The University of Edinburgh
2010

School of Mathematics
(U01457)

Geometry & Convergence

Problem Sheet 5

Assessment 5 due by 12.10 on Friday, 12 March 2010.
Tutorial 5 on Tuesday, 9 March 2010.

Pretutorial questions: 3, and 12.

Tutorial questions: 4, 5, 6, and 11.

Handin questions: 1, 2, 7, and 10.

(1†) Prove by induction that, for fixed $a \neq 1$ and $n = 1, 2, \ldots$

$$1 + a + a^2 + \ldots + a^{n-1} = \frac{a^n - 1}{a - 1}.$$

(2†) Define a_n $(n = 0, 1, 2, \ldots)$ by $a_0 = 1$ and $a_{n+1} = a_n + 2^n + 1$. Show by induction that

$$a_n = 2^n + n \qquad (n = 0, 1, 2, \ldots).$$

(3**) A certain algorithm takes time $T(n)$ to sort a set of 2^n elements, and time $T(n+1) = T(n) \times n^2$ to sort a set of 2^{n+1} elements. Show by induction that

$$T(n) = ((n-1)!)^2 T(1) \qquad (n = 1, 2, \ldots).$$

(4*) Prove by induction that $3^n - 2n^2 - 1$ is divisible by 8, for $n = 1, 2, \ldots$.

(5*) The Fibonacci numbers f_n are defined by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Prove by strong induction that

$$\phi^{n-2} \leq f_n \leq \phi^n \qquad (n = 1, 2, \ldots),$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$, the so-called *Golden Ratio*.

[Use the fact that $1 + \phi = \phi^2$.]

(6*) Let $\lfloor x \rfloor$ be the *floor* of x, i.e. the largest integer $\leq x$. Prove by induction that

$$n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor \qquad (n = 1, 2, \ldots).$$

(7†) What is wrong with the following inductive argument:

" **Theorem**. Let $x > 0$ be fixed. Then $x^{n-1} = 1$ for $n = 1, 2, \ldots$.

Proof. If $n = 1$ then $x^{1-1} = x^0 = 1$, so result true for $n = 1$. Assuming the result true for $1, 2, \ldots$, we have

$$x^{(n+1)-1} = x^n = x^{n-1} \times x^{n-1}/x^{n-2} = 1 \times 1/1 = 1,$$

so that the result holds for $n + 1$ as well." (Knuth)

(8) Show that if, for some proposition $P(n)$,

(a) $P(1)$ is true

(b) $P(n)$ true $\implies$ $P(2n)$ and $P(2n+1)$ both true $(n = 1, 2, \ldots)$

then $P(n)$ is true for $n = 1, 2, \ldots$.

[Use induction on the length of the binary representation of n].

(9) *(Esoteric variant of induction.)* Show that if, for some statement $P_2(n)$,

(a) $P_2(1)$ is true

(b) $P_2(n)$ true $\implies$ $P_2(2n)$ true $(n = 1, 2, \ldots)$

(c) $P_2(n+1)$ true $\implies$ $P_2(n)$ true (!) $(n = 1, 2, \ldots)$

then $P_2(n)$ is true for $n = 1, 2, \ldots$.

Convergence of sequences and series

(10†) Define a sequence $(t_n)_{n \in \mathbb{N}}$ by $t_n = \dfrac{2n+1}{n^3}$. Prove that this sequence tends to 0 as $n \to \infty$.

(11*) Define a sequence $(a_n)_{n\in\mathbb{N}}$ by $a_n = \dfrac{2n^2-1}{n^3-2}$. Prove that this sequence tends to 0 as $n \to \infty$.

(12**) Prove that the sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_n = \dfrac{2n+\sin n}{3n}$ tends to a limit as $n \to \infty$.

(13) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ tends to the limit A, while the sequence $(b_n)_{n\in\mathbb{N}}$ tends to B. Prove that the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ tends to $A + B$.

(14) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ tends to the limit A, while the sequence $(b_n)_{n\in\mathbb{N}}$ tends to B. Prove that the sequence $(a_n \cdot b_n)_{n\in\mathbb{N}}$ tends to AB.

(15) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ tends to a limit A, and the sequence $(b_n)_{n\in\mathbb{N}}$ tends to a limit B. Does the sequence $a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$ tend to a limit?

(16) Prove that the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2+1}$ converges.

(17) Prove that the series $\sum_{n=1}^{\infty} \dfrac{1}{n+100000}$ diverges.

(18) (Harder) You know (e.g. from the Group Theory course) that the rationals are countable. This means that there is a sequence $(t_n)_{n\in\mathbb{N}}$ that contains each rational number exactly once. (In fact there are many such sequences, obtained by re-ordering $(t_n)_{n\in\mathbb{N}}$ in any way you want to.)

- Prove that $(t_n)_{n\in\mathbb{N}}$ does not tend to a limit.
- On the other hand, prove that for every real number q there is a subsequence of $(t_n)_{n\in\mathbb{N}}$ that tends to q.

(19) Evaluate the recurring decimal $0.142857142857\ldots$ exactly as a rational.