The University of Edinburgh
2010

School of Mathematics
(U01457)

Geometry & Convergence

Problem Sheet 3

Assessment 3 due by 12.10 on Friday, 12 February 2010.
Tutorial 3 on Tuesday, 9 February 2010.

Pretutorial questions: 2, and 7.

Tutorial questions: 3, and 5.

Handin questions: 1, 4, and 8.

($1^\dagger$) (i) In the following question, $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{b} = -5\mathbf{i} + 2\mathbf{k}$ and $\mathbf{c} = -4\mathbf{k} + \mathbf{i}$.

(a) Let $\Pi = \{P \in \mathbb{R}^3 \;:\; \mathbf{p} = t\mathbf{a} + s\mathbf{b}, s, t \in \mathbb{R}\}$. Compute a normal vector to Π.

(b) Compute the distance from Π to the point C.

(ii) Compute the volume of the parallelepiped spanned by the vectors $\mathbf{a} = \langle -1, 0, 4\rangle$, $\mathbf{b} = \langle 3, 4, 0\rangle$ and $\mathbf{c} = \langle -5, 0, 2\rangle$.

(iii) Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. Is it true that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

for all $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$? If yes, demonstrate this; if no, give an example where the two differ.

(2^{**}) (i) Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ be a fixed vector in $\mathbb{R}^3$. Show that for $\mathbf{x} = \langle x, y, z\rangle$, the map f from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $f(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$ is a linear map.

(ii) Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle, \mathbf{b} = \langle b_1, b_2, b_3\rangle$ be fixed vectors in $\mathbb{R}^3$. Show that for $\mathbf{x} = \langle x, y, z\rangle$, the maps f and g from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $f(\mathbf{x}) = \mathbf{a} \times (\mathbf{b} \times \mathbf{x})$ and $g(\mathbf{x}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{x}$ are both linear maps.

(iii) Find the 3×3 matrices A, B such that $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{x}) = B\mathbf{x}$.

What are these matrices when $\mathbf{a} = \langle 1, 0, 3\rangle$ and $\mathbf{b} = \langle 2, 1, 1\rangle$?

(3^*) (i) Find a 3×3 orthogonal matrix all of whose entries are $\pm 1/3$ or $\pm 2/3$.

[Recall that rows have length 1, and are orthogonal to each other.]

(ii) Find a 4×4 orthogonal matrix all of whose entries are $\pm 1/2$.

($4^\dagger$) Suppose that A is an $n \times n$ matrix with the property that each row contains exactly one 1 and each column contains exactly one 1, and all other entries are 0. Prove that A is an orthogonal matrix.

(5^*) For the rotation map $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and the reflection map $M_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}$ verify by matrix multiplication that $M_\phi R_\theta = M_{\phi-\theta}$ and that $R_\theta M_\phi = M_{\phi+\theta}$. Use these identities to show, without any further matrix multiplication that $R_{-\theta} M_\phi R_\theta = M_{\phi-2\theta}$.

(6) Find the two points of intersection of the ellipse $x^2 + 2y^2 = 3$ with the line $4x - 3y = 1$.

(7^{**}) Find the four points of intersection of the two ellipses

$$E_1 : \frac{y^2}{81} + x^2 = 1, \qquad E_2 : y^2 + \frac{x^2}{16} = 1.$$

($8^\dagger$) For which values of $b > 0$ do the two ellipses E_1 and E_2 intersect?

$$E_1 : \frac{y^2}{b^2} + x^2 = 1 \qquad E_2 : y^2 + \frac{x^2}{16} = 1.$$

(9) For the ellipse $x^2/9 + y^2/4 = 1$, use the standard formulae to calculate its eccentricity e. Find both foci, and both directrixes. Check that for the point $(\frac{3}{2}\sqrt{3}, 1)$ on this ellipse:

- its distance from the left focus is e times its distance from the left directrix;
- its distance from the right focus is e times its distance from the right directrix;
- the sum of its distances to both foci is equal to the length of its major axis (i.e. $2a = 6$).

(10) Imagine the parabola $y^2 = 4x$ as a concave mirror. What is its focus? A light ray comes in from the right along the line $y = 3$. Find the slope of the tangent at the point where the light ray hits the mirror. Call this slope $\tan\phi$. Assuming that the light ray is reflected at this point in the usual way (i.e. so that the arriving and reflected ray make the same angle with this tangent), show that the slope of the reflected ray is $\tan(2\phi)$. Hence evaluate this slope, and so find the equation of the reflected ray. Show that this ray passes through the focus of the parabola.

(11) (Continuation of previous question.) Where does this ray hit the mirror again? Show that the tangent at this second point, of slope $\tan\phi'$ say, is such that $\tan(2\phi') = \tan(2\phi)$. Deduce without further calculation that, after being reflected again, the ray travels out to the right horizontally along the line $y = -4/3$.

(12) (Generalising the previous two questions.) Repeat these questions, but now for the general parabola $y^2 = 4ax$, and a horizontal ray $y = 2c$ coming in from the right. Show that the ray always passes through the focus, whatever the values of a and c. Show that after being reflected a second time it comes out along the line $y = -2a^2/c$.

(13) Use a suitable rotation transformation $\begin{bmatrix} x \\ y \end{bmatrix} = R_\theta \begin{bmatrix} x' \\ y' \end{bmatrix}$ to put the conic $2x^2 + 4xy + 5y^2 = 1$ into standard form. Give R_θ explicitly.

(14) Use the results of the previous question, and a suitable translation, to put the conic $2x^2 + 4xy + 5y^2 + 6x + 4y = 1$ into standard form. Give explicitly the mapping between the old and new coordinate variables x, y and x'', y''.

(15) Use a suitable rotation transformation $\begin{bmatrix} x \\ y \end{bmatrix} = R_\theta \begin{bmatrix} x' \\ y' \end{bmatrix}$ to put the conic $4x^2 - 4xy + 7y^2 = 1$ into standard form. Give R_θ explicitly.

(16) Use the results of the previous question, and a suitable translation, to put the conic $4x^2 - 4xy + 7y^2 + 4x + 2y = 1$ into standard form. Give explicitly the mapping between the old and new coordinate variables x, y and x'', y''.