The University of Edinburgh 2010

School of Mathematics (U01457)

Geometry & Convergence

Problem Sheet 4

Assessment 4 due by 12.10 on Friday, 26 February 2010.
Tutorial 4 on Tuesday, 23 February 2010.

Tutorial questions: 1, 3, and 5.

Handin questions: 2, and 4.

Conics

(1*) Put the following conics into standard form.

(i) $\mathcal{X}_0 : 7y^2 + 2xy + 7x^2 = 1$.
(ii) $\mathcal{X}_1 : 7y^2 + 2xy - y + 7x^2 + 11x = 1$.
(iii) What is the length of the semi-minor (resp. semi-majour) axis of $\mathcal{X}_0$?
(iv) What is the centre of $\mathcal{X}_1$?

(2†) Put the following conics into standard form.

(i) $\mathcal{X}_0 : 86y^2 - 96xy + 114x^2 = 1$.
(ii) $\mathcal{X}_1 : 86y^2 - 96xy + 45y + 114x^2 + 65x = 1$.

(3*) Put the following centred conics into standard form simultaneously.

$$\mathcal{X}_0 : 95y^2 + 216xy + 130x^2 = 1,$$
$$\mathcal{X}_1 : 222y^2 + 480xy + 278x^2 = 1.$$

Do these conics intersect?

(4†) Put the following centred conics into standard form simultaneously.

$$\mathcal{X}_0 : 30y^2 + 32xy + 9x^2 = 1,$$
$$\mathcal{X}_1 : 12y^2 + 20xy + 9x^2 = 1.$$

Do these conics intersect?

Induction

(5*) Prove by induction that $n^2 - n + 2$ is always even for $n = 1, 2, \ldots$.

(6) Prove by induction that, for $n = 1, 2, \ldots$

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(7) Prove by induction that $3^n > 5n^2$ for $n \geq 4$.

(8) Prove that $\cos(n\pi) = (-1)^n (n = 0, \pm 1, \pm 2, \ldots)$, by induction.

(9) In a computer memory, an arbitrary length vector v_n stores sequentially all the previous vectors $v_1, v_2, \ldots, v_{n-1}$, so the length ℓ_n of v_n satisfies

$$\ell_n = \ell_1 + \ldots + \ell_{n-1} \qquad (n = 2, 3, \ldots.$$

Given that $\ell_1 = 1$, prove by induction that $\ell_n = 2^{n-2}$ for $n = 2, 3, \ldots$.

What about $n = 1$?