The University of Edinburgh **School of Mathematics**
2010 **(U01457)**

Geometry & Convergence

Lecture Notes

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Documented generated at 13:16, Thursday 30th September, 2010

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Preface

Degree Examination. The degree examination for this course will be of the same format as the practice exams that will be distributed on the course web page in due course. The questions should be similar to the questions on the sample exams.

Words of Advice. There are a few pieces of advice for this course.

(1) Learn the Definitions—you *must* understand the meaning of the terms used in the course. Vagueness will be penalised.
(2) Learn the Theorems—again, you *must* understand the meaning of the theorems in this course, and how to apply them.
(3) Learn the Proofs—you *must* understand why certain things are true, not simply those facts. This should not be a burden, because a proof always explains more than just a fact.

How to Read These Notes. Mathematics is learned by doing exercises. Frequently, exercises reveal aspects of the definitions and theorems that you have learned which are not obvious.

It is recommended that you read these notes in parallel with the lectures. In addition, these notes should be useful for your reference. To aid this second purpose, the notes are equipped with an extensive index so that the reader can quickly locate relevant information.

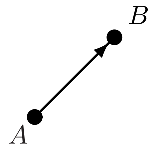
Books. There is no compulsory book for this course. Lecture notes will be provided from the course web page. If you feel the need for a textbook there are many books on linear algebra and calculus that may be useful. Anton's *Elementary Linear Algebra* and Adams' *Calculus, a complete course* have been recommended in the past.

CHAPTER 1

Vector geometry

1. Vector geometry in the plane

Definition. A *vector* is an oriented line segment; alternatively, it is a direction and a magnitude. We write $\overrightarrow{AB}$ for the vector that joins the point A to the point B; we call A the *tail* and B the *head* of the vector $\overrightarrow{AB}$. The *length* of the vector $\overrightarrow{AB}$ is denoted by $\left|\overrightarrow{AB}\right|$.

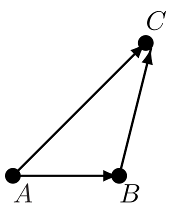


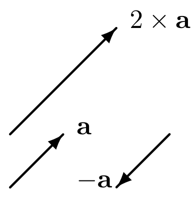
Two vectors are *equal* if they have the same direction and magnitude. The *zero vector* is the vector whose length is zero; the direction of the zero vector is undefined; $\overrightarrow{AA}$ represents the zero vector.

Notation. In addition to the notation $\overrightarrow{AB}$, we will also denote vectors using bold-face script $\mathbf{a}, \mathbf{b}, \ldots$ or underlined $\underline{a}, \underline{b}, \ldots$. The *zero vector* will be denoted by $\mathbf{0}$.

1.1. Vector operations. A number is also called a *scalar*. We *add* two vectors by the *parallelogram law*, which means that we place the tail of the second vector at the head of the first. This means that the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AC}$.

If k is a scalar and $\mathbf{a}$ is a vector, then $k\mathbf{a}$ is the vector whose magnitude is $|k|$ times the magnitude of $\mathbf{a}$ and the direction of $\mathbf{a}$ if $k > 0$ and opposite if $k < 0$.





Exercise. Use the definition of vector addition and scalar multiplication to show:

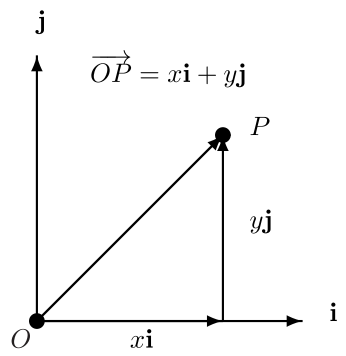
(1) If $\mathbf{a}$ is a vector, then $\mathbf{a} + \mathbf{0} = \mathbf{a}$.
(2) If $k \in \mathbb{R}$, then $k\mathbf{0} = \mathbf{0}$.

1.2. Coordinates. A *unit vector* is a vector $\mathbf{v}$ with unit length: $|\mathbf{v}| = 1$. Let O be a chosen point and let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors such that $\mathbf{j}$ is 90^o counterclockwise from $\mathbf{i}$.

If P is a point in the plane, then the vector $\overrightarrow{OP} = \mathbf{p}$ is called the *position vector* of P. We can write

$$\mathbf{p} = x\mathbf{i} + y\mathbf{j},$$

and we call (x, y) the *coordinates* of the point P.



The coordinates of $\mathbf{i}$ are $(1, 0)$, while that of $\mathbf{j}$ are $(0, 1)$.

Notation. When we wish to distinguish between a point P and its position vector $\mathbf{p}$, while using coordinates, we will write

$$P = (x, y) \qquad \mathbf{p} = \langle x, y\rangle, \text{ so}$$
$$\mathbf{i} = \langle 1, 0\rangle \qquad \mathbf{j} = \langle 0, 1\rangle.$$

For example, if

$$\mathbf{a} = \langle 2, 3\rangle \qquad \mathbf{b} = \langle -1, 4\rangle, \text{ then}$$

$$\mathbf{a} + \mathbf{b} = (2\mathbf{i} + 3\mathbf{j}) + (-1\mathbf{i} + 4\mathbf{j}) = (2 - 1)\mathbf{i} + (3 + 4)\mathbf{j} = \langle 1, 7\rangle,$$

and

$$5\mathbf{a} = 5(2\mathbf{i} + 3\mathbf{j}) = 10\mathbf{i} + 15\mathbf{j} = \langle 10, 15\rangle.$$

That is, we add vectors componentwise and multiply the scalar with each component.

1.3. Convention. We will assume that the origin O and unit vectors $\mathbf{i}$ and $\mathbf{j}$ are fixed for the rest of the discussion—unless stated to the contrary.

2. Lines

2.1. Parametric form of a line. Let A be a point in the plane with position vector $\mathbf{a}$. Let $\mathbf{u}$ be a non-zero vector. The *line* L with direction vector $\mathbf{u}$ that passes through the point A is the set of all points with position vectors:

$$L = \left\{ P \ : \ \overrightarrow{OP} = \mathbf{a} + t\,\mathbf{u} \text{ for some } t \in \mathbb{R} \right\}. \tag{1}$$

Notation. It is convenient to blur the distinction between points and their position vectors. We will do this with lines, writing

$$L = \{\mathbf{a} + t\,\mathbf{u} \ : \ t \in \mathbb{R}\}. \tag{2}$$

2.2. Intersection of two lines. This can be calculated if they are both in parametric form by solving two simultaneous equations for the two parameters (which must be given different names).

If one is in parameter form and one in equation form, substitute the parametrised line into the equation and solve for the parameter.

If both are in equation form, just solve two simultaneous equations. In all cases if there is no solution, the two lines must be parallel.

3. The section formula

3.1. Line through two points. The line through the points A, B with position vectors $\mathbf{a}, \mathbf{b}$ respectively is given by

$$\{t\mathbf{a} + (1-t)\mathbf{b}\}$$

(We will start to leave out the "$t \in \mathbb{R}$".)

3.2. The section formula. Let P be the point which divides the line segment AB (as above) in the ratio $n : m$ (meaning that P is a distance n from A and a distance m from B in suitable units). Then P has position vector

$$\frac{m\mathbf{a} + n\mathbf{b}}{m+n}.$$

(Remember the rule that for points on the line outside the segment AB, take one of m, n negative—the rule being that distances are positive when they are measured towards the other point and negative when in the other direction.)

4. Two theorems

4.1. Medians of a triangle. A *median* of a triangle is a line joining a vertex to the midpoint of the opposite side.

Theorem. The medians of a triangle meet at a point (the "centroid" or "centre of gravity" of the triangle). This point divides each median in the ratio $1 : 2$.

Theorem. The diagonals of a parallelogram bisect each other.

5. Scalar product

Notation. Henceforth we adopt the convention that the components of a vector $\mathbf{a}$ are given by a_1, a_2, etc. In other words,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

Definition. The *scalar product* $\mathbf{a} \cdot \mathbf{b}$ of two vectors is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$$

The scalar product is also called the *dot product* and the *inner product*.

Properties. The scalar product has the following properties:

(1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
(2) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) + (\mathbf{b} \cdot \mathbf{c})$
(3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$
(4) $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$
(5) $\mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$
(6) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
(7) $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$.

The last point (together with implicit use of the earlier ones) means that one can compute, for example,

$$(2\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} + 3\mathbf{j}) = 2(\mathbf{i} \cdot \mathbf{i}) - 3(\mathbf{j} \cdot \mathbf{j}) - (\mathbf{j} \cdot \mathbf{i}) + 6(\mathbf{i} \cdot \mathbf{j}) = 2 - 3 = -1.$$

Theorem. Let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

(Note that this enables one to compute the *angle* between two vectors easily.)

Corollary. The dot product of two vectors vanishes only if one of the vectors is zero or they are perpendicular.

5.1. Lines revisited. A vector perpendicular to the direction vector of a line is called a *normal* to the line. Let A be a point with position vector $\mathbf{a}$ and let $\mathbf{n}$ be a non-zero vector. Then the line through A with normal $\mathbf{n}$ has equation

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0.$$

(Putting $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ gives the equation in the familiar form $kx + ly = m$.) Conversely, given a line

$$kx + ly = m,$$

the vector $\mathbf{n} = k\mathbf{i} + l\mathbf{j}$ (or any non-zero multiple thereof) is a normal to the line.

Definition. If $\mathbf{v}$ is a vector, we define $J\mathbf{v}$ to be the result of rotating $\mathbf{v}$ anti-clockwise by a right-angle. Thus

$$J\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$$

5.2. Directions and normals. If $\mathbf{u}$ is a direction vector for a line, then $\mathbf{n} = J\mathbf{u}$ is a normal. Similarly, if $\mathbf{n}$ is a normal then $\mathbf{u} = J\mathbf{n}$ is a direction vector.

6. Areas

Definition. Let $\mathbf{a}, \mathbf{b}$ be two vectors. Then the number $[\mathbf{a}, \mathbf{b}]$ is defined by

$$[\mathbf{a}, \mathbf{b}] = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

This is the *determinant.*

Properties. The following all follow immediately from standard properties of determinants (which you should know from "Solving equations").

(1) $[\mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}]$.
(2) $[\mathbf{a} + \mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{c}] + [\mathbf{b}, \mathbf{c}]$ and similarly for the second argument.
(3) $[k\mathbf{a}, \mathbf{b}] = k[\mathbf{a}, \mathbf{b}]$ and similarly for the second argument.
(4) $[\mathbf{a}, \mathbf{b} + k\mathbf{a}] = [\mathbf{a}, \mathbf{b}]$.
(5) $[\mathbf{i}, \mathbf{i}] = [\mathbf{j}, \mathbf{j}] = 0, \quad [\mathbf{i}, \mathbf{j}] = 1, \quad [\mathbf{j}, \mathbf{i}] = -1$.

6.1. Geometric properties.

(1) $[\mathbf{a}, \mathbf{b}] = (J\mathbf{a}) \cdot \mathbf{b}$.
(2) $[\mathbf{a}, \mathbf{b}] = |\mathbf{a}||\mathbf{b}| \sin\theta$, where θ is the angle through which $\mathbf{a}$ must be rotated anti-clockwise to obtain the direction of $\mathbf{b}$.
(3) If $[\mathbf{a}, \mathbf{b}] \neq 0$ then it is positive if θ as defined above is less than π and negative if it is greater.
(4) $|[\mathbf{a}, \mathbf{b}]|$ is equal to the area of the parallelogram spanned by $\mathbf{a}, \mathbf{b}$.
(5) We say that the pair of vectors $\mathbf{a}, \mathbf{b}$ (not multiples of each other) is *positively oriented* (or just *oriented*) if $[\mathbf{a}, \mathbf{b}] > 0$. Equivalently, $\mathbf{a}, \mathbf{b}$ is oriented if the direction of $\mathbf{b}$ is less than π radians from that of $\mathbf{a}$ measured *anti-clockwise.*

CHAPTER 2

Vectors in $\mathbb{R}^3$

1. What carries over from the plane

Definitions. The definitions of a vector and its modulus, and of addition and scalar multiplication remain the same.

1.1. Coordinates. In 3-dimensional space we choose an origin O and we choose three mutually perpendicular unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. We do this in a way that satisfies the *right-hand rule:* rotating from $\mathbf{i}$ to $\mathbf{j}$ is a right-handed screw motion if you are looking in the direction of $\mathbf{k}$.

We then have that the position vector $\mathbf{p}$ of the point P with coordinates (x, y, z) is

$$\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and

$$|\mathbf{p}| = \sqrt{x^2 + y^2 + z^2}.$$

By convention, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ unless stated otherwise.

1.2. Lines. The parametric form for the line with direction vector $\mathbf{u}$ through the point with position vector $\mathbf{a}$ is

$$\{\mathbf{a} + t\mathbf{u} \ : \ t \in \mathbb{R}\}.$$

For example, the line through the point $(2, 1, 3)$ with direction vector $-3\mathbf{i} + \mathbf{j} - 8\mathbf{k}$ is

$$\{(2 - 3t, 1 + t, 3 - 8t) \ : \ t \in \mathbb{R}\}.$$

The formulae for the line through two points and also the Section Formula remain true.

1.3. Scalar product. We define the *scalar product* or *dot product* $\mathbf{a} \cdot \mathbf{b}$ to be the number

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

The algebraic properties are identical to the case of the plane. (We should add $\mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$, $\mathbf{k} \cdot \mathbf{k} = 1$.) We still have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

and the same proof works. (First, draw both vectors with a common start point, and restrict everything to the plane in which they both lie. The angle θ is the angle between the vectors in this plane.)

2. Planes

2.1. Parametric form for a plane. Let A be a point in space with position vector $\mathbf{a}$ and let $\mathbf{u}, \mathbf{v}$ be two non-zero vectors which are not multiples of each other. The *plane through A spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ is the set of all points with position vector

$$\{\mathbf{a} + s\mathbf{u} + t\mathbf{v} \ : \ s, t \in \mathbb{R}\}.$$

2.2. Equation of a plane. A plane has equation

$$px + qy + rz = c.$$

To obtain the equation from the parametric form, write the three expressions for x, y, z as functions of s, t and eliminate s and t between them.

2.3. Intersection of a line and a plane. Substitute the parametric form of the line into the equation of the plane and solve for the parameter.

2.4. Equation of a plane in vector form. A vector $\mathbf{n}$ perpendicular to a plane is called a *normal* to the plane. The plane through the point A with position vector $\mathbf{a}$ with normal $\mathbf{n}$ is

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0.$$

A normal to the plane

$$px + qy + rz = c$$

is $\mathbf{n} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$. Using this normal, the equation can be written

$$\mathbf{x} \cdot \mathbf{n} = c.$$

3. The cross product

Definition. The *cross product* of two vectors is the vector defined by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}. \tag{3}$$

This is more easily remembered as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

(One might argue that it is poor form to have vectors as entries in determinants, but since they all occur in the top row, there is no risk of having to take products of two vectors and everything makes perfect sense.)

Properties.

(1) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. (And so in particular $\mathbf{a} \times \mathbf{a} = 0$.)
(2) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ and similarly for the second argument.
(3) $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$ and similarly for the second argument.
(4) $|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$
(5) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.
(6)

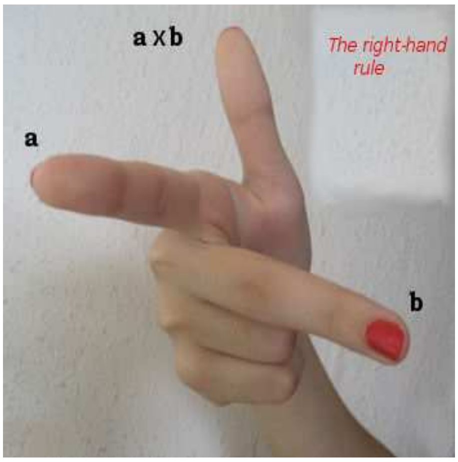
$$\begin{gathered} \mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j} \\ \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \end{gathered}$$

3.1. Geometric characterisation. If $\mathbf{a} = 0$ or $\mathbf{b} = 0$ or if $\mathbf{a}$ and $\mathbf{b}$ are scalar multiples of each other then $\mathbf{a} \times \mathbf{b} = 0$. Otherwise, let θ (where $0 < \theta < \pi$) be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then

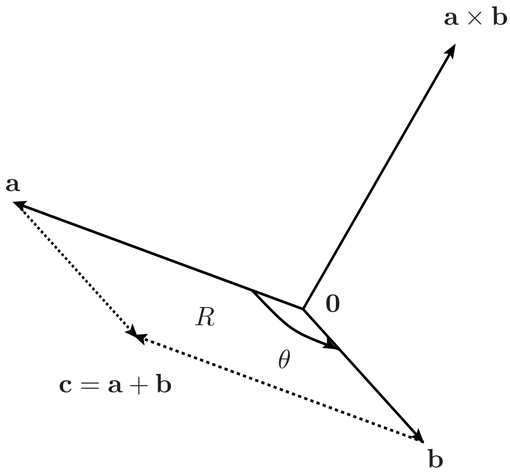
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

and $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and its direction is determined by the *right-hand rule*: rotation from $\mathbf{a}$ to $\mathbf{b}$ is right-handed when looking in the direction of $\mathbf{a} \times \mathbf{b}$ (see figure 1 for a different interpretation).

(This all follows from properties 4 and 5 above, except for the right-hand rule. For that, note that it is obeyed for $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$. Now any pair $\mathbf{a}, \mathbf{b}$ of vectors can be deformed continuously (without the cross product ever being zero) to the pair $\mathbf{i}, \mathbf{j}$, and the cross product can not jump between the two possible handednesses. Thus it must always be given by the right-hand rule. (This last part of the argument is not examinable.)



(a) The right-hand rule: $\mathbf{a}\times\mathbf{b}$. The direction of $\mathbf{a}\times\mathbf{b}$ is determined by pointing, with your right index finger, along $\mathbf{a}$ and your second finger along $\mathbf{b}$: your thumb points in the direction of $\mathbf{a}\times\mathbf{b}$.



(b) The magnitude of $\mathbf{a}\times\mathbf{b}$ is the area of the parallelogram R with vertices $\mathbf{0}, \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. Elementary trigonometry shows area $R = |\mathbf{a}|\ |\mathbf{b}| \sin\theta$ (see figure 2).

FIGURE 1. The *right-hand rule.*

Theorem. [An algebraic identity] Given three vectors, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we have

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

4. Geometric applications

4.1. Finding the equation of the plane through three points. Let A, B, C have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Assume that all three points do not lie on a single line. Then $\mathbf{b}-\mathbf{c}$ and $\mathbf{c}-\mathbf{a}$ are parallel to the plane containing A, B, C. Thus

$$\mathbf{n} = (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$$

is a normal to this plane. The (unique) plane containing A, B, C is thus

$$(\mathbf{x}-\mathbf{a})\cdot\mathbf{n} = 0.$$

(The $\mathbf{a}$ could be replaced by $\mathbf{b}$ or $\mathbf{c}$.)

4.2. Distance of a point from a plane. The distance of the point P with position vector $\mathbf{p}$ from the plane $\mathbf{x}\cdot\mathbf{n} = c$ is

$$\frac{|\mathbf{p}\cdot\mathbf{n} - c|}{|\mathbf{n}|}.$$

4.3. Distance between parallel planes. The distance between the parallel planes

$$\mathbf{x}\cdot\mathbf{n} = c, \quad \mathbf{x}\cdot\mathbf{n} = d$$

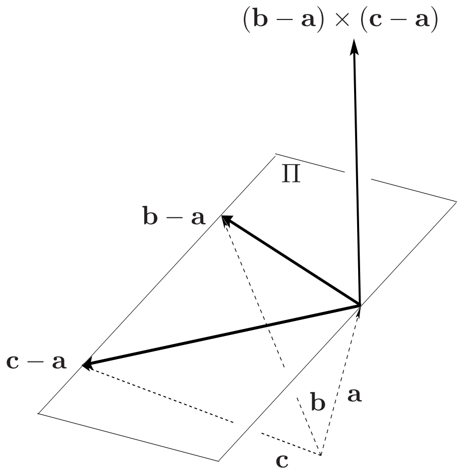
is

$$\frac{|c-d|}{|\mathbf{n}|}.$$

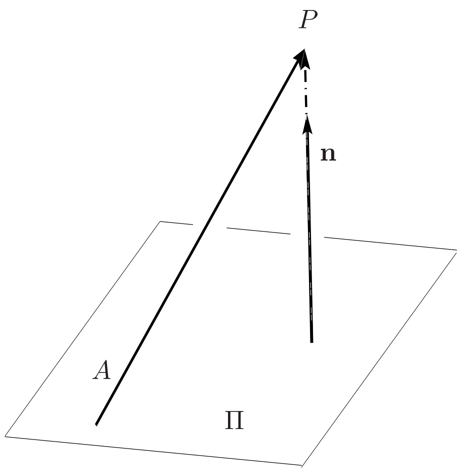
4.4. Distance between skew lines. Let $\ell_1 = \{\mathbf{a} + t\mathbf{u}\}$ and $\ell_2 = \{\mathbf{b} + s\mathbf{v}\}$ be two skew lines. (*Skew* means that the direction vectors are not multiples of each other.) The distance between the lines is

$$\frac{|(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.$$

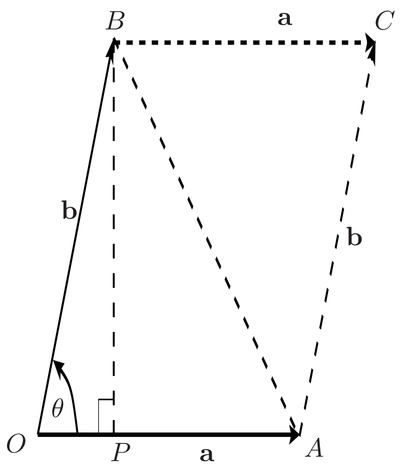
(Here, and analogously in the previous few results, the distance between two lines means the minimum possible distance between a point on one line and a point on the other. For example, it is zero if and only if the lines intersect.)



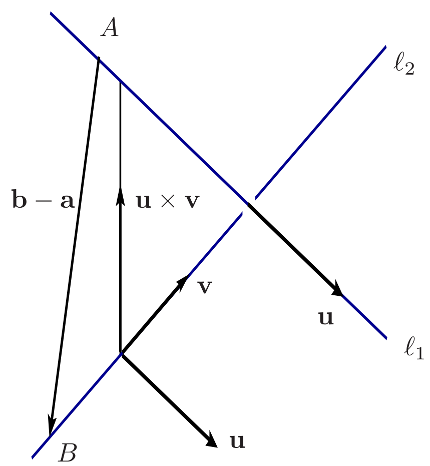
(a) The plane Π containing the points A, B, C and the plane's normal vector.



(b) Given a point A on the plane Π : $\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$, the distance to a point P is the length of the projection of the displacement vector $\mathbf{p} - \mathbf{a}$ onto the normal $\mathbf{n}$.



(c) The area of the parallelogram $OACB$ is twice the area of the triangle OAB. Area OAB equals $\frac{1}{2}\,|OA|\,|PB|$. which equals $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$.



(d) The distance between skew lines is the length of the projection of $\mathbf{b} - \mathbf{a}$ onto the normal $\mathbf{u} \times \mathbf{v}$.

FIGURE 2. Applications of the *cross product*.

5. The triple scalar product

Definition. The *triple scalar product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is the scalar

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The equality of $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ with the first determinant follows from the determinant form of the cross product. The equality of the two determinants follows from the fact that the determinant of a matrix and its transpose are equal.

Properties.

(1) $[\mathbf{u} + \mathbf{v}, \mathbf{a}, \mathbf{b}] = [\mathbf{u}, \mathbf{a}, \mathbf{b}] + [\mathbf{v}, \mathbf{a}, \mathbf{b}]$ and similarly for the other two entries.
(2) $[k\mathbf{u}, \mathbf{a}, \mathbf{b}] = k[\mathbf{u}, \mathbf{a}, \mathbf{b}]$ and similarly for the other two entries.
(3) The alternating property: the triple product is multiplied by -1 if any pair of arguments are exchanged. That is,

$$[\mathbf{b}, \mathbf{a}, \mathbf{c}] = [\mathbf{c}, \mathbf{b}, \mathbf{a}] = [\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

(4) The cyclic property: the triple scalar product is unchanged if the arguments are cyclically permuted:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}].$$

(5) If one vector in a triple product is a multiple of one of the other, then the triple product is zero.
(6) The triple product is unchanged if a multiple of one of the vectors is added to either of the others.

The first two properties (and the analogous statements for the dot and cross products) are often expressed by saying that the triple product is a "linear function of each of its arguments".

5.1. An algebraic identity. The cyclic property has many applications, for example in proving the following important identity.

Theorem.

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{a} \cdot \mathbf{u})(\mathbf{b} \cdot \mathbf{v}) - (\mathbf{a} \cdot \mathbf{v})(\mathbf{b} \cdot \mathbf{u}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{u} & \mathbf{a} \cdot \mathbf{v} \\ \mathbf{b} \cdot \mathbf{u} & \mathbf{b} \cdot \mathbf{v} \end{vmatrix}.$$

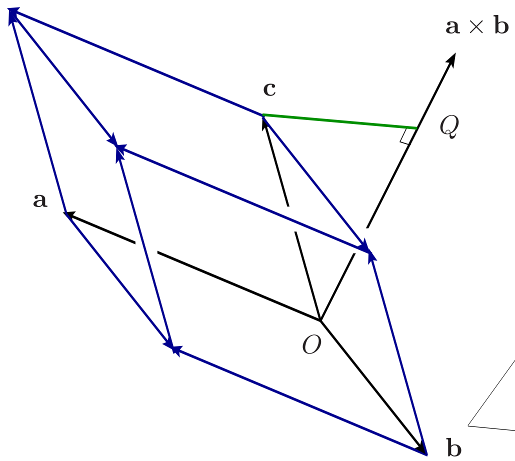
6. Areas and volumes

6.1. Parallelograms. The area of the parallelogram spanned by $\mathbf{a}, \mathbf{b}$ is given by $|\mathbf{a} \times \mathbf{b}|$.

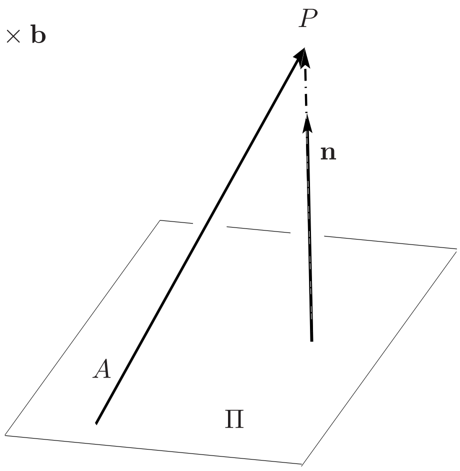
6.2. Parallelepiped. The *parallelepiped* spanned by the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is the set

$$\{s\mathbf{a} + t\mathbf{b} + u\mathbf{c} \ : \ 0 \le s, t, u \le 1\}.$$

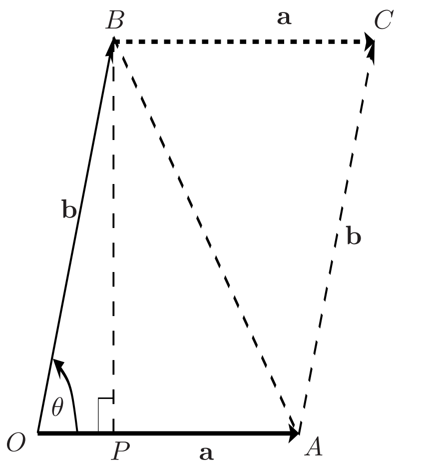
The volume of the parallelepiped is given by $|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$. The volume is zero if and only if all of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in a single plane. Otherwise, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] > 0$ if the rotation $\mathbf{a} \to \mathbf{b} \to \mathbf{c}$ is right-handed when looking in to the body from the origin.



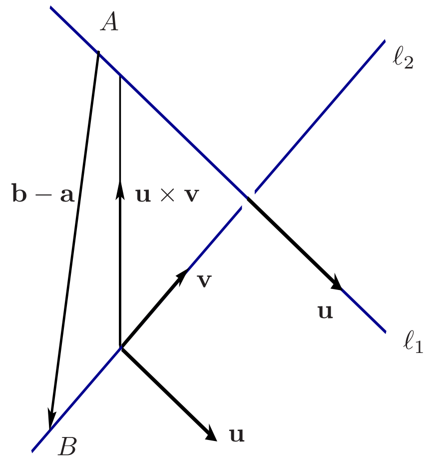
(a) The volume of the parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is the area of the base, $|\mathbf{a} \times \mathbf{b}|$, multiplied by the height $|OQ| = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| \, / \, |\mathbf{a} \times \mathbf{b}|$, so the volume is $|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$.



(b) Given a point A on the plane $\Pi : \mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$, the distance to a point P is the length of the projection of the displacement vector $\mathbf{p} - \mathbf{a}$ onto the normal $\mathbf{n}$.



(c) The area of the parallelogram $OACB$ is twice the area of the triangle OAB. Area OAB equals $\frac{1}{2} \, |OA| \; |PB|$. which equals $|\mathbf{a}| \; |\mathbf{b}| \, \sin\theta$.



(d) The distance between skew lines is the length of the projection of $\mathbf{b} - \mathbf{a}$ onto the normal $\mathbf{u} \times \mathbf{v}$.

FIGURE 3. Applications of the *cross product.*

CHAPTER 3

Linear maps

1. Definitions

1.1. Convention. In this section, we will (almost always) write vectors as column vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

rather than as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. We will also think of $\mathbf{a}$ as being a point in 3-dimensional space. The dot product of two vectors can be written using matrix multiplication as

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

1.2. Linear maps. Let A be an $n \times n$ matrix. The map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T_A : \mathbf{x} \mapsto A\mathbf{x}$$

is called the *linear map* defined by the matrix A. For us, n will always be 2 or 3, but in fact most of what we say is perfectly OK in any number of dimensions. Let T_A be a linear map as above. Then for all vectors $\mathbf{u}, \mathbf{v}$ and scalars k we have

- $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$
- $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$

(Indeed, it is is these two properties which are meant by the word "linear".)

1.3. Columns are the images of the basis vectors. Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ be the vectors which are the columns of the 3×3 matrix A. Then

$$T_A(\mathbf{i}) = T_A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{c}_1, \quad T_A(\mathbf{j}) = T_A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{c}_2, \quad T_A(\mathbf{k}) = T_A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{c}_3.$$

1.4. The determinant and expansion of volumes. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. Then the image under T_A of parallelepiped of oriented volume V is a parallelepiped with volume $(\det A)V$. The same applies to areas of parallelograms in the 2-dimensional case. Note in particular that if the determinant is negative then orientations are reversed.

2. Orthogonal matrices

Definition. The square matrix A is *orthogonal* if for all vectors $\mathbf{u}, \mathbf{v}$ we have

$$(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}.$$

Since distances between points and angles between vectors can be defined in terms of dot products, linear maps given by orthognal matrices preserve distances and angles.

2.1. Characterisation of orthogonal matrices. The following are equivalent:

(1) A is an orthogonal matrix;
(2) $A^T A = \mathrm{I}$;
(3) $AA^T = \mathrm{I}$;
(4) $A^T = A^{-1}$;
(5) The columns of A are orthogonal unit vectors.

2.2. Determinant of an orthogonal matrix. If A is an orthogonal matrix, then $\det A = \pm 1$.

3. Orthogonal 2×2 matrices

3.1. Rotations and reflections. Let θ be a real number. The linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotates the plane by an angle θ about the origin and that defined by the matrix

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

reflects the plane in the line through the origin making an angle of $\theta/2$ (measured anti-clockwise) with the positive x-axis.

3.2. Composition of reflections. For all θ, ϕ we have

$$M_\theta M_\phi = R_{\theta-\phi}.$$

Thus the composition of two reflections of the plane in lines through the origin is a rotation of the plane about the origin. (Note that the left-hand side corresponds to doing M_ϕ first, then M_θ.)

Theorem. Let A be an orthogonal 2×2 matrix. If $\det A = 1$ then $A = R_\theta$ for some θ. If $\det A = -1$ then $A = M_\theta$ for some θ.

CHAPTER 4

Conics

1. Standard conics

Definition. A *conic* is a curve in the plane defined by an equation of the form

$$lx^2 + 2mxy + ny^2 + px + qy = c$$

where l, m, n, p, q, c are real numbers.

We assume that the solution set is not empty (as it would be in the case $x^2 + y^2 = -1$, for example) nor a single point (as it would be for $x^2 + y^2 = 0$) nor a single line (as it would be for $x + y = 1$ or $x^2 = 0$).

1.1. Standard conics. A *standard ellipse* is a curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a \geq b > 0$ are constants.

A *standard hyperbola* is a curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a > 0, b > 0$ are constants.

A *standard parabola* is a curve

$$y^2 = 4ax$$

where $a > 0$ is a constant.

2. Classification of central conics

Definition. A conic is *central* if it is of the form

$$lx^2 + 2mxy + ny^2 = c.$$

A central conic can be written as

$$\mathbf{x}^T S \mathbf{x} = c$$

where

$$S = \begin{pmatrix} l & m \\ m & n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

2.1. Rotation of coordinates. Let P be a 2×2 rotation matrix. Change coordinates according to

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix} = P\mathbf{u}.$$

Then the central conic $\mathbf{x}^T S \mathbf{x} = c$ becomes $\mathbf{u}^T K \mathbf{u} = c$ where the symmetric matrix $K = P^T S P$.

2.2. Revision. If S is a symmetric 2×2 matrix, then its eigenvalues are real. Let λ_1, λ_2 be eigenvalues with corresponding *unit-length* orthogonal eigenvectors $\mathbf{c}_1, \mathbf{c}_2$. Let P be the 2×2 matrix with $\mathbf{c}_1, \mathbf{c}_2$ as columns. Then P is orthogonal and

$$P^T S P = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Theorem. By rotation of the axes, a central conic $\mathbf{x}^T S \mathbf{x} = c$ can be transformed into one of the following:

(1) A standard ellipse;
(2) A standard hyperbola;
(3) A conic

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

which is the pair of lines $bx \pm ay = 0$.

If $\det S > 0$ then we have an ellipse, and if $\det S < 0$ it is a hyperbola or line pair.

Example. Transform the central conic $3x^2 + 3y^2 + 10xy = 2$ to standard form. The matrix is

$$S = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}.$$

($\det S = -16$ and so this is a hyperbola or line pair.) Calculating *unit* eigenvectors and eigenvalues we get

$$\lambda_1 = -2, \quad \mathbf{c}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \quad \lambda_2 = 8, \quad \mathbf{c}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

So

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Substituting

$$x = \frac{1}{\sqrt{2}}(u + v), \quad y = \frac{1}{\sqrt{2}}(-u + v)$$

we get

$$4v^2 - u^2 = 1.$$

(This is a "standard hyperbola with $a = 1/2, b = 1$".) Note that the eigenvectors are the symmetry axes of the standard form.

3. Classification of general conics

3.1. Diagonalisation. By rotation of the axes as for central conics, a general conic can be transformed to

$$lx^2 + ny^2 + px + qy = c.$$

3.2. Classification. If neither of l, n are zero, then a translation of coordinates

$$x = u - \frac{p}{2l}, \quad y = v - \frac{q}{2n}$$

converts the conic to a central conic which is an ellipse, hyperbola or line pair,

$$lu^2 + nv^2 = f, \qquad f = c + \frac{p^2}{4l} + \frac{q^2}{4n}.$$

We can not have both of l, n zero (otherwise we have a line). If one of l, n are zero then a translation of coordinates transforms the conic to a standard parabola.

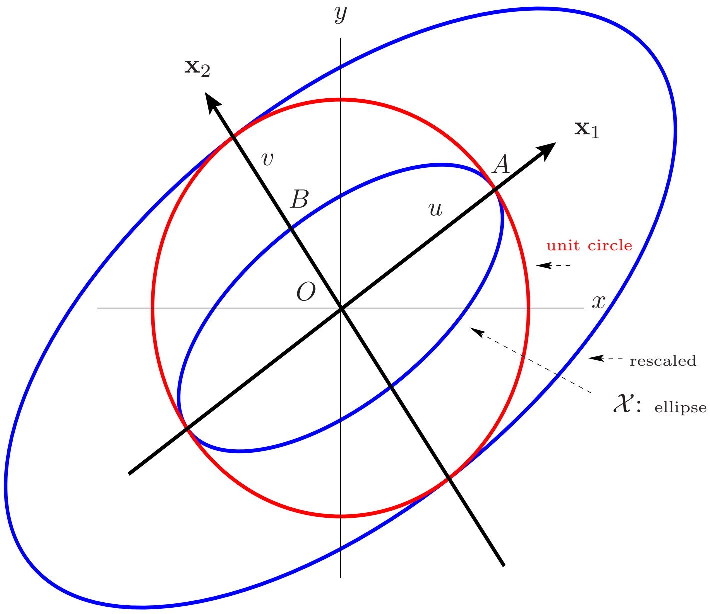


FIGURE 1. The principal axes of a central conic $\mathcal{X}$. In the coordinates (u, v), the ellipse $\mathcal{X}$ is $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$ where $a = |OA|$ and $b = |OB|$. Angle $\angle AOB$ is a right angle.

Example. Let us put the conic

$$85y^2 + 240xy + 8y + 15x^2 + 9x - 100 = 0$$

into standard form. We first compute the eigenvalues of the symmetric matrix

$$S = \begin{pmatrix} 15 & 120 \\ 120 & 85 \end{pmatrix}.$$

The characteristic polynomial is $\det(S - \lambda I) = (15 - \lambda)(85 - \lambda) - 120^2 = \lambda^2 - 100\lambda - 13125$, with roots $\lambda_1 = 175, \lambda_2 = -75$.

To find an eigenvector associated to λ_1, we must solve $(S - \lambda I)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$ with $\lambda = \lambda_1$. We find

$$S - 175I = \begin{pmatrix} -160 & 120 \\ 120 & -90 \end{pmatrix} \mapsto \begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{x}_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}.$$

Since $\mathbf{x}_2 \perp \mathbf{x}_1$, we find the second eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}.$$

When we substitute

$$\mathbf{x} = U\mathbf{u} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5} \times \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

we get

$$175u^2 - 75v^2 + 12v/5 + 59u/5 - 100 = 0,$$

and when we substitute $r = u - \frac{59}{5}/(2 \times 175)$, $s = v - \frac{12}{5}/(2 \times (-75))$, we get

$$175r^2 - 75s^2 - 350629/3500 = 0.$$

4. Geometric properties of conics

4.1. Eccentricity. In the standard ellipse, the *eccentricity* e is defined by the equation

$$b^2 = a^2(1 - e^2).$$

Since $a \geq b > 0$ we have $0 \leq e < 1$ and $e = 0$ only when the ellipse is a circle.

4.2. Focus and directrix. A *focus* of the standard ellipse is one of the points $(\pm ae, 0)$. A *directrix* is one of the lines $x = \pm a/e$.

Theorem. The standard ellipse is the set of all points P in the plane such that

$$\frac{\text{distance of } P \text{ from the focus } (ae, 0)}{\text{distance of } P \text{ from the directrix } x = a/e} = e.$$

By symmetry, the same holds for the other focus and the other directrix.

Theorem. The sum of the distances from a point on an ellipse to the two foci is a constant.

5. Hyperbolas and parabolas

5.1. Hyperbolas. The *eccentricity* of the standard hyperbola is defined by the equation

$$b^2 = a^2(e^2 - 1).$$

Thus, $e > 1$.

A *focus* of the standard hyperbola is one of the points $(\pm ae, 0)$. A *directrix* of the standard hyperbola is one of the lines $x = \pm a/e$.

The standard hyperbola is the set of all points P in the plane such that

$$\frac{\text{distance of } P \text{ from the focus } (ae, 0)}{\text{distance of } P \text{ from the directrix } x = a/e} = e.$$

5.2. Parabolas. The *focus* of the standard parabola is the point $(a, 0)$. The *directrix* of the standard hyperbola is the line $x = -a$.

The standard parabola is the set of all points P in the plane such that

$$\frac{\text{distance of } P \text{ from the focus } (a, 0)}{\text{distance of } P \text{ from the directrix } x = -a} = 1.$$

6. Intersection problems

6.1. Intersection of a line with a conic. To find the intersection of a line $y = mx + c$ with a conic, substitute y into the equation of the conic and solve the resulting quadratic equation for x. The quadratic will have 0,1 or 2 real solutions. This corresponds to the the line missing the conic, being tangent to the conic and intersecting the conic twice.

6.2. Intersection of a conic with a conic. Two distinct conics have at most 4 real points of intersection. To determine the points of intersection, it is useful to have the conics in standard form. In general, this can only be done for one of the two conics. For centred conics, though, we can do better.

6.2.1. *Intersection of a centred conic with a centred conic.* Let

$$\mathcal{X}_0 : a_0x^2 + 2_0b_0x_0y + c_0y^2 = f \qquad \mathbf{x}'S\mathbf{x} = f$$

$$\mathcal{X}_1 : a_1x^2 + 2_1b_1x_1y + c_1y^2 = g \qquad \mathbf{x}'R\mathbf{x} = g$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

be two centred conics. We know that to solve the equations

$$\mathcal{X}_0 : \alpha u^2 + \beta v^2 = f$$

$$\mathcal{X}_1 : \gamma u^2 + \delta v^2 = g$$

is straightforward. So let's see if we can find coordinates that do this.

7. The standard form of two centred conics

In figure 2, we see there are two distinguished "*axes*" determined by the condition that a rescaling of $\mathcal{X}_1$ is tangent to $\mathcal{X}_0$. The condition is that at the point $\mathbf{x}$, the normal vector $R\mathbf{x}$ to $\mathcal{X}_1$ should be a scalar multiple of the normal vector $S\mathbf{x}$ of $\mathcal{X}_0$:

(4) $$R\mathbf{x} = \lambda S\mathbf{x} \qquad \text{(generalised } \textit{eigenvector} \text{ equation)}$$

This condition implies that

(5) $$\det(R - \lambda S) = 0, \qquad \text{(generalised } \textit{eigenvalue} \text{ equation)}$$

which is a quadratic equation in λ. If $(\lambda_1, \mathbf{x}_1)$ and $(\lambda_2, \mathbf{x}_2)$ are two non-trivial solutions to these equations, then we introduce the coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} = U\,\mathbf{u} \qquad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad U = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}2 \end{bmatrix}$$

so that

$$\mathcal{X}_0 : \mathbf{u}'U'SU\mathbf{u} = f \qquad \mathcal{X}_1 : \mathbf{u}'U'RU\mathbf{u} = g,$$

and

(6)

$$U'SU = \begin{bmatrix} \mathbf{x}_1'S\mathbf{x}_1 & \mathbf{x}_1'S\mathbf{x}_2 \\ \mathbf{x}_2'S\mathbf{x}_1 & \mathbf{x}_2'S\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \quad U'RU = \begin{bmatrix} \mathbf{x}_1'R\mathbf{x}_1 & \mathbf{x}_1'R\mathbf{x}_2 \\ \mathbf{x}_2'R\mathbf{x}_1 & \mathbf{x}_2'R\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha\lambda_1 & 0 \\ 0 & \beta\lambda_2 \end{bmatrix}$$

which gives

$$\mathcal{X}_0 : \alpha u^2 + \beta v^2 = f \qquad \mathcal{X}_1 : \lambda_1\alpha u^2 + \lambda_2\beta v^2 = g,$$

the desired form with $\gamma = \lambda_1\alpha$ and $\delta = \lambda_2\beta$. As a final step, if $\alpha\beta \neq 0$, then we can introduce a final change of variables

$$u = \frac{r}{\sqrt{|\alpha|}} \qquad v = \frac{s}{\sqrt{|\beta|}}$$

giving

(7) $$\left.\begin{aligned} \mathcal{X}_0 :&\quad r^2 + s^2 = f \\ \mathcal{X}_1 :&\quad \lambda_1 r^2 + \lambda_2 s^2 = g, \end{aligned}\right\} \qquad \text{if } \alpha, \beta > 0$$

(8) $$\left.\begin{aligned} \mathcal{X}_0 :&\quad r^2 - s^2 = f \\ \mathcal{X}_1 :&\quad \lambda_1 r^2 - \lambda_2 s^2 = g, \end{aligned}\right\} \qquad \text{if } \alpha > 0 > \beta$$

and the case where $0 > \alpha, \beta$ can be dealt with by reversing the sign of α, β and f, g and using (7).

Remark. In 6, we have used the fact, proven in class, that if $\lambda_1 \neq \lambda_2$, then $\mathbf{x}_2' S \mathbf{x}_1 = 0$ and similarly for R. To see this, observe

$$\begin{aligned}
0 &= \mathbf{x}_2' R \mathbf{x}_1 - \mathbf{x}_1' R \mathbf{x}_2 && \text{since } R' = R \\
&= \mathbf{x}_2'(\lambda_1 S \mathbf{x}_1) - \mathbf{x}_1'(\lambda_2 S \mathbf{x}_2) && \text{due to (4)} \\
&= (\lambda_1 - \lambda_2)\mathbf{x}_2' S \mathbf{x}_1 && \text{since } S' = S \\
&\implies \mathbf{x}_2' S \mathbf{x}_1 = 0 && \text{since } \lambda_1 - \lambda_2 \neq 0, \\
&\implies \mathbf{x}_2' R \mathbf{x}_1 = 0 && \text{since one of } \lambda_1, \lambda_2 \neq 0.
\end{aligned}$$

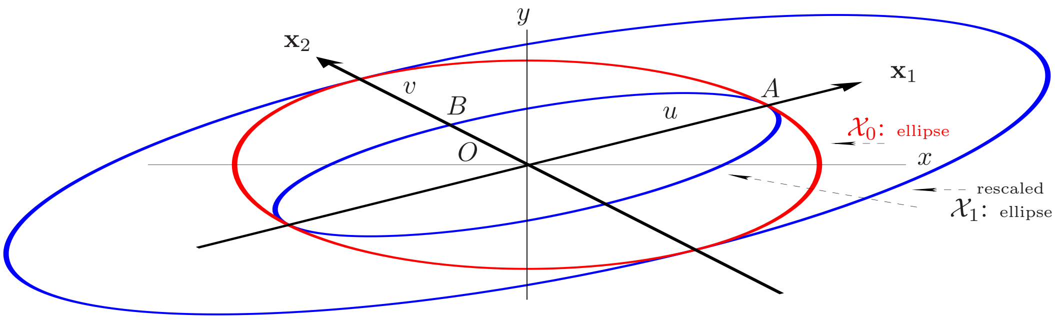


FIGURE 2. The centred *conics* $\mathcal{X}_0$ and $\mathcal{X}_1$. In the coordinates (u, v), the ellipse $\mathcal{X}_0$ is $u^2 + v^2 = 1$ and $\mathcal{X}_1$ is $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ where $a = |OA|$ and $b = |OB|$. Angle $\angle AOB$ is not necessarily a right angle.

Example. Let us do a worked example to illustrate the theory. Let

$$\mathcal{X}_0 : 23\,y^2 + 24\,x\,y + 2\,x^2 = 1 \qquad \mathcal{X}_1 : 123\,y^2 + 216\,x\,y + 102\,x^2 = 1$$

with the associated symmetric matrices

$$S = \begin{bmatrix} 2 & 12 \\ 12 & 23 \end{bmatrix} \qquad R = \begin{bmatrix} 102 & 108 \\ 108 & 123 \end{bmatrix}.$$

We compute that the generalised characteristic polynomial is

$$\det(R - \lambda S) = (123 - 23\lambda)(102 - 2\lambda) - (108 - 12\lambda)^2 = 98(\lambda^2 - 9)$$
$$\implies \text{roots } \lambda = 3, -3.$$

When we solve the generalised eigenvector equation $(R - \lambda S)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$, we get

$$\lambda_1 = 3 : \mathbf{x}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \qquad \lambda_2 = -3 : \mathbf{x}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

We let $\mathbf{x} = U\mathbf{u}$, as above, and deduce that

$$\mathcal{X}_0 : -49v^2 + 98u^2 = 1 \qquad \mathcal{X}_1 : 147v^2 + 294u^2 = 1,$$

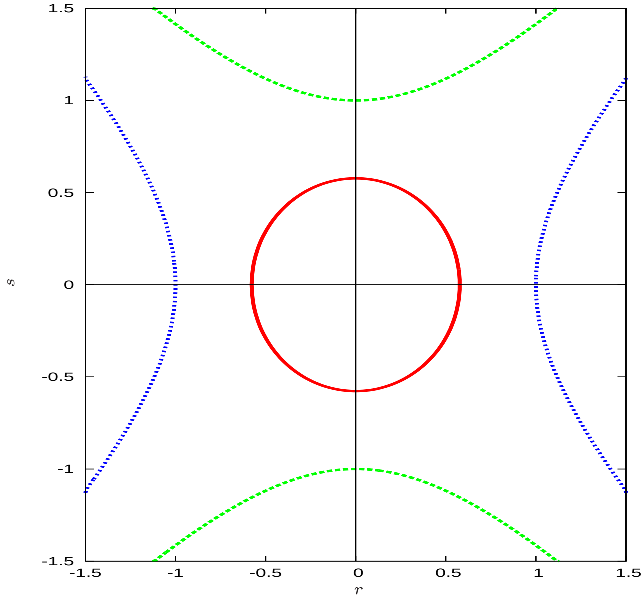
(so $\alpha = 98$, $\beta = -49$) and with the substitution $u = r/(7\sqrt{2}), v = s/7$,

$$\mathcal{X}_0 : -s^2 + r^2 = 1 \qquad \mathcal{X}_1 : 3s^2 + 3r^2 = 1.$$

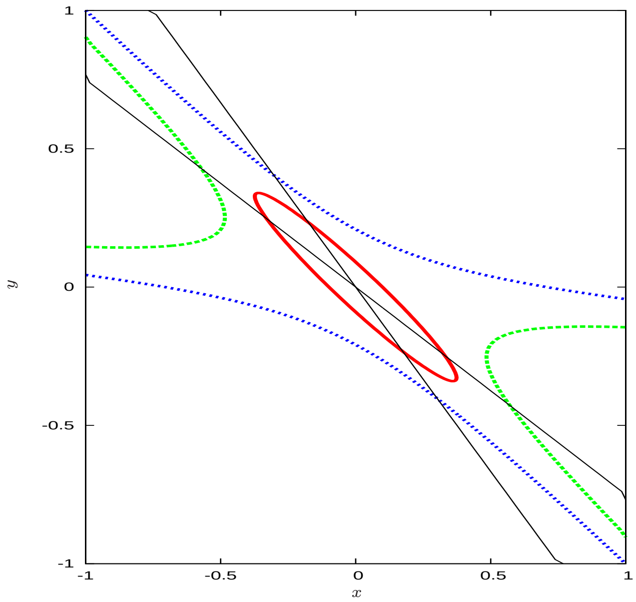
To sketch the two conics in the (r, s) coordinates, we note that we can parameterise each

$$\mathcal{X}_0 : r = \pm\cosh(t), s = \sinh(t) \qquad \mathcal{X}_1 : r = \cos(t)/\sqrt{3}, s = \sin(t)/\sqrt{3}.$$

To get the sketch in the original (x, y) coordinates, we simply use the transformations $(r, s) \to (u, v) \to (x, y)$ that we defined above. We know, in particular, that the r-axis (resp. s-axis) gets mapped to the line $\mathbb{R}\mathbf{x}_1$ (resp. $\mathbb{R}\mathbf{x}_2$).



(a) $\mathcal{X}_0$ (in blue) and $\mathcal{X}_1$ (in red). The conic in green is $\mathcal{X}_0$ with constant -1 instead of $+1$.



(b) $\mathcal{X}_0$ (in blue) and $\mathcal{X}_1$ (in red). The conic in green is $\mathcal{X}_0$ with constant -1 instead of $+1$. The black lines are the generalised eigenspaces. One can see that by rescaling $\mathcal{X}_1$, it becomes tangent to the two hyperbola along these lines.

FIGURE 3.

CHAPTER 5

Induction

1. Sigma notation

Definition. We use the notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

which defines the Σ operator. Clearly, the sum does not need to start at 1 or finish at n. Thus

$$\sum_{k=-1}^{2} k^2 = (-1)^2 + 0^2 + 1^2 + 2^2 = 6.$$

2. Induction

Definition. *Induction* (or "mathematical induction" or "the principle of mathematical induction") is the the observation that if one wishes to prove that a statement involving a natural number n is true for *all* n, it suffices to prove firstly that it holds when $n = 1$ and secondly that *if it holds for* n*, then it holds for* $n+1$.

Example. Show that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Solution:

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}$$

and so the claimed statement certainly holds when $n = 1$. Now let us assume that the statement holds for a particular n. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{(by the result for } n\text{)} \\
&= (n+1)\left(\frac{n}{2} + 1\right) \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}
\end{aligned}$$

and so the statement is true for $n+1$, and hence for all n by induction.

The result for n is often called the *inductive hypothesis*. Thus, the annotation "by the result for n" might be replaced by "by the inductive hypothesis".

2.1. Variations. There are numerous variations. One (sometimes called "strong induction") is that instead of proving truth for n implies truth for $n+1$, you prove truth for all values up to and including n implies truth for $n+1$.

A simpler variation is that one can prove (for example) that something is true for all natural numbers $n \geq 4$ by proving that it is true for $n = 4$ and proving that provided $n \geq 4$, truth for n implies truth for $n+1$. Another minor variation is given in the following example.

Example. Prove that for every natural number $n \geq 2$ we have $3^n > 3n$.

Solution: This is true for $n = 2$ since $9 > 6$. Now suppose it is true for a particular n then

$$\begin{aligned} 3^{n+1} = 3.3^n &> 3.3n \quad \text{by the inductive hypothesis} \\ &= 3n + 6n \\ &> 3n + 3 = 3(n+1) \quad \text{since } 6n > 3 \text{ for } n \geq 2. \end{aligned}$$

Hence the result is true by induction.

CHAPTER 6

Sequences and Series

1. Sequences

Definition. A *sequence* is a function

$$n \mapsto f(n) \qquad\qquad n \in \mathbb{Z}, n \geq b,$$

where b is some "base case."

Notation. Although sequences are functions, they are special functions, and we use special notation to indicate sequences. Amongst the notations for a sequence are

enumeration:	$a_1, a_2, \ldots, a_n, \ldots,$	$1, 1, 2, 3, 5, \ldots,$
formula:	$a_n = f(n),$	$a_n = 1/n,$
set-like:	$\{a_n\}_{n=1}^{\infty}, (a_n)_{n=1}^{\infty}$	$\{1/n\}_{n=1}^{\infty},$
terse:	$a_n, (a_n)$	$1/n, (1/n).$

We will favour the first two notations, but be aware that the last two are frequently used as shorthand.

2. Arithmetic and geometric sequences

Definition. The *arithmetic sequence* with first term a and common difference d is the sequence

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

In other words, it is the sequence (a_k) where

$$a_k = a + (k-1)d.$$

Definition. The *geometric sequence* with first term a and common ratio r is the sequence

$$a,\ ar,\ ar^2,\ ar^3, \ldots$$

In other words, it is the sequence (a_k) where

$$a_k = ar^{k-1}.$$

3. Convergence of sequences

We have talked about the idea of sequences tending to some sort of "limiting value". For example, arithmetic sequences do not have a limit (unless $d = 0$). On the other hand, geometric sequences with $a \neq 0$ have a limit of 0 if $|r| < 1$, a limit of a if $r = 1$ and no limit if $r > 1$ or $r \leq -1$. It took many years for people to see that the best way of rigorously encapsulating the idea of a sequence tending to a limit is the following definition.

Definition. The sequence (a_k) converges to 0 if, for each $\epsilon > 0$, there are only finitely many k such that $|a_k| > \epsilon$.

Equivalently, here is a more standard definition

Definition. The sequence (a_k) converges to 0 if, for each $\epsilon > 0$, there is an $N = N(\epsilon)$ such that $k \geq N$ implies $|a_k| < \epsilon$.

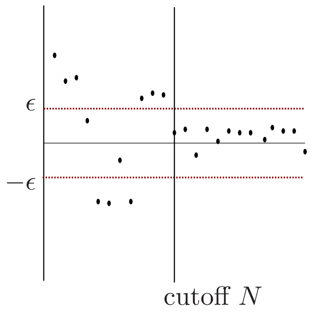


FIGURE 1. Convergence to 0 and ϵ-corridors. For each $\epsilon > 0$, there is a cutoff N such that all terms to the right of N are within ϵ of 0.

Example. Let's show that $a_k = 1/k$ converges to 0.

`Rough Work.` Let $\epsilon > 0$ be fixed. If $|a_k| > \epsilon$, then $1/k > \epsilon$ so $k < 1/\epsilon$.
`End of Rough Work`

Proof. Let $\epsilon > 0$ be given. Our rough work shows that there are only finitely (at most $1/\epsilon$) terms in the sequence $1/k$ such that $1/k > \epsilon$. Therefore, from definition (3), $1/k$ converges to 0.

Let's define a *limit.*

Definition. The sequence (a_k) *converges to the limit* l (or "has limit l" or "converges to l" or "tends to l") if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - l| < \epsilon$ if $n \geq N$.

In other words, (a_k) converges to l iff $(a_k - l)$ converges to 0.

Notation. The notation

$$\lim_{n\to\infty} a_n = l$$

means that the sequence (a_m) converges to the limit l.

Example. Does the sequence (a_n) where

$$a_n = n/(n+1)$$

have a limit? If so, give a proof.

`Rough Work.` When faced with such a question, first try and decide if there is a limit. Here, it is clear that when n is very large, a_n is very close to 1. Therefore one guesses the limit is 1. Next, you have to try and see how large n has to be in order to have

$$\left|\frac{n}{n+1} - 1\right| < \epsilon.$$

To this end, we calculate

$$\frac{n}{n+1} - 1 = \frac{n-(n+1)}{n+1} = \frac{-1}{n+1}.$$

Thus

$$|a_n - 1| = \frac{1}{n+1} < \frac{1}{n}.$$

So we want $1/n < \epsilon$, so it is OK provided $n > 1/\epsilon$. Thus, we must choose N to be a natural number $> 1/\epsilon$. We now have everything we need, and the trick is to write up the proof that follows *and then throw away the calculations we have just* *done which are not part of it!*. Before we do that, just notice how I "backed off" a little and just used $|a_n - l| < 1/n$ rather than the stricter $|a_n - l| = 1/(n+1)$. This makes the calculation of N a shade simpler. The point is *you don't need to find the smallest possible N, you just need one that does the job.*

`End of Rough Work`

Proof. Here is the proof that (a_n) as above has limit one. Let $\epsilon > 0$ be given. Let N be a natural number larger than $1/\epsilon$. If $n \geq N$ then

$$|a_n - 1| = \left| \frac{n}{n+1} - 1 \right| = \left| \frac{-1}{n+1} \right| < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Theorem. The sequence (a_n) where

$$a_n = \frac{1}{n^k}$$

converges to zero for all real numbers $k > 0$.

Definition. A sequence (a_n) is *bounded* above (resp. below) if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ (resp. $a_n \geq M$) for all n. A sequence is *bounded* if it is bounded above and below. A sequence (a_n) is *increasing* if $a_{n+1} \geq a_n$ for all n.

Theorem. A bounded, increasing sequence converges. (This theorem, for which we give no proof, depends on quite subtle properties of the real numbers. A proof will be given in "Foundations of Calculus" next year.)

With the aid of the above theorem, one can prove

Theorem.

$$\begin{aligned} e &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n, \\ &= \lim_{n\to\infty} x_n \end{aligned}$$

where $x_{n+1} = x_n + \dfrac{1}{n!}$ and $x_0 = 0$.

Exercises.

In the following exercises, "prove" means "show, using the definition."

(1) Let $k > 0$ be a fixed real number. Prove that $a_n = n^{-k}$ converges to 0.

(2) Prove that $\displaystyle\lim_{n\to\infty} \frac{3n^2 - 1}{5n^2 + 2n + 1} = \frac{3}{5}$.

(3) Prove that $\displaystyle\lim_{n\to\infty} \frac{n^2 + 3n - 1}{5n^4 + 2n^3 + 9} = 0$.

(4) Let $x_k = \dfrac{1}{k!}$. Prove that $\lim_{k\to\infty} x_k = 0$.

(5) Prove that $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

4. Sequences that tend to infinity

Definition. A sequence $(a_n)_{n=1}^{\infty}$ is *unbounded* if it is not bounded (definition (3)). Equivalently, it is bounded if, for each $B > 0$ there are infinitely many $n \in \mathbb{N}$ such that $|a_n| > B$.

Example. Let

$$a_n = (-1)^n \ln n.$$

Prove that a_n is an unbounded sequence.

`Rough Work.` Similar to proofs of convergence, let us first intuit the proof. Recall that if $y = e^x$, then $\ln y = x$. We know therefore that ln is a monotone increasing function and that as y grows larger, so must x—without bound.

`End of Rough Work`

Proof. Let $B > 0$ be given. We know that if $N > e^B$, then since ln is an increasing function, $\ln N > \ln(e^B) = B$. Therefore, choose $N > e^B$. If

$$n \geq N \text{ then} \qquad |a_n| = \ln n \geq \ln N \geq B.$$

This proves that all a_n with $n \geq N$ have modulus larger than B, which proves that a_n is unbounded.

Definition. A sequence $(a_n)_{n=1}^{\infty}$ *tends to* $+\infty$ (resp. *tends to* $-\infty$) if, for each $B > 0$ there is an $N = N(B)$ such that if $n \geq N$ then $a_n > B$ (resp. $a_n < -B$).

Example. Convince yourself that $\ln n$ tends to $+\infty$.

Example. Let

$$x_n = \frac{5n^2 - 4}{4n - 1}.$$

Prove that x_n tends to $+\infty$.

`Rough Work.` We see that $x_n \sim 5n/4 > n$ as n gets large, so x_n should tend to ∞ as n tends to ∞. Indeed, since we want to show that x_n gets large, we will find a *lower bound* for its numerator and an *upper bound* for its denominator.

$$\begin{aligned} 5n^2 - 4 &\geq 4n^4 + (n^2 - 2) & \qquad 4n - 1 < 4n \\ &\geq 4n^2 & \qquad \text{if } n \geq 2. \end{aligned}$$

Therefore

$$\begin{aligned} x_n &> \frac{4n^2}{4n} & \qquad \text{if } n \geq 2, \\ &= n. \end{aligned}$$

`End of Rough Work`

Proof. Let $B > 0$ be given. Choose the cutoff N to be at least B and 2; we can do this by choosing $N > B + 2$. Then

$$\begin{aligned} n \geq N \text{ implies} \qquad x_n &> n & \qquad \text{since } n \geq N > 2 \\ &\geq N & \qquad \text{since } n \geq N \\ &\geq B & \qquad \text{since } N > B. \end{aligned}$$

This proves that x_n tends to $+\infty$.

5. Convergence of series

Definition. A *series* is an infinite sum

$$\sum_{k=1}^{\infty} a_k.$$

The n-th *partial sum* of the series is

$$s_n = \sum_{k=1}^{n} a_k.$$

The series is said to *converge to* S if

$$\lim_{n\to\infty} s_n = S.$$

A series that does not converge is said to *diverge*.

A series whose partial sums s_n are increasing (resp. decreasing) either converges to a limit S or the partial sums tend to $+\infty$ (resp. $-\infty$). In the latter case, where the partial sums increase (resp. decrease) without bound, we say that the series diverges.

Theorem. Let (a_n) be the geometric series $a_k = ar^{k-1}$ with $a \neq 0$ and $r \neq 1$. The partial sums are given by

$$s_n = \sum_{k=1}^{n} a_k = a + ar + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}.$$

The series $\sum_{k=1}^{\infty} a_k$ diverges if $|r| \geq 1$. If $|r| < 1$ then the series converges and

$$\sum_{k=1}^{\infty} a_k = \frac{a}{1-r}.$$

Proof. Let us observe that $rs_n = ar + \cdots + ar^n = a + ar + \cdots + ar^{n-1} + ar^n - a = s_n + ar^n - a$. Solving for s_n, we get $s_n = a(r^n - 1)/(r-1)$ for $r \neq 1$.

If $|r| > 1$, then it is clear that $|s_n| = \frac{a}{|r-1|} \times |r^n - 1| \geq \frac{a}{|r-1|} \times (|r|^n - 1)$. Since $|r| > 1$, this sequence tends to $+\infty$, so the series diverges.

If $r = 1$, then $|s_n| = |a| \times n$, which also tends to $+\infty$.

If $r = -1$, then $s_n = a$ if n is even and 0 if n is odd. Since $a \neq 0$, the sequence s_n does not converge. Therefore, the series diverges.

Theorem. The *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges.

Proof. Let us observe that if $0 < k \leq x$, then $1/k \geq 1/x$. Therefore, integrating over the interval $[k, k+1]$,

$$\frac{1}{k} \geq \int_{k}^{k+1} \frac{dx}{x} = \ln(k+1) - \ln(k).$$

Summing from $k = 1, \ldots, n$, we get that the n-th partial sum s_n satisfies

$$s_n = \sum_{k=1}^{n} \frac{1}{k} \sum_{k=1}^{n} \int_{k}^{k+1} \frac{dx}{x} = \int_{1}^{n+1} \frac{dx}{x} = \ln(n+1).$$

Now, we showed above that $a_n = \ln n$ tends to $+\infty$ as $n \to \infty$. Therefore, s_n must also tend to $+\infty$ as $n \to \infty$.

Remark. From the bound $1/x \geq 1/(k+1)$ for $x \leq k+1$, one can integrate over $[k, k+1]$ and sum from $k = 1, \ldots, n$ to obtain $\int_1^{n+1} \frac{dx}{x} \geq \sum_{k=2}^{n+1} \frac{1}{k} = s_{n+1} - 1$. This means that $s_n \geq \ln(n+1) \geq s_n - \frac{n}{n+1}$, so the divergence of the harmonic series is slow (logarithmic).

It is a deep fact that the difference $s_n - \ln(n+1)$ converges to a constant, commonly called the Euler-Mascheroni constant, denoted by γ. It is approximately 0.5772. Despite the fact that Euler first defined γ in the 18-th century, it is currently unknown if γ is a rational number.

Theorem. The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges.

Proof. Let $s_n = \sum_{k=1}^n k^{-2}$ be the n-th partial sum of our series. Since $s_{n+1} = s_n + (n+1)^{-2} > s_n$, the sequence of partial sums is increasing. If we prove this sequence is bounded above, then we conclude it converges. Let us observe

$$\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)} \qquad \text{so}$$

$$r_n = \sum_{k=1}^n \frac{1}{(k+1)^2} \leq \sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1},$$

since

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \frac{1}{k} - \frac{1}{(k+1)}.$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and all but the first and last terms are cancelled. Therefore $r_n \leq 1$ for all n.

This leads us to conclude that our partial sums $s_{n+1} = r_n + 1 \leq 2$ for all n. Therefore, the partial sums s_n are increasing and bounded above and hence converge.

Remark. In fact, $\sum_{k=1}^\infty 1/k^2 = \pi^2/6$, but that is not quite so easy to show!

Theorem. Let a_n, b_n be sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges.

Proof. Let $s_n = \sum_{k=1}^n a_k$ and $r_n = \sum_{k=1}^n b_k$ be the partial sums for the sequences a_k and b_k, respectively. Let $r = \lim_{n\to\infty} r_n$. We know that

$$a_k \leq b_k \qquad \text{for all } k\text{, so, summing}$$
$$s_n \leq r_n \qquad \text{for all } n.$$

And, since a_k, $b_k \geq 0$, we know both s_n and r_n are non-dereasing and

$$r_n \leq r \qquad \text{for all } n.$$

Therefore, s_n is an non-dereasing sequence that is bounded above. Therefore it converges.

Example. Let us prove that $\sum_{k=1}^\infty k^{-p}$ converges for $p \geq 2$ (in fact, it converges for $p > 1$).

Since $p \geq 2$, $k^p \geq k^2$ so $0 < k^{-p} \leq k^{-2}$ for $k \in \mathbb{N}$. Since the series $\sum_{k=1}^\infty k^{-2}$ converges, the comparison theorem shows that $\sum_{k=1}^\infty k^{-p}$ converges.

Exercises.

(1) Let $p > 1$. Show that $\displaystyle\sum_{k=1}^n k^{-p}$ converges.
[Hint: Let $r_n = 1 + \int_1^{n+1} x^{-p}\,dx$ for all $n \in \mathbb{N}$; now mimic the proof of the comparison theorem.]

(2) Show that the series $\displaystyle\sum_{n=1}^\infty \frac{3n+1}{4n^2+1}$ diverges.

CHAPTER 7

Taylor-Maclaurin Series

1. Taylor series

Let $f : (a, b) \to \mathbb{R}$ be a real-valued function of one variable that is infinitely differentiable. Let $x_0 \in (a, b)$.

Definition. The *formal series*

$$T(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) + \cdots$$
$$= \sum_{n=0} \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0)$$

is called the *Taylor series* of f about x_0. If $x_0 = 0$, we call T the *Maclaurin series* of f.

Examples.

(1) Let $f(x) = \exp(x)$. Since $f'(x) = f(x)$, we see that $f^{(n)}(x) = \exp(x)$ for all $n \geq 0$ and all x. In particular, $f^{(n)}(0) = 1$ for all $n \geq 0$. Therefore, the Taylor series of f about $x = 0$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

(2) Let $f(x) = \ln(1 - x)$. Let us compute the Taylor series of f about $x = 0$. Observe that

$$f(0) = 0 \qquad f'(0) = -\frac{1}{(1-x)^1}\bigg|_{x=0} = -1 = -0!$$

$$f''(0) = -\frac{1}{(1-x)^2}\bigg|_{x=0} = -1! \qquad f'''(0) = -\frac{1 \cdot 2}{(1-x)^3}\bigg|_{x=0} = -2!$$

and, by induction, the k-th derivative of f at $x = 0$ is

$$f^{(k)}(0) = -\frac{(k-1)!}{(1-x)^k}\bigg|_{x=0} = -(k-1)! \qquad \text{for } k \geq 1\text{, so}$$

$$T(x) = \sum_{k=1} -\frac{x^k}{k}$$
$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots$$

Theorem. Let f be function on the real line such that all derivatives of f exist. Consider the interval $[0, x]$ and suppose that there exists a real number M such that $|f^{(n)}(t)| \leq M^n$ for all $n \in \mathbb{N}$ and $t \in [0, x]$. Then the Taylor-Maclaurin series for f

$$\sum_{k=1}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

converges to $f(x)$.

Remark. The proof of this theorem is beyond this course; you will see it in Foundations of Calculus next year.

Corollary. The conditions of the Theorem apply for all x for the functions $\exp, \sin, \cos$. Thus it is really true that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Remark. Taylor series are *not* guaranteed to converge even for the most "reasonable" functions. The Taylor series of

$$f(x) = \frac{1}{1+x^2}$$

about zero is

$$1 - x^2 + x^4 - x^6 + x^8 \dots.$$

This is a geometric series of common ratio $-x^2$ and it converges to $f(x)$ for $|x| < 1$ but outside of that it diverges.

CHAPTER 8

Sequences, lists, etc, in Maple

See also the Maple/Maxima worksheet for this section.

1. Iteration

Let $f : \mathbb{R} \to \mathbb{R}$ be a scalar-valued function of 1 variable. Define a sequence

$$x_{n+1} = f(x_n) \qquad \text{for } n \geq 1, \tag{9}$$
$$x_0 = \text{given.}$$

We say that the sequence x_n is obtained by *iterating* with a given *initial condition.*

Example. A *logistic map* is defined as $f(x) = 4x(1-x)$ for $x \in \mathbb{R}$. For $x_0 = x$, we define the sequence x_n by (9).

When $x_0 < 0$, $1 - x_0 > 1$, so $x_1 = 4x_0(1-x_0) < 4x_0$. By an easy induction argument (do it!), it follows that $x_n < 4^n x_0$, so x_n tends to $-\infty$ as $n \to \infty$.

When $x_0 > 1$, $x_1 < 0$, so as in the first case, x_n tends to $-\infty$ as $n \to \infty$.

When $x_0 \in [0,1]$, $1 - x_0 \in [0,1]$, and so $0 \leq 4x_0(1-x_0) \leq 1$. Therefore $x_1 \in [0,1]$. By a simple induction argument (do it!), it follows that $x_n \in [0,1]$ for all $n \in \mathbb{N}$.

The sequence $(x_n)_{n\in\mathbb{N}}$ can be graphed by plotting (x_n, x_{n+1}) for $n \geq 0$. If we add line segments from (x_n, x_{n+1}) to (x_{n+1}, x_{n+1}) to (x_{n+1}, x_{n+2}), we end up with a *cobweb diagram.*

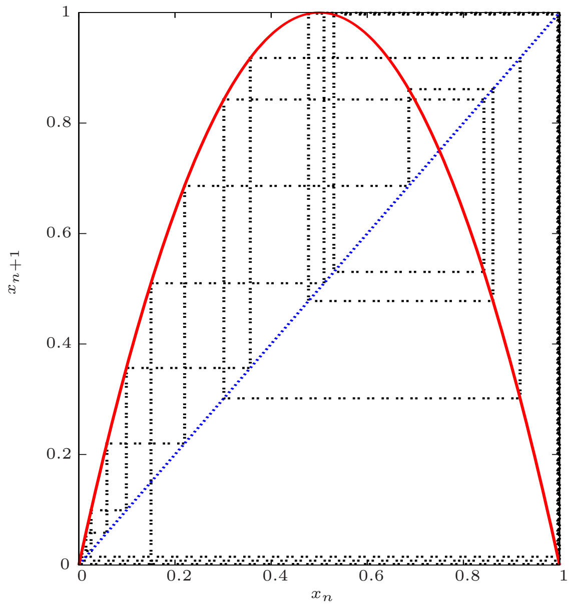


FIGURE 1. A *cobweb diagram* for the logistic map, with $x_0 = 0.15$.

2. Fixed points

2.1. Fixed points and iteration. A *fixed point* of a function f is a point a such that $f(a) = a$. Consider sequences $x_0, x_1, x_2, \ldots$ generated as in (9). Let a be a fixed point of f. Then a is said to be *attracting* if when you choose x_0 sufficiently close to a, the sequence $x_0, x_1, x_2, \ldots$ approaches a. Otherwise it is said to be *repelling*.

Theorem. Let a be a fixed point of f. If $|f'(a)| < 1$, then the fixed point is attracting; if $|f'(a)| > 1$, then the fixed point is repelling.

Example. Let $f(x) = 4x(1-x)$ be the logistic map of the previous example. It has a fixed point at $a = 0$, where $f'(0) = 4$, and a second fixed point at $a = 3/4$ with $f'(a) = -2$. Therefore both fixed points are *repelling*.

2.2. Solving equations by iteration. To solve the equation $f(x) = x$, one can guess a solution x_0 and generate a sequence as above. If the sequence approaches a fixed value, that value is a solution of the equation. The Theorem above then tells us that if there is a solution $x = a$ of the equation and $|f'(a)| < 1$ then this method will find it (to as good an accuracy as we desire), at least if our initial guess is good enough. On the other hand, if $|f'(a)| > 1$ then it can not work.

Example. Let us solve the equation

$$x^2 = 2$$

using iteration. To do this, let

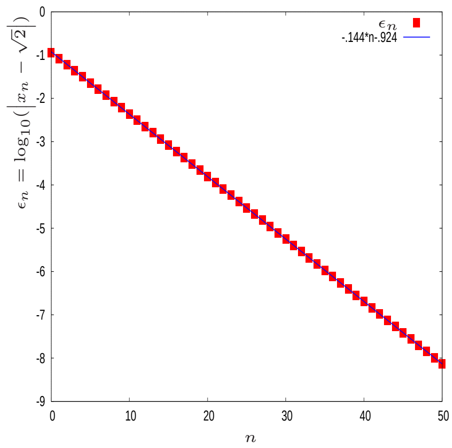
$$f_c(x) = x + c(x^2 - 2). \tag{10}$$

We see that $f(\sqrt{2}) = \sqrt{2}$ and $f'(\sqrt{2}) = 1 + 2c\sqrt{2}$ for any value of c. Since we want $\left|f'(\sqrt{2})\right| < 1$, we can choose any c between $-1/\sqrt{2}$ and 0, so we choose $c = -1/10$. Table 2 shows a sample of the sequence with $x_0 = 1.3$. Figure 3 shows that this sequence converges to $\sqrt{2}$ roughly like $10^{-0.14n-0.92}$.

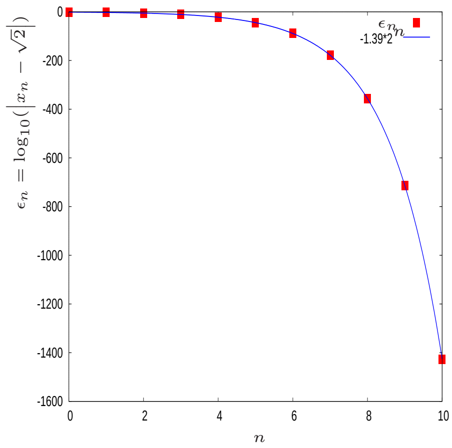
n	x_n	n	x_n
0	1.30000000	40	1.41421336
1	1.33100000	41	1.41421342
2	1.35384390	42	1.41421346
3	1.37055457	43	1.41421349
4	1.38271259	44	1.41421351
5	1.39152318	45	1.41421352
6	1.39788950	46	1.41421353
7	1.40248000	47	1.41421354
8	1.40578498	48	1.41421355
9	1.40816184	49	1.41421355
10	1.40986986	50	1.41421356

FIGURE 2. The sequence x_n versus n for selected values. This sequence converges slowly to $\sqrt{2} \cong 1.414213562$.

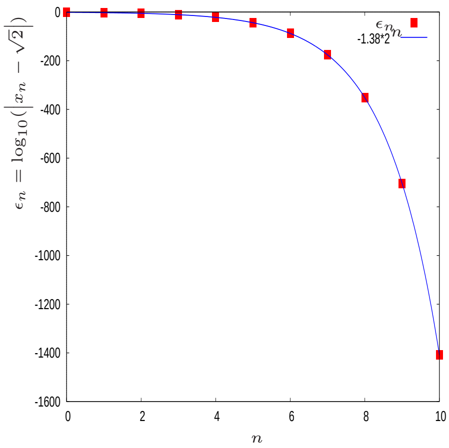
Example. Let us continue with the previous example, but let us observe that we cheated when we chose $c = -1/(2 \times \sqrt{2})$: this requires us to know the value of $\sqrt{2}$! We can avoid this cheat, by noting that $x \cong \sqrt{2}$ and substituting $c = -1/(2x)$ to



(a) The sequence $x_{n+1} = f_c(x_n)$, with $c = -1/10$, converges to $\sqrt{2}$. The astute reader will observe that $\log_{10}\left|f'(\sqrt{2})\right| \cong -0.144$ and that $\log_{10}(\left|x_0 - \sqrt{2}\right|) \cong -0.924$.



(b) The sequence $x_{n+1} = f_c(x_n)$, with $c = -2^{-\frac{3}{2}}$, converges to $\sqrt{2}$ quite quickly. The astute reader will note that $f_c'(\sqrt{2}) = 0$ for this value of c. It is worth noting that after 10 iterations there are over 1400 significant digits.



(c) The sequence $x_{n+1} = f_c(x_n)$, with $c = -1/(2x)$, converges to $\sqrt{2}$. The astute reader will observe that the error behaves very similarly to (b).

FIGURE 3. The sequence $\epsilon_n = \log_{10}(\left|x_n - \sqrt{2}\right|)$ measures the approximate number of significant digits in the sequence x_n.

get the map

$$g(x) = f_c(x)|_{c=-1/(2x)} = \frac{x}{2} + \frac{1}{x} \tag{11}$$

$$x_{n+1} = g(x_n), \qquad x_0 = \text{given}. \tag{12}$$

Figure 3.c shows how quickly this sequence converges to $\sqrt{2}$.

One derives that the error $e_n = \left|x_n - \sqrt{2}\right|$ equals approximately b^{2^n}, where $b = e_0$, by the following reasoning:

$$\begin{aligned} e_{n+1} &= \left|x_{n+1} - \sqrt{2}\right| && \text{by definition} \\ &= \left|g(x_n) - \sqrt{2}\right| \\ &= \left|g(\sqrt{2} + e_n) - \sqrt{2}\right| \\ &= \left|g(\sqrt{2}) - \sqrt{2} + g'(\sqrt{2})\, e_n + \frac{1}{2} g''(\sqrt{2})\, e_n^2 + \cdots\right| && \text{by Taylor expanding } g \\ &= a\, e_n^2 + \cdots && a = \frac{1}{2\sqrt{2}}, \end{aligned}$$

where it has been used that g fixes $\sqrt{2}$ and g' vanishes at $\sqrt{2}$. To solve the recurrence relation $e_{n+1} = a e_n^2$, with $e_0 = b$ given, one guesses a solution in the form

$$\begin{aligned} e_n &= b^{p_n} && \text{and derives that} \\ p_{n+1} &= 2p_n + d && \text{where } d = \log(a)/\log(b), p_0 = 1. \end{aligned}$$

One solves this recurrence and gets

$$p_n = (d+1) \times 2^n - d, \qquad \log_{10} e_n \cong \log_{10}(b) \times (d+1) \times 2^n \qquad \text{for large } n.$$

When one computes $\log_{10}(b) \times (d+1)$ with $x_0 = 1.3$, one arrives at a figure of -1.3938, which is quite close to that estimated in figure 3.c.

3. Maple and Maxima

3.1. Maxima CAS. Maxima is a *free* and *open source* computer algebra package that can be downloaded from its homepage at `http://maxima.sourceforge.net/`.

The cobweb diagram in 1 is created with the following code:

```
load(dynamics);
staircase(4*x*(1-x), .15, 11, [x, 0, 1.01]);
```

while the data used to generate the graphs in figure 3 is created from

```
c2bf(x) := rectform(expand(bfloat(expand(x))));

iter(f,x0,N,[opt]) := block([x1,l],
  l : if opt=[] then [] else append([[0,x0]],create_list(0,i,1,N)),
  for i:1 thru N do (
    x1 : c2bf(apply(f,[x0])),
    x0 : x1,
    if opt#[] then l[i+1] : [i,x1]
    ),
  return(if opt=[] then x1 else l));

f(x) := x + c*(x^2-2);
define(g(x), subst(c=-1/(2*x),f(x)));

fpprec : 18;
iter(f,1.3,50,true),c=-1/10;

fpprec : 2000;
iter(f,1.3,10,true),c=-1/2^(3/2);
iter(g,1.3,10,true);
```

The function `iter` computes the sequence x_n for $n = 0, \ldots, N$ given x_0 =`x0`. The functions `f` and `g` are defined as in (10) and (11). The constant `fpprec` sets the precision of the floating point computations–since after 50 iterations, the first sequence is only within 10^{-8} of $\sqrt{2}$, we choose a relatively small number of significant digits for the first and then a large number for the other computations.

3.2. Maple. Maple is a commercial computer algebra package that is widely used in the School of Mathematics.

LISTING 8.1. Maple code for a cobweb diagram

```
f := x -> 4*x*(1-x);
n1 := 15;
a := 0;
b := 1;

pic := plot(x ,x=a..b, color=black):
pic := pic union plot(f, a..b, color=black):
for i from 1 to n1 do
        x1:= f(x0):
        pic := pic union plot([[x0,x0],[x0,x1],[x1,x1]], color=blue):
        x0:=x1:
od:
display(pic);
```