The University of Edinburgh School of Mathematics
2010 (U01457)

Geometry & Convergence

Problem Sheet 1

Assessment 1 due by 12.10 on Friday, 15 January 2010.

Tutorial questions: 4, 6, and 14.

Handin questions: 1, and 3.

(1†) Sine and cosine rules. Consider the standard triangle with angles A, B, C opposite sides a, b, c. Drop a perpendicular from the angle A onto the side a. (Draw this.)

(a) Show that $b \sin C = c \sin B$. Deduce the sine rule $a/\sin A = b/\sin B = c/\sin C$.

(b) Show that the right-angled triangle with hypotenuse c has other sides $b \sin C$ and $a - b \cos C$. Deduce the cosine rule $c^2 = a^2 + b^2 - 2ab \cos C$.

(2) *"Angle at centre is twice that at circumference."*

(a) Draw a circle with centre O, and a triangle ABC whose vertices A, B, C lie on the circle. Join A and O, and let the angle $\angle OAB = \beta$, $\angle OAC = \gamma$. Find $\angle AOB$ in terms of β and $\angle AOC$ in terms of γ.
Deduce that $\angle BOC = 2(\beta + \gamma) = 2\angle BAC$.
What happens when AC is a diagonal of the circle?

(b) Let BC be a fixed line segment. A point A moves on the plane so that the angle $\angle BAC$ stays constant. It starts at B and ends at C. Show that A traces out an arc of a circle.

(3†) Let $\mathbf{a} = \langle -1, -1 \rangle, \mathbf{b} = \langle -2, 3 \rangle$. Calculate, in radians to 2 decimal places, the angles between

- $\mathbf{a}$ and $\mathbf{b}$;
- $\mathbf{a}$ and $\mathbf{a} - \mathbf{b}$;
- $\mathbf{b}$ and $\mathbf{b} - \mathbf{a}$.

What is the sum of these angles? Why?

(4*) Find the component of $\mathbf{c} = \langle 2, 1 \rangle$ in the direction $\mathbf{v} = \langle 4, 1 \rangle$. Hence write $\mathbf{c}$ in the form $\mathbf{c} = \lambda \mathbf{v} + \mathbf{w}$, where $\mathbf{v} \cdot \mathbf{w} = 0$.

Check also that $\mathbf{w}$ itself is the component of $\mathbf{c}$ in the direction $\mathbf{w}$.

Draw a picture to illustrate the question.

(5) Let $\mathbf{a} = \langle 1, 2 \rangle, \mathbf{b} = \langle 2, -3 \rangle$. Find the equation of the line given parametrically by
$(1 - t)\mathbf{a} + t\mathbf{b}$.

Which values of t describe the set of points on the line that are nearer to $\mathbf{a}$ than to $\mathbf{b}$?

Does the point $(6, 7)$ lie on the line?

Give examples of points on the line that lie

- between $\mathbf{a}$ and $\mathbf{b}$;
- on the side of $\mathbf{a}$ away from $\mathbf{b}$;
- on the side of $\mathbf{b}$ away from $\mathbf{a}$.

Find the two points on the line that are each twice as far from $\mathbf{a}$ as from $\mathbf{b}$.

(6*) Let $\mathbf{u} = \langle u_1, u_2 \rangle, \mathbf{v} = \langle v_1, v_2 \rangle$ be two independent vectors in $\mathbb{R}^2$ (i.e. neither is a multiple of the other), and put $\mathbf{u}^\perp = \langle -u_2, u_1 \rangle, \mathbf{v}^\perp = \langle -v_2, v_1 \rangle$.

Show that $\mathbf{u}^\perp \cdot \mathbf{u} = \mathbf{v}^\perp \cdot \mathbf{v} = 0$.

Writing $\mathbf{x} \in \mathbb{R}^2$ as a linear combination $\lambda \mathbf{u} + \mu \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$, show by taking appropriate dot products that

- $\lambda = (\mathbf{v}^\perp \cdot \mathbf{x})/(\mathbf{v}^\perp \cdot \mathbf{u})$;
- $\mu = (\mathbf{u}^\perp \cdot \mathbf{x})/(\mathbf{u}^\perp \cdot \mathbf{v})$.

Now try this procedure with the example $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle 2, -1\rangle$ and $\mathbf{x} = \langle 1, 1\rangle$. Check that indeed $\mathbf{x} = (\mathbf{v}^\perp \cdot \mathbf{x})/(\mathbf{v}^\perp \cdot \mathbf{u})\mathbf{u} + (\mathbf{u}^\perp \cdot \mathbf{x})/(\mathbf{u}^\perp \cdot \mathbf{v})\mathbf{v}$ for this example.

($6\frac{1}{2}$) In Q6, explain why $\mathbf{v}^\perp \cdot \mathbf{u}$ (and, similarly, $\mathbf{u}^\perp \cdot \mathbf{v}$) is nonzero.

(7) Consider three simultaneous equations

$$ax + by + cz = d,$$
$$kx + ly + mz = n,$$
$$px + qy + rz = s$$

for x, y and z. Each equation describes a plane. Assume that each equation describes a different plane.
Give a sketch of a possible arrangement of the three planes in each of the following cases:

(a) There is a unique solution of all three equations together;

(b) Each pair of equations has a solution, but there is no solution of all three together;

(c) No pair of equations has a solution;

(d) There are infinitely many solutions of all equations together;

(e) The fifth case!

(8) Consider the plane Π which contains the three points $(1, 0, -2)$, $(2, -1, 3)$ and $(0, 1, 1)$. Find the parametric form for Π and the equation for Π. *[hint: Pick one of the points as a base point and find two vectors in the plane relative to this base point.]*

(9) Consider the points $P = (1, 2, 3)$, $Q = (-1, 1, 3)$ the lines $\ell_1 = (2 + t, 1 + 2t, -t)$, $\ell_2 = (1 - 3t, 2 + 2t, 1 + t)$ and the planes Π_1 given by $2x - y - z = 2$, Π_2 given by $z - x = 3$. Find

(a) the distance between P and Q,

(b) the distance between ℓ_1 and ℓ_2, *[hint: Use the formula from lectures.]*

(c) the distance between Π_1 and Π_2,

(d) the distance from P to Π_1,

(e) the point where ℓ_1 meets Π_1,

(f) the line $\Pi_1 \cap \Pi_2$,

(10) Consider the line given by $\mathbf{a} + t\mathbf{d}$ and a point P with position vector $\mathbf{c}$. Let $f(t) = |\mathbf{a} + t\mathbf{d} - \mathbf{c}|^2$ be the given function of t. What is the geometrical interpretation of f?
Find the stationary point of f and deduce that the closest point on the line to P is given by $\mathbf{a} + t\mathbf{d}$ for this stationary value of t.
Show that for this value of t, $\mathbf{a} + t\mathbf{d} - \mathbf{c}$ is perpendicular to $\mathbf{d}$.

(11) Suppose $\mathbf{a}$ and $\mathbf{b}$ are two vectors in the xy-plane. Find the two real numbers λ and μ satisfying the equation $\mathbf{a} + \lambda\mathbf{a} \times \mathbf{k} = \mathbf{b} + \mu\mathbf{b} \times \mathbf{k}$. *[hint: Apply $\times\mathbf{v}$, with an appropriate vector $\mathbf{v}$, to both sides to eliminate the term with μ.]*

(12) Simplify $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.

(13) Suppose that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ in fact lie in a plane. Prove that

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0.$$

(14*) Consider the points $A = (1, 1, 1)$, $B = (1, -1, -1)$, $C = (-1, 1, -1)$, $D = (-1, -1, 1)$ in $\mathbb{R}^3$.

(a) Show that they are all equidistant from the origin $O = (0, 0, 0)$. What is this distance?

(b) Show that they are equidistant from each other, and so form the vertices of a regular tetrahedron. What is this distance?

(c) Find the angle $\angle AOB$.
(This is the so-called *tetrahedral angle*, and is e.g. the angle

subtended at the carbon atom by two hydrogen atoms in a methane molecule CH_4.)