc of a circle.

Solution.

(a) The triangle AOB is isoceles, so $\angle OBA$ also $= \beta$ and $\angle OCA$ also $= \gamma$. As sum of angles of a triangle is π, we have $\angle BOA = \pi - 2\beta$ and $\angle COA = \pi - 2\gamma$. Hence $\angle BOC = 2\pi - \angle BOA - \angle COA = 2(\beta + \gamma) = 2\angle BAC$. When AC is a diagonal of the circle, $\angle AOC = \pi$, so $\angle BAC = \pi/2$. Thus triangle ABC is a right-angled triangle.

(b) Take A at any point on its locus (the path of its travels), not at B or C, and draw the circle through A, B and C, with centre O. Then for any point A' on the circle (on the same side of BC as A), the angle $\angle BA'C$ must be half of the fixed angle $\angle BOC$, by (a). Thus it's equal to $\angle BAC$, and the arc of the circle that contains A is the required locus.

($3^{\dagger}$) Let $\mathbf{a} = \langle -1, -1 \rangle, \mathbf{b} = \langle -2, 3 \rangle$. Calculate, in radians to 2 decimal places, the angles between

- $\mathbf{a}$ and $\mathbf{b}$;
- $\mathbf{a}$ and $\mathbf{a} - \mathbf{b}$;
- $\mathbf{b}$ and $\mathbf{b} - \mathbf{a}$.

What is the sum of these angles? Why?

Solution.

The angle θ between $\mathbf{a}$ and $\mathbf{b}$ given by $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, we have that the three angles $\theta_1, \theta_2, \theta_3$ are given by $-1 = \sqrt{2}.\sqrt{13}\cos\theta_1, -3 = \sqrt{2}.\sqrt{17}\cos\theta_2, -14 = \sqrt{13}.\sqrt{17}\cos\theta_3$ or $\theta_1 = \cos^{-1}(-1/\sqrt{26}) \simeq 1.77, \theta_2 = \cos^{-1}(3/\sqrt{34}) \simeq 1.03, \theta_3 = \cos^{-1}(14/\sqrt{221}) \simeq 0.34$ **[marks: 3]**, totalling π **[mark: 1]**, because we're summing the three angles of the triangle with vertices $0, (-1,-1), (-2,3)$ **[mark: 1]**. **Total Marks for Question: 5.**

(4*) Find the component of $\mathbf{c} = \langle 2,1\rangle$ in the direction $\mathbf{v} = \langle 4,1\rangle$. Hence write $\mathbf{c}$ in the form $\mathbf{c} = \lambda\mathbf{v} + \mathbf{w}$, where $\mathbf{v}\cdot\mathbf{w} = 0$.

Check also that $\mathbf{w}$ itself is the component of $\mathbf{c}$ in the direction $\mathbf{w}$.

Draw a picture to illustrate the question.

Solution.

The component of $\mathbf{c} = \langle 2,1\rangle$ in the direction $\mathbf{v} = \langle 4,1\rangle$ is $(\mathbf{c}\cdot\mathbf{v})/|\mathbf{v}|^2)\mathbf{v} = (9/17)\mathbf{v}$. So $\mathbf{c} = \lambda\mathbf{v} + \mathbf{w}$, where $\lambda = 9/17$ and $\mathbf{w} = \mathbf{c} - \lambda\mathbf{v} = \langle -2/17, 8/17\rangle$.

Finally $\mathbf{w}\cdot\mathbf{c}/|\mathbf{w}|^2 = (4/17)/(68/17^2) = 1$, so the component of $\mathbf{c}$ in the direction $\mathbf{w}$ is $1.\mathbf{w} = \mathbf{w}$.

[Picture not included]

(5) Let $\mathbf{a} = \langle 1,2\rangle, \mathbf{b} = \langle 2,-3\rangle$. Find the equation of the line given parametrically by
$(1-t)\mathbf{a} + t\mathbf{b}$.

Which values of t describe the set of points on the line that are nearer to $\mathbf{a}$ than to $\mathbf{b}$?

Does the point $(6,7)$ lie on the line?

Give examples of points on the line that lie

- between $\mathbf{a}$ and $\mathbf{b}$;
- on the side of $\mathbf{a}$ away from $\mathbf{b}$;
- on the side of $\mathbf{b}$ away from $\mathbf{a}$.

Find the two points on the line that are each twice as far from $\mathbf{a}$ as from $\mathbf{b}$.

Solution.

The equation of the line through $(x_1,y_1),(x_2,y_2)$ is $\left|\begin{bmatrix} x-x_1 & y-y_1 \\ x_2-x_1 & y_2-y_1\end{bmatrix}\right| = 0$, giving $\left|\begin{bmatrix} x-1 & y-2 \\ 1 & -5\end{bmatrix}\right| = 0$, or $y = -5x+7$.

As $7 \neq -5.6+7$, the point $(6,7)$ is not on the line.

For a point between between $\mathbf{a}$ and $\mathbf{b}$, take $0 < t < 1$, e.g. $t = 1/2$, giving the point $(1/2)\mathbf{a} + (1/2)\mathbf{b} = \langle 3/2, -1/2\rangle$.

For a point on the side of $\mathbf{a}$ away from $\mathbf{b}$, take $t < 0$, e.g. $t = -1$, giving the point $2\mathbf{a} - 1\mathbf{b} = \langle 0,7\rangle$.

For a point on the side of $\mathbf{b}$ away from $\mathbf{a}$, take $t > 1$, e.g. $t = 2$, giving the point $-\mathbf{a} + 2\mathbf{b} = \langle 3,-8\rangle$.

The two points on the line that are each twice as far from $\mathbf{a}$ as from $\mathbf{b}$ are given by $t = 2/3$ and $t = 2$, giving the points $(1/3)\mathbf{a} + (2/3)\mathbf{b} = \langle 5/3, -4/3\rangle$ and $\langle 3,-8\rangle$ found earlier.

(6*) Let $\mathbf{u} = \langle u_1,u_2\rangle, \mathbf{v} = \langle v_1,v_2\rangle$ be two independent vectors in $\mathbb{R}^2$ (i.e. neither is a multiple of the other), and put $\mathbf{u}^\perp = \langle -u_2,u_1\rangle, \mathbf{v}^\perp = \langle -v_2,v_1\rangle$.

Show that $\mathbf{u}^\perp\cdot\mathbf{u} = \mathbf{v}^\perp\cdot\mathbf{v} = 0$.

Writing $\mathbf{x}\in\mathbb{R}^2$ as a linear combination $\lambda\mathbf{u} + \mu\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$, show by taking appropriate dot products that

- $\lambda = (\mathbf{v}^\perp\cdot\mathbf{x})/(\mathbf{v}^\perp\cdot\mathbf{u})$;
- $\mu = (\mathbf{u}^\perp\cdot\mathbf{x})/(\mathbf{u}^\perp\cdot\mathbf{v})$.

Now try this procedure with the example $\mathbf{u} = \langle 2,3\rangle, \mathbf{v} = \langle 2,-1\rangle$ and $\mathbf{x} = \langle 1,1\rangle$. Check that indeed $\mathbf{x} = (\mathbf{v}^\perp\cdot\mathbf{x})/(\mathbf{v}^\perp\cdot\mathbf{u})\mathbf{u} + (\mathbf{u}^\perp\cdot\mathbf{x})/(\mathbf{u}^\perp\cdot\mathbf{v})\mathbf{v}$ for this example.

Solution.

Now $\mathbf{u}^\perp \cdot \mathbf{u} = u_1(-u_2) + u_2 u_1 = 0$, and similarly for $\mathbf{v}^\perp \cdot \mathbf{v}$.
Taking the dot product of $\mathbf{x} = \lambda\mathbf{u} + \mu\mathbf{v}$ with $\mathbf{v}^\perp$ and then with $\mathbf{u}^\perp$ we get

$$\mathbf{v}^\perp \cdot \mathbf{x} = \lambda \mathbf{v}^\perp \mathbf{u},$$

using $\mathbf{v}^\perp \cdot \mathbf{v} = 0$, and then

$$\mathbf{u}^\perp \cdot \mathbf{x} = \mu \mathbf{u}^\perp \mathbf{v},$$

using $\mathbf{u}^\perp \cdot \mathbf{u} = 0$, giving the formulas of λ and μ.
In the example we get, using the formulas of λ and μ, $\mathbf{x} = (3/8)\mathbf{u} + (1/8)\mathbf{v} = \langle 1, 1\rangle$, which checks.

($6\frac{1}{2}$) In Q6, explain why $\mathbf{v}^\perp \cdot \mathbf{u}$ (and, similarly, $\mathbf{u}^\perp \cdot \mathbf{v}$) is nonzero.

Solution.

Now if $\mathbf{v}^\perp \cdot \mathbf{u} = 0$ then $v_2 u_1 = v_1 u_2$, so that, if $u_1 \neq 0$ then $\langle v_1, v_2\rangle = (v_1/u_1)\langle u_1, u_2\rangle$, contradicting the fact that neither of $\mathbf{u}, \mathbf{v}$ is a multiple of the other. Similar identities hold if one of the other components u_2, v_1, v_2 is nonzero; and if they are all zero, then $\mathbf{u} = \mathbf{v}$, giving the same contradiction.

(7) Consider three simultaneous equations

$$ax + by + cz = d,$$
$$kx + ly + mz = n,$$
$$px + qy + rz = s$$

for x, y and z. Each equation describes a plane. Assume that each equation describes a different plane.
Give a sketch of a possible arrangement of the three planes in each of the following cases:

(a) There is a unique solution of all three equations together;

Solution.

This is the general case where two planes intersect in a line and the third cuts this line in a point.

(b) Each pair of equations has a solution, but there is no solution of all three together;

Solution.

The planes form the sides of a prism. Each pair of planes intersects in a line but the lines are parallel. This follows because each pair of intersection lines lies in one of the planes and so they must be parallel in order not to intersect.

(c) No pair of equations has a solution;

Solution.

In this case the planes are all parallel (ie. they have the same normal vectors).

(d) There are infinitely many solutions of all equations together;

Solution.

The three planes meet in a common line.

(e) The fifth case!

Solution.

Two planes are parallel, but the third plane isn't parallel to those two. In this case there is common point to all three planes. Two of the pairs of equations have a solution, but the parallel pair clearly doesn't.

(8) Consider the plane Π which contains the three points $(1, 0, -2)$, $(2, -1, 3)$ and $(0, 1, 1)$. Find the parametric form for Π and the equation for Π. *answer:* $(1 + s - t, -s + t, -2 + 5s + 3t)$ *and* $x + y = 1$.

Solution.

We use the first point as a base point P on the plane and then we have two vectors in the plane by subtracting the position vector from the other two positions: $\mathbf{a} = \langle 2, -1, 3\rangle - \langle 1, 0, -2\rangle = \langle 1, -1, 5\rangle$ and $\mathbf{b} = \langle 0, 1, 1\rangle - \langle 1, 0, -2\rangle = \langle -1, 1, 3\rangle$. Then the parametric form is given by $\langle 1, 0, -2\rangle + s\langle 1, -1, 5\rangle + t\langle -1, 1, 3\rangle = \langle 1 + s - t, -s + t, -2 + 5s + 3t\rangle$.

The equation of the line is given by $\left|\begin{bmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{bmatrix}\right| = 0$, giving $x+y=1$.
Alternative way of finding the equation: The normal is given by $\mathbf{a}\times\mathbf{b} = \langle -3+5, -5-3, 1-1\rangle = \langle -8,-8,0\rangle = -8\langle 1,1,0\rangle$ and the equation is $\langle 1,1,0\rangle\cdot\mathbf{x} = \langle 1,1,0\rangle\cdot\langle 1,0,-2\rangle = 1$ ie. $x+y=1$. Note that if we had not divided the normal vector by -8 then we would get the equation $-8x-8y=-8$ which we could then divide by -8.

(9) Consider the points $P=(1,2,3)$, $Q=(-1,1,3)$ the lines $\ell_1 = (2+t, 1+2t, -t)$, $\ell_2 = (1-3t, 2+2t, 1+t)$ and the planes Π_1 given by $2x-y-z=2$, Π_2 given by $z-x=3$. Find

(a) the distance between P and Q, *answer:* $\sqrt{5}$

(b) the distance between ℓ_1 and ℓ_2, *answer:* $\sqrt{21}/7$

(c) the distance between Π_1 and Π_2, *answer:* 0

(d) the distance from P to Π_1, *answer:* $5/\sqrt{6}$

(e) the point where ℓ_1 meets Π_1, *answer:* $(1,-1,1)$

(f) the line $\Pi_1\cap\Pi_2$, *answer:* $(5,0,8)+t(2,1,4)$

(10) Consider the line given by $\mathbf{a}+t\mathbf{d}$ and a point P with position vector $\mathbf{c}$. Let $f(t) = |\mathbf{a}+t\mathbf{d}-\mathbf{c}|^2$ be the given function of t. What is the geometrical interpretation of f?
Find the stationary point of f and deduce that the closest point on the line to P is given by $\mathbf{a}+t\mathbf{d}$ for this stationary value of t.
Show that for this value of t, $\mathbf{a}+t\mathbf{d}-\mathbf{c}$ is perpendicular to $\mathbf{d}$. *answer: f is the distance (squared) from P to a general point of the line.*
$f' = 2t|\mathbf{d}|^2 + 2\mathbf{a}\cdot\mathbf{d} - 2\mathbf{c}\cdot\mathbf{d}$. *Stationary when*

$$t = \frac{(\mathbf{c}-\mathbf{a})\cdot\mathbf{d}}{|\mathbf{d}|^2}$$

(11) Suppose $\mathbf{a}$ and $\mathbf{b}$ are two vectors in the xy-plane. Find the two real numbers λ and μ satisfying the equation $\mathbf{a}+\lambda\mathbf{a}\times\mathbf{k} = \mathbf{b}+\mu\mathbf{b}\times\mathbf{k}$.
answer: $\lambda = (\mathbf{b}-\mathbf{a})\cdot\mathbf{b}/[\mathbf{a},\mathbf{k},\mathbf{b}]$ *and* $\mu = (\mathbf{a}-\mathbf{b})\cdot\mathbf{a}/[\mathbf{b},\mathbf{k},\mathbf{a}]$.

(12) Simplify $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$.

Solution.

On expanding, the expression $= \mathbf{u}\times\mathbf{u} - \mathbf{u}\times\mathbf{v} + \mathbf{v}\times\mathbf{u} - \mathbf{v}\times\mathbf{v}$, which is 0, using $\mathbf{u}\times\mathbf{u} = 0$ and $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$.

(13) Suppose that the vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{R}^3$ in fact lie in a plane. Prove that

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = 0.$$

Solution.

Both $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are in the direction of the normal to the plane, so their cross product is 0.

(14*) Consider the points $A=(1,1,1)$, $B=(1,-1,-1)$, $C=(-1,1,-1)$, $D=(-1,-1,1)$ in $\mathbb{R}^3$.

(a) Show that they are all equidistant from the origin $O=(0,0,0)$. What is this distance?

(b) Show that they are equidistant from each other, and so form the vertices of a regular tetrahedron. What is this distance?

(c) Find the angle $\angle AOB$.
(This is the so-called *tetrahedral angle*, and is e.g. the angle subtended at the carbon atom by two hydrogen atoms in a methane molecule CH_4.)

Solution.

(a) All distance $\sqrt{3}$ from O.

(b) All sides of length $2\sqrt{2}$.

(c) Relevant triangle has sides of lengths $\sqrt{3},\sqrt{3},2\sqrt{2}$, so cosine rule gives $\angle AOB = \cos^{-1}(-1/3) = 109.47^o$.

Total Marks for Paper: 10.