The University of Edinburgh
2010

School of Mathematics
(U01457)

Geometry & Convergence

Problem Sheet 2

Assessment 2 due by 12.10 on Friday, 29 January 2010.
Tutorial 2 on Tuesday, 26 January 2010.

Pretutorial questions: 2, and 4.

Tutorial questions: 7, 8, and 16.

Handin question: 3.

(1) (a) For the line $7x - 3y = 1$, write down a normal to the line, and a point on the line. Find the distance of the point $(3, 4)$ from the line.

(b) More generally, for the line $ax + by = c$, write down a normal, and show that the point $(ac/(a^2 + b^2), bc/(a^2 + b^2))$ lies on the line. Deduce that the distance of the point (x_0, y_0) from the line is $|ax_0 + by_0 - c|/\sqrt{a^2 + b^2})$.

(2**) (a) Which points in the plane $\mathbb{R}^2$ are equidistant from the x- and y-axes?

(b) Find the point that is equidistant from the x-axis, the y-axis and the line $3x + 4y = 36$.

(c) What is the largest radius of a circle that will fit inside the triangle specified by the three lines in (b)? (This circle is called the *incircle* of the triangle, and its centre is the *incentre* of the triangle).

($3^\dagger$) (a) Given a line in $\mathbb{R}^3$ in parametric form $\mathbf{a} + t\mathbf{v}$, and a point $\mathbf{b}$ in $\mathbb{R}^3$, sketch the plane containing $\mathbf{b}$ and the line.

(b) Deduce that the distance of $\mathbf{b}$ from the line is $|\mathbf{a} - \mathbf{b} - \lambda\mathbf{v}|$, where $\lambda\mathbf{v}$ is the component of $\mathbf{a} - \mathbf{b}$ in the direction $\mathbf{v}$.

(c) How far is the point $(1, 2, 3)$ from the line $(1, 0, -1)+t(-1, 3, 2)$?

(d) This method works for a point and a line in $\mathbb{R}^n$ for any n. In $\mathbb{R}^4$, how far is the point $(1, -1, 1, 0)$ from the line $(0, 1, 1, 0) + t(1, 1, 0, -1)$?

(4**) If $\mathbf{a} \neq 0$ and $\mathbf{a} \times \mathbf{b} = 0$, prove that $\mathbf{b}$ is a scalar multiple of $\mathbf{a}$.

(5) Verify by expanding both sides that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

(6) Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

(7*) Solving the equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$, for given vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$.

(a) Show that if $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ then $\mathbf{a} \cdot \mathbf{b} = 0$. We assume that this condition holds for the rest of the question.

(b) Use Q4 above to show that $-(\mathbf{a} \times \mathbf{b})/|\mathbf{a}|^2$ is a solution of $\mathbf{a} \times \mathbf{x} = \mathbf{b}$.

(c) If $\mathbf{x} = \mathbf{u} - (\mathbf{a} \times \mathbf{b})/|\mathbf{a}|^2$ is another solution, show that $\mathbf{u} \times \mathbf{a} = 0$. Hence use Q3 above to write down the general solution $\mathbf{x}$ to $\mathbf{a} \times \mathbf{x} = \mathbf{b}$.

[In this question we've seen that the solution set $\{\mathbf{x} : \mathbf{a} \times \mathbf{x} = \mathbf{b}\}$ is either empty (when $\mathbf{a} \cdot \mathbf{b} \neq 0$) or a line. Note the contrast with the equation $\mathbf{a} \cdot \mathbf{x} = b$, whose solution set is a plane.]

(8*) (Converse to question 6). Show that a given line $\mathbf{w} + t\mathbf{a}$ in $\mathbb{R}^3$ is the solution set of the equation $\mathbf{a} \times \mathbf{x} = \mathbf{a} \times \mathbf{w}$.

(9) Consider the points $P = (7, 0, -1)$, $Q = (2, 5, 4)$, $R = (2, -4, -2)$. Find

(a) the angle between $\overrightarrow{QP}$ and $\overrightarrow{QR}$;

(b) the parametric form for the line joining the points P and Q;

(c) the equation of the plane containing P, Q and R.

(10) If $\mathbf{a}$ and $\mathbf{b}$ lie in the xy-plane show that $|[\mathbf{a}, \mathbf{b}, \mathbf{k}]|$ equals the area of the parallelogram generated by $\mathbf{a}$ and $\mathbf{b}$.

(11) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of three points P, Q and R. Show that P, Q and R lie on a line (then we say that they are *collinear*) if and only if $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = \mathbf{0}$.

(12) Suppose $\mathbf{n} \cdot \mathbf{x} = d$ and $\mathbf{n}' \cdot \mathbf{x} = d'$ are two non-parallel planes.

(a) Show that $[\mathbf{n}, \mathbf{n}', \mathbf{n} \times \mathbf{n}'] \neq 0$.

(b) Let A be the matrix with rows given by $\mathbf{n}$, $\mathbf{n}'$ and $\mathbf{n} \times \mathbf{n}'$ respectively. Explain why A is invertible.

(c) Consider the vector $\mathbf{b} = (d, d', 0)$ and let $\mathbf{a} = A^{-1}\mathbf{b}$. Show that $\mathbf{a} + t\mathbf{n} \times \mathbf{n}'$ gives the intersection line of the two planes.

(d) Can we choose other vectors for $\mathbf{b}$? Explain.

(13) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be two linear maps. Show that the function h defined by $h(\mathbf{a}) = f(\mathbf{a}) + g(\mathbf{a})$ is also linear. *[hint: Consider $h(\lambda\mathbf{a} + \mu\mathbf{b})$.]*

(14) For the following maps (a) identify a suitable domain and range, and (b) state with reasons if the map is linear or not ($\mathbf{b}$ and $\mathbf{c}$ are fixed non-zero vectors and λ is a scalar):

(a) $f(\mathbf{x}) = \mathbf{0}$

(b) $f(\mathbf{x}) = \mathbf{x} + \mathbf{b}$

(c) $f(\mathbf{x}) = \lambda\mathbf{x}$

(d) $f(\mathbf{x}) = (\mathbf{b} \cdot \mathbf{x})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{x}$

(e) $f(\mathbf{x}) = (\mathbf{b} \cdot \mathbf{x})\mathbf{c} + (\mathbf{b} \cdot \mathbf{x})\mathbf{x}$

(f) $f(\mathbf{x}) = [\mathbf{x}, \mathbf{b}, \mathbf{c}]$

(g) $f(\mathbf{x}) = |\mathbf{x}|$

(15) For those functions in the previous question which are linear maps find their corresponding matrices and decide whether they are invertible or not.

(16*) For the plane given parametrically by $\imath(t,u) = (3,0,0) + t(-3,4,0) + u(-3,0,6)$ write down three points on the plane that don't lie on one line. Use these to find a normal to the plane. Find the distance of the point $(1,2,3)$ from the plane.

(17) Show directly that for the projection matrix $P_\theta = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

(a) $P_\theta^2 = P_\theta$ and

(b) $(P_\theta P_\phi)^2 = \lambda P_\theta P_\phi$, where $|\lambda| \leq 1$ is some real number depending on θ and ϕ.

Explain these facts geometrically.

(18) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map which is not the identity and satisfies $f \circ f = f$. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the 2×2 matrix corresponding to f. Show that

(a) $\det P = 0$,

(b) the eigenvalues of P are either 0 or 1,

(c) $0 \leq a \leq 1$ and $0 \leq d \leq 1$.

(d) Either

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } P = \begin{bmatrix} \cos^2\theta & \alpha \\ \frac{\cos^2\theta\sin^2\theta}{\alpha} & \sin^2\theta \end{bmatrix}$$

for some angle $0 \leq \theta \leq \pi/2$ and non-zero real number α.

(e) Find the eigenvectors of P and show that the eigenvectors are orthogonal to each other if and only if P is the projection to a line through the origin.