MANIFOLDS OF INFINITE TOPOLOGICAL TYPE WITH INTEGRABLE GEODESIC FLOWS

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Abstract. Let $(M, \mathbf{k})$ be a complete surface of constant negative curvature (resp. an $\widetilde{SL}(2;\mathbf{R})$-geometric 3-manifold). This paper constructs a complete riemannian 8-manifold (resp. 9-manifold) $(\Sigma, \mathbf{h})$ such that Σ is homotopy equivalent to M, the geodesic flow of $\mathbf{h}$ is completely integrable and there is a riemannian embedding $(M, \mathbf{k}) \hookrightarrow (\Sigma, \mathbf{h})$. This embeds the geodesic flow of $(M, \mathbf{k})$ as a subsystem of an integrable geodesic flow. Amongst the manifolds Σ is an 8-dimensional manifold whose fundamental group is the free group on countably many generators.
AMS Subject Classification (2000): 58F17, 53D25, 37D40
Keywords: geodesic flows, integrable systems, $\widetilde{SL}(2;\mathbf{R})$, geometric manifolds

1. Introduction

A smooth flow $\varphi_t : M \to M$ is *integrable* if there is a dense subset L whose components are fibred by φ_t-invariant toroidal cylinders and the fibre-bundle trivializations conjugate φ_t to a translation-type flow on the toroidal cylinders; the flow is *tamely integrable* if $M - L$ is a tamely embedded polyhedron. It seems obvious that integrable flows cannot be 'interesting' from a dynamical point-of-view, and 'interesting' dynamical systems are generally non-integrable. Two prototypical 'interesting' dynamical systems are hyperbolic toral automorphisms and the geodesic flow of a compact surface of constant negative curvature. In [6], Bolsinov and Taĭmanov show that a hyperbolic 2-torus automorphism embeds as a subsystem of the time-1 map of an integrable geodesic flow on a compact 3-manifold. The present paper embeds the geodesic flow of a complete surface of constant negative curvature as a subsystem of an integrable geodesic flow on a non-compact 8-manifold. It is an unknown which symplectic diffeomorphisms embed as a subsystem of the time-1 map of an integrable geodesic flow. However, [13] shows that each symplectic diffeomorphism embeds as a subsystem of the time-1 map of an integrable hamiltonian flow.

The topology of the riemannian manifolds in this paper is also appreciably more complicated than previously-constructed examples. The Bolsinov-Taĭmanov example is on a compact manifold whose Betti numbers are dominated by those of $\mathbf{T}^3$ and whose fundamental group is polycyclic. The present paper's examples include (non-compact) manifolds whose first k Betti numbers are infinite, and whose fundamental group is a free group on countably many generators. Integrable geodesic flows are also constructed on manifolds homotopy equivalent to any surface of genus greater than

Thanks to Keith Burns and Leo Jonker for comments. Research partially supported by the Natural Sciences and Engineering Research Council of Canada.

one. The topology of these manifolds is compelling in light of a topological obstruction to integrability in the class of real-analytic integrals on compact manifolds [32]: the fundamental group must be almost abelian.

Results: $\mathrm{O}(2,1)$ has a subgroup of index two, denoted by $\mathrm{O}_+(2,1) =: \mathbf{G}$, that leaves invariant the upper-half space $\mathbb{H}^2 = \{\mathbf{v} \in \mathbf{R}^3 \ : \ v_1^2 + v_2^2 - v_3^2 = -1, v_3 > 0\}$. Let $\mathbf{H} = \mathbf{G}^3$, $\delta(g) = (g,g,g)$ be the "diagonal" embedding of $\mathbf{G}$ in $\mathbf{H}$ and p_i be the projection of $\mathbf{H}$ onto the i-th copy of $\mathbf{G}$. For each left-invariant metric $\mathbf{g}$ on $\mathbf{G}$, there is a unique left-invariant metric $\mathbf{h}$ on $\mathbf{H}$ such that for all i

$$\begin{array}{ccccc} (\mathbf{G},\mathbf{g}) & \xrightarrow{\delta} & (\mathbf{H},\mathbf{h}) & \xrightarrow{p_i} & (\mathbf{G},\frac{1}{3}\mathbf{g}) \\ \downarrow & & \downarrow & & \downarrow \\ (\Gamma\backslash\mathbf{G},\mathbf{g}) & \longrightarrow & (\Delta\backslash\mathbf{H},\mathbf{h}) & \longrightarrow & (\Gamma\backslash\mathbf{G},\frac{1}{3}\mathbf{g}) \end{array}$$

commutes in the category of riemannian manifolds. Γ is a lattice subgroup of $\mathbf{G}$ and $\Delta = \delta(\Gamma)$.

Theorem 1.1. *The geodesic flow of* $(\Sigma = \Delta\backslash\mathbf{H},\mathbf{h})$ *is tamely integrable, has positive topological entropy and admits the geodesic flow of* $(\Gamma\backslash\mathbf{G},\mathbf{g})$ *as a subsystem.*

Although $(\Sigma,\mathbf{h})$ is non-compact and has infinite riemannian volume, the non-wandering set of its geodesic flow has a non-empty interior. Thus the geodesic flow's first integrals do not originate from a boundary at infinity – it does not exist – or similar *léger de main*. For the definition of topological entropy, see [7].

Assume $\mathbf{g}$ is $\mathbf{K} = O(2)$-invariant and $\Gamma \times \mathbf{K}$ acts uniformly discretely on $\mathbf{G}$. There are unique metrics such that for all i

$$\begin{array}{ccccc} (\mathbf{G},\mathbf{g}) & \xrightarrow{\delta} & (\mathbf{H},\mathbf{h}) & \xrightarrow{p_i} & (\mathbf{G},\frac{1}{3}\mathbf{g}) \\ \downarrow & & \downarrow & & \downarrow \\ (\Gamma\backslash\mathbf{G}/\mathbf{K},\mathbf{g}_s) & \longrightarrow & (\Delta\backslash\mathbf{H}/\mathbf{L},\mathbf{h}_s) & \longrightarrow & (\Gamma\backslash\mathbf{G}/\mathbf{K},\frac{1}{3}\mathbf{g}_s) \end{array}$$

commutes, where $\mathbf{L} = \delta(\mathbf{K})$. Each complete surface $(M,\mathbf{k})$ of constant negative curvature is isometric to a $(\Gamma\backslash\mathbf{G}/\mathbf{K},\mathbf{g}_s)$ so

Theorem 1.2. *For each complete surface* $(M,\mathbf{k})$ *of constant negative curvature there is a riemannian covering space* $(\Lambda,\mathbf{l})$ *of* $(\Delta\backslash\mathbf{H}/\mathbf{L},\mathbf{h}_s)$ *such that* Λ *is homotopy equivalent to* M*, the geodesic flow of* $\mathbf{l}$ *is tamely integrable, and the the geodesic flow of* $\mathbf{k}$ *is a subsystem of* $\mathbf{l}$*'s.*

Applied to $M = \mathbf{R}^2 - \mathbf{Z}^2$, Theorem 1.2 implies that

Corollary 1.1. *there is an* 8*-dimensional real-analytic manifold* Λ *such that:*

- $\pi_1(\Lambda)$ *is a free group on a countably infinite number of generators;*
- $H_1(\Lambda;\mathbf{Z}) = \oplus_{k=1}^{\infty}\mathbf{Z}$*; and*
- Λ *admits a real-analytic, tamely integrable geodesic flow.*

The manifolds Λ^m (product of m copies) admit integrable geodesic flows and have $H_s(\Lambda^m;\mathbf{Z}) = \oplus_{k=1}^{\infty}\mathbf{Z}$ for $s = 1,\ldots,m$.

$(M, \mathbf{k})$ is a *geometric manifold* if the isometry group of its universal riemannian covering acts transitively. Geometric 3-manifolds have a universal riemannian covering space isometric to $\mathbb{E}^3$, S^3, $S^2 \times \mathbb{E}^1$, Nil, Sol, $\widetilde{\mathrm{SL}}(2;\mathbf{R})$, $\mathbb{H}^2 \times \mathbb{E}^1$ or $\mathbb{H}^3$ [31, 36]. Since $\widetilde{\mathrm{SL}}(2;\mathbf{R})$ is locally isomorphic to the connected component of $O_+(2,1)$ containing the identity, Theorems 1.1 and 1.2 along with [12, 6, 14], imply

Corollary 1.2. *Let $(M, \mathbf{k})$ be a compact geometric 3-manifold whose universal riemannian cover is not $\mathbb{H}^3$. There is a real-analytic riemannian manifold $(\Sigma, \mathbf{h})$ such that Σ is homotopy equivalent to M, the geodesic flow of $\mathbf{h}$ is tamely integrable and the geodesic flow of $\mathbf{k}$ is a subsystem of $\mathbf{h}$'s.*

Except for the geometric $\widetilde{\mathrm{SL}}(2;\mathbf{R})$- and $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds, the geodesic flow of $(M, \mathbf{k})$ is itself completely integrable. It is unknown if the compact geometric $\mathbb{H}^3$-manifolds have a representative in their homotopy class that admits an integrable geodesic flow.

A 3-manifold S is a Seifert manifold if each point has a neighbourhood homeomorphic to the solid torus modulo the action of a finite group; S fibres over a 2-orbifold M, with fibres $\mathbf{T}^1$. Compact Seifert manifolds admit the structure of a geometric manifold, and the geometry is determined by two invariants: the Euler characteristic of M, $\chi(M)$, and the Euler class of ξ, $e(\xi)$ (see Table 1). Corollary 1.2 implies

Corollary 1.3. *If $\xi : S \to M$ is a compact Seifert manifold, then there is a real-analytic riemannian manifold $(\Sigma, \mathbf{h})$ such that Σ is homotopy equivalent to S and the geodesic flow of $\mathbf{h}$ is tamely integrable.*

	$\chi(M) > 0$	$\chi(M) = 0$	$\chi(M) < 0$
$e(\xi) = 0$	$S^2 \times \mathbb{E}^1$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{E}^1$
$e(\xi) \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}}(2;\mathbf{R})$

TABLE 1. Compact Seifert manifolds and their geometries [31].

Theorems 1.1 and 1.2 are stated solely for the geodesic flow of a riemannian metric, but the techniques of the present paper yield integrable geodesic flows of sub-riemannian and non-riemannian Finsler metrics on each of the manifolds. See section 5.4.

A continuing question: If the geodesic flow of $(\Sigma, \mathbf{g})$ is tamely integrable, let $r(\mathbf{g})$ be the minimum homological dimension of the geodesic flow's invariant toroidal cylinders; otherwise it is $-\infty$. Let $\mathbf{r}(\Sigma)$ equal $\max_{\mathbf{g}} r(\mathbf{g})$. Define $\kappa(\Sigma) = \min\ \dim \Sigma' - \mathbf{r}(\Sigma')$, taken over all manifolds Σ' homotopy equivalent to Σ; $\kappa(\Sigma) = 0$ iff a homotopy equivalent manifold Σ' admits a tamely integrable geodesic flow with compact lagrangian tori and $\kappa(\Sigma) = \infty$ iff there are no tamely integrable geodesic flows on any manifold in Σ's homotopy class. In [14], it is shown that if Σ is a 3-manifold whose fundamental group is not almost polycyclic, then $\mathbf{r}(\Sigma) \leq 2$. We are led to ask:

Question: if Σ is a compact $\widetilde{\mathrm{SL}}(2;\mathbf{R})$, $\mathbb{H}^2 \times \mathbb{E}^1$ or $\mathbb{H}^3$-manifold, is $\mathbf{r}(\Sigma) = 2$? $\kappa(\Sigma) < \infty$?

Related Work: Topologically necessary conditions for integrability of geodesic flows are known, at least when the first-integral map is sufficiently well-behaved. Kozlov [24] shows that the only compact surfaces with real-analytically integrable geodesic flows have genus 0 or 1, i.e. only $S^2, \mathbf{R}P^2, \mathbf{T}^2$ or the Klein bottle admit real-analytically integrable geodesic flows. Bolotin generalizes Kozlov's argument to non-compact surfaces, with some additional hypotheses on the behaviour of the metric at infinity [2, 25]. Bolotin and Bolotin & Negrini show that if $(\Sigma, \mathbf{g})$ is a compact real-analytic surface of genus greater than 1, then in a neighbourhood of any non-trivial minimal periodic orbit there is a horseshoe [3, 4]. Taĭmanov generalizes Kozlov's theorem to higher dimensions and shows that if Σ is a compact real-analytic manifold that admits a real-analytically integrable geodesic flow, then $\pi_1(\Sigma)$ contains a finite index abelian subgroup, and $H^*(\Sigma)$ contains a subalgebra isomorphic to $H^*(\mathbf{T}^b)$ where b is the first Betti number of Σ [32, 33, 34]. Paternain has also obtains interesting results using an *entropy* approach [28, 29, 30]. Paternain's approach is quite independent the preceding work, and does not give necessary conditions for real-analytic integrability except in dimension two. Indeed, as far as this author knows, it is an open question if real-analytic integrability implies vanishing topological entropy. The present author and Bolsinov and Taĭmanov [10, 12, 11, 13, 15, 6] have found real-analytic geodesic flows on compact manifolds that are only C^∞ integrable. The topology of these manifolds is complicated — in the examples of [6, 15] the fundamental groups have exponential word growth — *but* still quite controlled. As mentioned above, for example, the fundamental groups of all manifolds in these examples are *polycyclic* or almost polycyclic,[1] and $\dim H^k(\Sigma) \leq \dim H^k(\mathbf{T}^n)$ for all k ($n = \dim \Sigma$). In addition, in section 4 of [14] we show that, for an appropriate choice of algebra of first integrals $\mathcal{F}$, the singular sets of these flows are real-analytic varieties. Despite this, the flows on the singular sets of the examples in [6, 15] contain an Anosov subsystem and have positive topological entropy like the examples in the present paper.

Additional papers that construct integrable geodesic flows include those by Thimm, Guillemin and Sternberg, Brailov, Paternain and Spatzier, Kiyohara, Bazaikin, and Dullin and Matveev and Topalov [35, 18, 19, 8, 9, 27, 22, 23, 1, 17]. These papers deal with geodesic flows on manifolds with a finite fundamental group and the techniques do not immediately generalize to the present setting.

2. Background

Let P be an analytic manifold. A *Poisson bracket* on P is a skew-symmetric bracket $\{,\}$ that makes $(C^\infty(P), \{,\})$ into a Lie algebra of derivations of $C^\infty(P)$. This bracket is induced by a skew-symmetric bundle map $\mathcal{P} : T^*P \to TP$. A smooth function H naturally induces a hamiltonian vector field $Y_H = \{., H\}$. If $\mathcal{F} \subset C^\infty(P)$, let $d\mathcal{F}_p = \mathrm{span}\,\{df_p \ : \ f \in \mathcal{F}\}$ and $Z(\mathcal{F}) = \{f \in \mathcal{F} \ : \ \{\mathcal{F}, f\} \equiv 0\}$. When $\mathcal{F}$ is a Lie subalgebra, $Z(\mathcal{F})$ is the centre of $\mathcal{F}$.

[1] Γ is almost $\mathcal{S}$ if it contains a finite index subgroup $H \in \mathcal{S}$.

Say that $c \in N$ is a *strongly regular value* of the smooth map $F : M \to N$ if there is an open neighbourhood $U \ni c$ such that $F : F^{-1}(U) \to U$ is a trivial fibre bundle; the points in $F^{-1}(U)$ are *strongly regular points*. Let $k = \sup \dim d\mathcal{F}_p$, $l = \sup \dim \mathcal{P}_p\, dZ(\mathcal{F})_p$. Say $p \in P$ is *$\mathcal{F}$-regular* if there exist $f_1, \ldots, f_k \in \mathcal{F}$ such that p is a strongly regular point for the map $F = (f_1, \ldots, f_k)$, $f_1, \ldots, f_l \in Z(\mathcal{F})$ and $\{Y_{f_i}(p) \ : \ i = 1, \ldots, l\}$ is linearly independent; if p is not $\mathcal{F}$-regular then it is *$\mathcal{F}$-critical*. Let $L(\mathcal{F})$ be the set of $\mathcal{F}$-regular points.

Definition 2.1. *$H \in C^\infty(P)$ is* **integrable** *if there is a Lie subalgebra $\mathcal{F} \subset C^\infty(P)$ such that*

(I1) *$H \in Z(\mathcal{F})$;*
(I2) *$k + l = \dim P$ and $L(\mathcal{F})$ is an open and dense subset of P; and*
(I3) *for each $f \in Z(\mathcal{F})$ the vector field Y_f is complete.*

In this case, $\mathcal{F}$ is an **integrable subalgebra**. *If $\mathcal{F}^\omega = \mathcal{F} \cap C^\omega(T^*\Sigma)$ is also an integrable subalgebra and $H \in \mathcal{F}^\omega$, then H is real-analytically integrable.*
H is **tamely integrable** *if $P - L(\mathcal{F})$ is contained in a nowhere-dense tamely-embedded polyhedron.*

See [5] for an analogous definition and explanation.
Note that the common level sets of $\mathcal{F}$ are not necessarily compact.

Here is why the integrability of H in the sense of Definition 2.1 implies that the flow Φ_t of Y_H is integrable in the sense of the first sentence of the present paper: Let G denote the abelian group of C^∞ diffeomorphisms of P generated by the complete flows of Y_f, $f \in Z(\mathcal{F})$. $Z(\mathcal{F})$ induces a nonsingular distribution on $L(\mathcal{F})$, and by the Sussman-Stefan orbit theorem [21], the G-orbits are immersed C^∞ submanifolds. I2 implies that these orbits are embedded submanifolds. I2 and I3 imply that each $p \in L(\mathcal{F})$ has a G-invariant open neighbourhood U and there is an action of $\mathbf{R}^l$ on U whose orbits coincide with G's. So $G.p \simeq \mathbf{R}^l/\mathbf{P}_p$ where $\mathbf{P}_p$ is a discrete subgroup of rank $r \leq l$. Since the topology of the G-orbits in $L(\mathcal{F})$ is locally constant, a continuation-type argument implies that rank $\mathbf{P}_p$ is constant on the connected components of $L(\mathcal{F})$ (*c.f.* [26, 16]). So there is a C^∞ atlas $\mathcal{A} = \{\varphi : V \to \mathbf{R}^l/\mathbf{Z}^r \times \mathbf{D}^k\}$ of $L(\mathcal{F})$, where $\mathbf{D}^k$ is an open disk in $\mathbf{R}^k$, and $r = r(V)$ is a constant. $\mathcal{A}$ satisfies the universal property that, for each chart (φ, V) and 1-parameter subgroup g^t of G, $\varphi \circ g^t \circ \varphi^{-1}(x, y) = (x + t\xi(y), y)$ for all $x \in \mathbf{R}^l/\mathbf{Z}^r$, $y \in \mathbf{D}^k$ where $\xi : \mathbf{D}^k \to \mathbf{R}^l$ is C^∞. By I1, Φ_t is a 1-parameter subgroup of G, so the flow Φ_t is integrable.

A hamiltonian vector field is Liouville (or completely) integrable (resp. non-commutatively integrable) if it is integrable in the sense of Definition 2.1 with $\mathcal{F} = \operatorname{span}\{f_1, \ldots, f_k\}$ and $l = k$ (resp. $l \leq k$) and the regular-point set of $F = (f_1, \ldots, f_k)$ is dense. We utilize Definition 2.1 as our definition of integrability because it properly stresses the Lie-algebraic structure of an integrable hamiltonian system.

3. The construction

The principal tool used to construct integrals is the *momentum map*.

3.1. **Poisson geometry and the momentum map.** Let $\mathbf{G}$ be a real semisimple linear Lie group and $\mathfrak{g}$ its Lie algebra. The dual space $\mathfrak{g}^*$ is naturally identified with $\mathfrak{g}$ via the $\mathrm{Ad}_{\mathbf{G}}$-invariant Cartan-Killing form $\kappa = \langle , \rangle$. On $\mathfrak{g}^* \equiv \mathfrak{g}$, there is a canonically defined Poisson structure. For $f \in C^\infty(\mathfrak{g})$, let $\nabla_x f \in \mathfrak{g}$ denote the gradient of f with respect to κ at $x \in \mathfrak{g}$. The Poisson bracket is defined by:

$$\{f, h\}_{\mathfrak{g}}(x) := -\langle x, [\nabla_x f, \nabla_x h]\rangle, \tag{1}$$

for all $f, h \in C^\infty(\mathfrak{g})$ and $x \in \mathfrak{g}$. A function $f \in C^\infty(\mathfrak{g})$ is a *Casimir* if $E_f = \{., f\}$ is trivial; $C(x) = \kappa(x, x)$ is a Casimir.

The action of $\mathbf{G}$ on $\mathfrak{g}$ by conjugation is called the (co)adjoint action. The coadjoint action preserves the Poisson bracket. Trivialize $T^*\mathbf{G} = \mathbf{G} \times \mathfrak{g}$ via the left action. The map

$$\psi(g, x) := \mathrm{Ad}_g x = gxg^{-1} \tag{2}$$

is well-defined. One knows that
$\psi : T^*\mathbf{G} \to \mathfrak{g}^*$ *is the momentum map of* $\mathbf{G}$*'s left-action on* $T^*\mathbf{G}$*. The map* $\omega(g, x) = x$ *is the momentum map of* $\mathbf{G}$*'s right action on* $T^*\mathbf{G}$*. Both maps are submersions.*

The canonical Poisson structure on $T^*\mathbf{G}$, $\{,\}_{T^*\mathbf{G}}$, is related to $\{,\}_{\mathfrak{g}}$ as follows: $\{,\}_{T^*\mathbf{G}}$ is right (resp. left) invariant, so the Poisson bracket of right (resp. left) invariant functions is again right (resp. left) invariant. If $\mathcal{R}$ (resp. $\mathcal{L}$) denotes the right (resp. left) invariant functions smooth functions on $T^*\mathbf{G}$, then $\mathcal{R} = \psi^* C^\infty(\mathfrak{g})$ (resp. $\mathcal{L} = \omega^* C^\infty(\mathfrak{g})$), and ψ^* (resp. ω^*) is a Lie algebra isomorphism (resp. anti-isomorphism). For this reason, it is convenient to drop the subscript on the bracket. In addition, because right and left multiplication commute these two subalgebras commute: $\{\mathcal{R}, \mathcal{L}\} \equiv 0$.

Consequently, to prove integrability of Y_H on $T^*\mathbf{G}$, where $H = \omega^* h$, one must: (1) find sufficiently many functions in $\mathcal{R}$; and (2) find integrals of E_h on $\mathfrak{g}$. With luck, the sum of these two subalgebras of integrals is sufficient for integrability. To study Y_H on $T^*(\Gamma\backslash\mathbf{G})$, one needs to find sufficiently many functions in $\mathcal{R}^\Gamma$. That is the central task of the following sections. Note that $\mathcal{R}^\Gamma$, the set of smooth functions on $T^*\mathbf{G}$ that are invariant under left translation by Γ and right translation by $\mathbf{G}$, equals $\psi^* C^\infty(\mathfrak{g})^\Gamma$.

3.2. **Notation.** Let $\mathbf{G} = \mathrm{O}_+(2, 1)$, $\mathfrak{g} = o(2, 1)$, $\mathbf{H} = \mathbf{G}^3$ (Cartesian product of three copies of $\mathbf{G}$), $\mathfrak{h} = \mathfrak{g}^3$. From the KAN-decomposition theorem, $\mathbf{G}$ is analytically diffeomorphic to $\mathbf{K} \times \mathbf{A} \times \mathbf{N}$ where

$$\mathbf{K} = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ s\sin\theta & s\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta \in \mathbf{R}, s = \pm 1 \right\},$$

$$\mathbf{A} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} : \theta \in \mathbf{R} \right\},$$

$$\mathbf{N} = \left\{ \begin{bmatrix} 1 & b & b \\ -b & 1-\frac{1}{2}b^2 & -\frac{1}{2}b^2 \\ b & \frac{1}{2}b^2 & 1+\frac{1}{2}b^2 \end{bmatrix} \ : \ b \in \mathbf{R} \right\}.$$

and the diffeomorphism is $(k, a, n) \mapsto kan$ [20]. Let $\mathbf{K}_o$ be the connected component of $\mathbf{K}$ containing the identity. A $G \in \mathbf{H}$ equals $(g_1, g_2, g_3) \in \mathbf{G}^3$ and similarly $X \in \mathfrak{h}$ equals $(x_1, x_2, x_3) \in \mathfrak{g}^3$. Let $\kappa(x, y) = -\frac{1}{2}\mathrm{Trace}\, xy$ for all $x, y \in \mathfrak{g}$ be the Cartan-Killing form. If $x \in \mathfrak{g}$, write

$$x = \begin{bmatrix} 0 & -c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} = a\alpha + b\beta + c\gamma, \tag{3}$$

so that $\kappa(x, x) = c^2 - b^2 - a^2$. $C(x) = \frac{1}{2}\kappa(x, x)$ generates the Casimirs of $\mathfrak{g}$.

4. Orbits in $\mathfrak{g}$

Let $\mathcal{O}_x = \mathrm{Ad}_{\mathbf{G}} x$ be the coadjoint orbit through $x \in \mathfrak{g}$. Let $J = \mathrm{diag}(-1, 1, 1)$ and let $\mathbf{C}_2$ be the group of order 2 generated by J. The KAN-decomposition of $\mathbf{G}$ implies

Lemma 4.1. *There are three orbit types in* $\mathfrak{g} - \{0\}$*:*

(i) $C(x) > 0 \implies \mathrm{stab}(x)$ *is conjugate to* $\mathbf{K}_o = \mathrm{SO}(2)$
$\implies \mathcal{O}_x \simeq \mathbf{G}/\mathbf{K}_o = \mathrm{O}_+(2,1)/\mathrm{SO}(2)$;

(ii) $C(x) = 0 \implies \mathrm{stab}(x)$ *is conjugate to* $\mathbf{N}$
$\implies \mathcal{O}_x \simeq \mathbf{G}/\mathbf{N} = \mathrm{O}_+(2,1)/\mathbf{N}$;

(iii) $C(x) < 0 \implies \mathrm{stab}(x)$ *is conjugate to* $\mathbf{AC}_2 = \mathrm{SO}(1,1) \times \mathbf{Z}_2$
$\implies \mathcal{O}_x \simeq \mathbf{G}/\mathbf{AC}_2 = \mathrm{O}_+(2,1)/\left(\mathrm{SO}(1,1) \times \mathbf{Z}_2\right)$,

where $\simeq$ *means "*$\mathbf{G}$*-equivariantly diffeomorphic to."*

$\mathbb{H}^2$ is $\mathbf{G}$-equivariantly diffeomorphic to $\mathbf{G}/\mathbf{K}$. Type (i) coadjoint orbits are 2-fold covers of $\mathbb{H}^2$ and each component is $\mathbf{G}$-equivariantly diffeomorphic to $\mathbb{H}^2$. Type (iii) coadjoint orbits are cylinders.

For $\Gamma < \mathbf{G}$, let $C^\infty(\mathfrak{g})^\Gamma$ be the set of smooth Ad_Γ-invariant functions. The previous lemma, plus the well-known ergodicity of the $\mathbf{A}$- and $\mathbf{N}$-actions on $\Gamma\backslash\mathbf{G}$ when Γ is a lattice, imply

Lemma 4.2. *Let* $\Gamma < \mathbf{G}$ *be a lattice subgroup and* $f \in C^\infty(\mathfrak{g})^\Gamma$*. Then*

(i) $C(x) > 0 \implies f|_{\mathcal{O}_x} \in C^\infty(\mathcal{O}_x)^\Gamma \equiv C^\infty(\Gamma\backslash\mathbf{G})^{\mathbf{K}_o}$

(ii) $C(x) \le 0 \implies f|_{\mathcal{O}_x} \equiv$ *const.*

Thus, if Γ is a lattice there is only one independent function in $C^\infty(\mathfrak{g})^\Gamma$ on $\{C < 0\}$ while, on $\{C > 0\}$ there are three independent functions. This simple observation determines the entire course of this paper's construction.

Let's give an explicit construction of elements in $C^\infty(\mathfrak{g})^\Gamma$. To do so, define

Definition 4.3. $\mathbf{F}$ *to be the set of* $\varphi \in C^\infty(\mathbf{R})$ *such that* $\exists c > 0$ *and* $\varphi|_{(-\infty,c]} \equiv 0$*.*

$\Gamma\backslash\mathbf{G}$ is a real-analytic manifold with a real-analytic action of $\mathbf{K}_o$ on the right. For $F \in C^\infty(\Gamma\backslash\mathbf{G})^{\mathbf{K}_o}$ let $\tilde{F} \in C^\infty(\mathbf{G}/\mathbf{K}_o)^\Gamma$ be the induced smooth function on $\mathbf{G}/\mathbf{K}_o$.

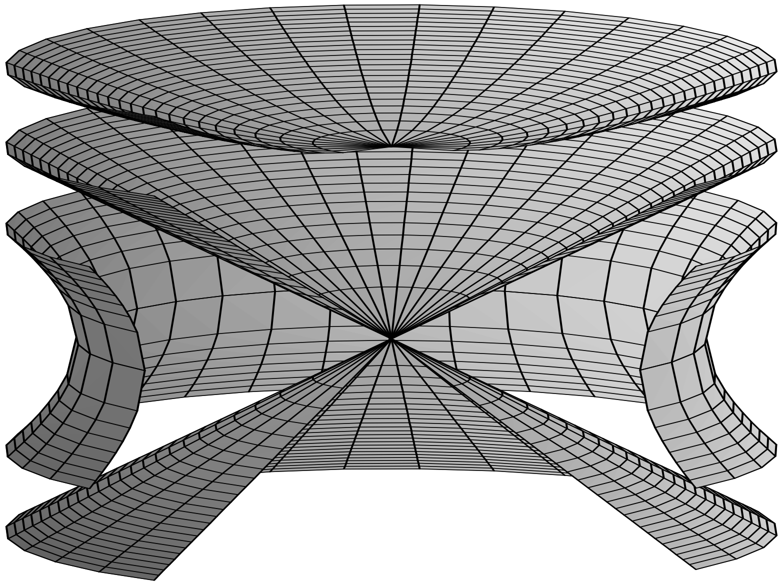

FIGURE 1. A cutaway of coadjoint orbits in $o(2,1)$.

Define $f \in C^\infty(\mathfrak{g})^\Gamma$ as follows: Let $\mathcal{C}_+ = \{x \in \mathfrak{g} : C(x) > 0\}$ and $\pi : \mathbf{G}/\mathbf{K}_o \to \mathcal{O}_\gamma$ denote the $\mathbf{G}$-equivariant diffeomorphism (γ defined in Equation 3). Extend π to a diffeomorphism $\Pi : \mathbf{R}^+ \times \mathbf{G}/\mathbf{K}_o \to \mathcal{C}_+$ by $\Pi(c,q) = \sqrt{c}\pi(q)$. By construction $C(\Pi(c,q)) = c$. Then define $f_0(c,q) = \varphi(c)\tilde{F}(q)$ where $\varphi \in \mathbf{F}$. The function $f(x) := f_0 \circ \Pi^{-1}(x)$ for $x \in \mathcal{C}_+$ and 0 elsewhere is Γ-invariant and smooth.

Definition 4.4. *Let* $C_0^\infty(\mathfrak{g})^\Gamma$ *denote the image of the map* $(F,\varphi) \to f$ *described in the previous paragraph.*

$\Gamma g\mathbf{K}_o$ is a *cone point* if $\mathrm{stab}_\Gamma(g\mathbf{K}_o) \neq 1$; let $\mathcal{K}$ be the set of cone points. Since $\Gamma\backslash\mathbf{G}/\mathbf{K}_o - \mathcal{K}$ is a real-analytic manifold, there are 3 independent smooth functions at all points $(c,q) \in \mathbf{R}^+ \times (\Gamma\backslash\mathbf{G}/\mathbf{K}_o - \mathcal{K})$. $\mathcal{K}$ is a nowhere dense real-analytic subset of $\Gamma\backslash\mathbf{G}/\mathbf{K}_o$ so

Lemma 4.5. $\mathcal{N} = \{x \in \mathcal{C}_+ : dC_0^\infty(\mathfrak{g})_x^\Gamma = \mathfrak{g}^*\}$ *is an open, dense real-analytic subset of* $\mathcal{C}_+$. *If* Γ *is torsion-free, then* $\mathcal{N} = \mathcal{C}_+$.

5. ORBITS IN $\mathfrak{h}$

Let $\mathfrak{h}^{reg} = \{X \in \mathfrak{h} : \Pi_{i=1}^3 C(x_i) \neq 0\}$, and $\mathcal{B} = \{X \in \mathfrak{h}^{reg} : \mathrm{span}\{x_1,x_2,x_3\} = \mathfrak{g}\}$. $\mathcal{B}$ is an open dense analytic subset of $\mathfrak{h}$. Let $\mathbf{D} = \delta(\mathbf{G})$ be the diagonal copy of $\mathbf{G}$ in $\mathbf{H}$ and let $\mathrm{GL}(\mathfrak{h})^\mathbf{D}$ be the set of invertible linear maps $L : \mathfrak{h} \to \mathfrak{h}$ that commute with the coadjoint action of $\mathbf{D}$. To construct an $L \in \mathrm{GL}(\mathfrak{h})^\mathbf{D}$, let $[a_{ij}]$ be an invertible 3×3 matrix; for each $X \in \mathfrak{h}$ define the i-th component of $L(X)$, $L_i(X) = L(X)_i$, to be $\sum_{j=1}^3 a_{ij}x_j$.

Lemma 5.1. $\mathcal{B} \subset \bigcup_{L \in \mathrm{GL}(\mathfrak{h})^\mathbf{D}} L^{-1}(\mathcal{C}_+^3)$.

Proof. Let $X \in \mathcal{B}$. Since $\mathcal{C}_+^3$ is open and $\mathcal{B}$ is dense, there is a $Y \in \mathcal{C}_+^3 \cap \mathcal{B}$. Since the components of both X and Y give a basis of $\mathfrak{g}$, there are constants a_{ij} and b_{ij} such that $y_i = \sum_{j=1}^3 a_{ij} x_j$, and $x_i = \sum_{j=1}^3 b_{ij} y_j$ for $i = 1, 2, 3$. Since $\mathbf{A} = [a_{ij}]$ is in $\mathrm{GL}(\mathbf{R}^3)$, the linear map L (resp. M) defined as in the previous paragraph by the coefficients a_{ij} (resp. b_{ij}) is $\mathbf{D}$-equivariant, invertible and $L(X) = Y \in \mathcal{C}_+^3$. ☐

Clearly,

Lemma 5.2. *$\mathcal{B}$ is an $\mathrm{Ad}_{\mathbf{D}}$-invariant set.*

Let $L \in \mathrm{GL}(\mathfrak{h})^{\mathbf{D}}$ and let $\varphi \in \mathbf{F}$. Define:

$$\Phi(X) := \Pi_{i=1}^3 \ \varphi(C(L_i(X))) \tag{4}$$

It is straightforward to verify that $\Phi \in C^\infty(\mathfrak{h})^{\mathbf{D}}$.
For $f \in C^\infty(\mathfrak{g})^\Gamma$, $i = 1, 2, 3$ and $X \in \mathfrak{h}$ define

$$F_i(X) := \begin{cases} \Phi(X)\, f \circ L_i(X) & \text{if } X \in L^{-1}(\mathcal{C}_+^3), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

F_i is clearly smooth and Ad_Δ-invariant where $\Delta = \delta(\Gamma)$.

Lemma 5.3. *$\mathcal{B}$ is contained in $\{X \in \mathfrak{h} \ : \ dC^\infty(\mathfrak{h})^\Delta_X = \mathfrak{h}^*\}$.*

Proof. Fix $X \in \mathcal{B}$. Since $\mathcal{N}^3$ is open in $\mathfrak{h}$ and $\mathcal{B}$ is open and dense, $\mathcal{N}^3 \cap \mathcal{B}$ is open and non-empty. Therefore, let $Y \in \mathcal{N}^3 \cap \mathcal{B}$ and let $L \in \mathrm{GL}(\mathfrak{h})^{\mathbf{D}}$ map $X \to Y$. Let $B \subset \mathcal{B}$ be a compact neighbourhood of X such that $L(B) \subset \mathcal{N}^3 \cap \mathcal{B}$. By compactness of B, there exists $c_1 > 0$ such that for all $Z \in L(B)$, $2c_1 < C(z_i)$. Let $\varphi \in \mathbf{F}$ be such that $\varphi|_{(-\infty, c_1]} \equiv 0$ and $\varphi|_{[2c_1, \infty)} \equiv 1$. Define Φ (Equation 4) with this L and φ. Then $F_i(U) = f(z_i)$ for all $U \in B$ and $Z = L(U)$ (Equation 5). Since $f \in C^\infty(\mathfrak{g})^\Gamma$ was arbitrary

$$dC^\infty(\mathfrak{g})^\Gamma_{z_1} \oplus dC^\infty(\mathfrak{g})^\Gamma_{z_2} \oplus dC^\infty(\mathfrak{g})^\Gamma_{z_3} \quad \subseteq \quad L^{-1} dC^\infty(\mathfrak{h})^\Delta_U, \tag{6}$$

for all $U \in B$. Since $L(B) \subset \mathcal{N}^3$, the left-hand side of (6) equals $\mathfrak{h}^*$. This proves the lemma. ☐

Corollary 5.4. *$\Delta \backslash \mathcal{B} \equiv \mathrm{Ad}_\Delta \backslash \mathcal{B}$ is a real-analytic 9-manifold.*

Proof. $C^\infty(\mathfrak{h})^\Delta$ provides a smooth atlas for $\Delta \backslash \mathcal{B}$, and the functions whose germ at a given point are real-analytic provides a real-analytic subatlas. ☐

5.1. **The Algebras $\mathcal{A}$ and $\mathcal{H}$.** Let Ψ (resp. Ω) be the momentum map of $\mathbf{H}$'s right (resp. left) action on $T^*\mathbf{H}$ and let

$$\mathcal{A} = \Psi^* C^\infty(\mathfrak{h})^\Delta = \mathcal{R}^\Delta.$$

Let $\hat{l}_i \in C^\omega(\mathfrak{g})$ be a non-central hamiltonian. The diagram

$$T^*\mathbf{H} \xrightarrow{\ \Omega\ } \mathfrak{h} \xrightarrow{\ p_i\ } \mathfrak{g} \xrightarrow{\ \hat{l}_i\ } \mathbf{R}$$

yields

$$\begin{array}{ccc} T^*\mathbf{H} & \stackrel{\Omega}{\longrightarrow} & \mathfrak{h} \\ & L \quad \searrow & \downarrow \hat{l} \\ & & \mathbf{R}^3, \end{array}$$

and $\mathcal{H} = L^* C^\infty(\mathbf{R}^3)$ is an abelian subalgebra of left-invariant functions on $T^*\mathbf{H}$.

5.2. **Locally-Trivial Fibre Bundle.** Let $J = (\Psi, L)$. From $T^*\mathbf{H} \stackrel{J}{\longrightarrow} \mathfrak{h} \times \mathbf{R}^3$ one sees that

$$\mathcal{A} + \mathcal{H} = J^*(C^\infty(\mathfrak{h})^\Delta \oplus C^\infty(\mathbf{R}^3)) \subset J^* C^\infty(\mathfrak{h} \times \mathbf{R}^3)^\Delta = \mathcal{F},$$

and that $d(\mathcal{A}+\mathcal{H})_p = d\mathcal{F}_p$ and $dZ(\mathcal{A}+\mathcal{H})_p = dZ(\mathcal{F})_p$ for all $p \in T^*\mathbf{H}$. It is equally clear that

Lemma 5.5. *$\mathcal{H} \subset Z(\mathcal{F})$ and $\mathcal{F} \subset C^\infty(T^*\mathbf{H})^\Delta$.*

Let $h = \hat{l}^* s$, $s(a_1, a_2, a_3) = a_1 + a_2 + a_3$. $\mathcal{F}$ is *proper* if $h \in C^\infty(\mathfrak{h})$ is proper. Henceforth, $\mathcal{F}$ is assumed to be proper.

Lemma 5.6. *If $\mathcal{F}$ is proper, then for all $f \in Z(\mathcal{F})$, the hamiltonian vector field Y_f on $T^*\mathbf{H}$ is complete.*

Proof. Because $f \in Z(\mathcal{F})$ is left-invariant, Y_f descends to $T^*(D\backslash\mathbf{H})$. If D is cocompact, then $H = \Omega^* h$ is proper on $T^*(D\backslash\mathbf{H})$. Since Y_f is tangent to the level sets of H, Y_f is complete. □

Let $\mathrm{Reg}(J)$ be the regular-value set of J. Since Ψ is an $\mathbf{H}$-equivariant submersion and L is left-invariant, (G, X) is critical for J iff X is critical for the map $X \stackrel{\mathcal{X}}{\mapsto} (\chi_1(x_1), \chi_2(x_2), \chi_3(x_3))$ where $\chi_i(\bullet) = (\hat{l}_i(\bullet), C(\bullet))$. Thus $\mathrm{Reg}(J)$ is contained in $\mathcal{B} \times \mathbf{R}^3$. Let $\mathbf{M} = J^{-1}(\mathrm{Reg}(J))$ be the regular-point set of J. Since J is Δ-equivariant, $\mathbf{M}$ is a Δ-invariant, open and dense analytic subset of $T^*\mathbf{H}$. The commutative diagram

$$\begin{array}{ccc} T^*\mathbf{H} & \stackrel{J}{\longrightarrow} & \mathfrak{h} \times \mathbf{R}^3 \\ {\scriptstyle \text{inclusion}}\uparrow & & \uparrow{\scriptstyle \text{inclusion}} \\ \mathbf{M} & \stackrel{J|\mathbf{M}}{\longrightarrow} & \mathcal{B} \times \mathbf{R}^3, \end{array}$$

and the Δ-invariance of $\mathcal{B}$ implies that there is an analytic submersion J_Δ defined such that

$$\begin{array}{ccccc} T^*\mathbf{H} & \stackrel{\text{inclusion}}{\longleftarrow} & \mathbf{M} & \stackrel{J|\mathbf{M}}{\longrightarrow} & \mathcal{B} \times \mathbf{R}^3, \\ \downarrow & & \downarrow & & \downarrow \\ T^*(\Delta\backslash\mathbf{H}) & \stackrel{\text{inclusion}}{\longleftarrow} & \Delta\backslash\mathbf{M} & \stackrel{J_\Delta}{\longrightarrow} & \Delta\backslash\mathcal{B} \times \mathbf{R}^3 \end{array}$$

commutes.

Theorem 5.1. *J_Δ is a locally-trivial fibre bundle map. If $(X, u) \in \mathrm{Reg}(J)$, then each connected component of $J_\Delta^{-1}(\Delta X, u)$ is analytically diffeomorphic to $\mathbf{T}^{3+m} \times \mathbf{R}^{3-m}$ where $m = \#\{i : C(x_i) > 0\}$.*

Let $\bar{\bullet}$ denote the object on $T^*(\Delta\backslash\mathbf{H})$ induced by $\bullet$ on $T^*\mathbf{H}$. Theorem 5.1 implies that

Corollary 5.7. *$L(\bar{\mathcal{F}})$ contains an open dense real-analytic set. If $H \in \mathcal{F}$, then the flow of $Y_{\bar{H}}$ has a non-wandering set with a non-empty interior.*

Theorem 5.2. *The hamiltonian $H = L^*s \in \mathcal{H}$ induces a tamely integrable hamiltonian $\bar{H} \in C^\omega(T^*(\Delta\backslash\mathbf{H}))$.*

Proof. Since the covering map $T^*\mathbf{H} \to T^*(\Delta\backslash\mathbf{H})$ is a Poisson map, Lemmas 5.5 and 5.6 and Corollary 5.7 show that $\bar{\mathcal{F}}$ satisfies conditions (I1—I3) of Definition 2.1. Since $L(\bar{\mathcal{F}})$ contains a dense analytic set, $T^*(\Delta\backslash\mathbf{H}) - L(\bar{\mathcal{F}})$ is contained in a nowhere dense tamely-embedded polyhedron. □

Theorem 1.1. To complete the proof of Theorem 1.1, for each i let $\hat{l}_i$ be the positive-definite quadratic form on $\mathfrak{g}$ induced by the left-invariant riemannian metric $\mathbf{g}$ on $\mathbf{G}$. The hamiltonian $H = L^*s$ is the hamiltonian of the riemannian metric $\mathbf{h}$ on $\mathbf{H}$ (see diagram preceding Theorem 1.1). Theorem 5.2 proves that the hamiltonian flow of $\bar{H}$ is tamely integrable on $T^*(\Delta\backslash\mathbf{H})$ so the geodesic flow of $(\Delta\backslash\mathbf{H}, \mathbf{h})$ is integrable.

Since $(\Gamma\backslash\mathbf{G}, \mathbf{g}) \to (\Delta\backslash\mathbf{H}, \mathbf{h})$ is a riemannian embedding, the geodesic flow of $(\Gamma\backslash\mathbf{G}, \mathbf{g})$ is a subsystem of the geodesic flow of $(\Delta\backslash\mathbf{H}, \mathbf{h})$. Since Γ is a lattice subgroup of a semi-simple Lie group, it has exponential word growth. Hence the geodesic flow of $\mathbf{g}$ has positive topological entropy; since it is a subsystem of the geodesic flow of $\mathbf{h}$, the latter has positive topological entropy (*c.f.* [7]). □

5.3. **Proof of Theorem 5.1.** Theorem 5.1 is proven by constructing local trivializations of J_Δ. To simplify matters, $\hat{l}_i \in C^\omega(\mathfrak{g})$ is a proper, $\mathbf{C}_2$-invariant, non-central hamiltonian. Let R be $\mathcal{X}$'s regular-value set and if $V \subset \mathfrak{h}$, then let $\mathcal{O}_V = \cup_{v \in V}\ \mathcal{O}_v$.

Lemma 5.8. *If $r \in R$, then there are open neighbourhoods $U \ni r$, $\chi^{-1}(U) \subset V$ and an analytic map $k : \mathbf{K}^3 \times \mathcal{O}_V \times U \to \mathbf{G}$ such that for all $u \in U$ and $X \in \mathcal{O}_V$*

$$\hat{l}^{-1}(u) \cap \mathcal{O}_X = \{\mathrm{Ad}_{k(\theta,X,u)}X \ : \ \theta \in \mathbf{K}^3\},$$

and the map $\theta \to \mathrm{Ad}_{k(\theta,X,u)}X$ is an analytic diffeomorphism.

Proof. This follows from the KAN-decomposition, the description of the orbits in $\mathbf{G}$, and the properness of $\mathcal{X}$. □

Let $\mathbf{H}_m = \mathbf{K}_o^m \times (\mathbf{AC}_2)^{3-m}$.

Lemma 5.9. *Let $(X, u) \in \mathrm{Reg}(J)$ and $m = \#\{i : C(x_i) > 0\}$. Then there are open neighbourhoods $V \ni X$ and $U \ni u$ and a real-analytic map f such that*

$$\begin{array}{ccc} \mathbf{K}^3 \times \mathbf{H}_m \times \mathcal{O}_V \times U & \xrightarrow{f} & \mathbf{M} \\ \downarrow & & \downarrow J|\mathbf{M} \\ \mathcal{O}_V \times U & \xrightarrow{\text{inclusion}} & \mathcal{B} \times \mathbf{R}^3 \end{array}$$

commutes and f is a diffeomorphism with $J^{-1}(\mathcal{O}_V \times U)$. Hence $J|\mathbf{M}$ is a locally-trivial fibre bundle map.

Proof. Since $\mathcal{B} \subset \mathfrak{h}^{reg}$, $X \in \mathfrak{h}^{reg}$. Let $\mathbf{H}_X = \text{stab}_{\mathbf{H}}(X)$ and $\mathbf{G}_{x_i} = \text{stab}_{\mathbf{G}}(x_i)$. Then $\mathbf{H}_X = \mathbf{G}_{x_1} \times \mathbf{G}_{x_2} \times \mathbf{G}_{x_3}$ and $\mathbf{G}_{x_i}$ is conjugate to $\mathbf{K}_o$ (resp. $\mathbf{AC}_2$) if $C(x_i) > 0$ (resp. $C(x_i) < 0$). By the KAN decomposition, there is an analytic map $Z : \mathfrak{h}^{reg} \to \mathbf{H}$ such that $Z(Y)\mathbf{H}_Y Z(Y)^{-1} = \mathbf{H}_m$ for all Y in the connected component of $\mathfrak{h}^{reg}$ containing X.

Since (X, u) is a regular value of J, u is a regular value of $\hat{l}|\mathcal{O}_X$. Let $k : \mathbf{K}^3 \times \mathcal{O}_V \times U \to \mathbf{G}$ be the analytic map from Lemma 5.8, where $U \ni u$ and $V = \hat{l}^{-1}(U)$ are open sets.

Fix $(Y, s) \in \mathcal{O}_V \times U$. If $(G, \text{Ad}_{G^{-1}}Y) \in J^{-1}(Y, s)$, then by Lemma 5.8 there is a unique $\theta \in \mathbf{K}^3$ such that $\text{Ad}_{k(\theta,Y,s)}Y = \text{Ad}_{G^{-1}}Y$ so $W' = Gk(\theta, Y, s) \in \mathbf{H}_Y$. Thus, let $f : \mathbf{K}^3 \times \mathbf{H}_m \times \mathcal{O}_V \times U \to T^*\mathbf{H}$ be defined by

$$f(\theta, W, Y, s) = (Z(Y)^{-1} W Z(Y) k(\theta, Y, s)^{-1},\ \text{Ad}_{k(\theta,Y,s)}Y),$$

which equals $(G, \text{Ad}_{G^{-1}}Y)$. Clearly f is analytic, $J \circ f(\theta, W, Y, s) = (Y, s)$, and since f^{-1} is also a well-defined analytic map, f is an analytic diffeomorphism. ☐

Lemma 5.10. *$J_\Delta : \Delta\backslash\mathbf{M} \to \Delta\backslash\mathcal{B} \times \mathbf{R}^3$ is a locally-trivial fibre bundle.*

Proof. It suffices to find a local trivializations of J_Δ about any point $(\Delta X, u) \in \text{Reg}(J_\Delta)$. Let f, U and $\mathcal{O}_V$ be defined as above. Since J is Δ-equivariant, there is an analytic action of Δ on the domain of f that makes f a Δ-equivariant diffeomorphism. So, f induces an analytic diffeomorphism f_Δ such that

$$\begin{array}{ccc} \Delta\backslash(\mathbf{K}^3 \times \mathbf{H}_m \times \mathcal{O}_V \times U) & \xrightarrow{f_\Delta} & \Delta\backslash\mathbf{M} \\ \downarrow & & \downarrow J_\Delta \\ \Delta\backslash\mathcal{O}_V \times U & \xrightarrow{\text{inclusion}} & \Delta\backslash\mathcal{B} \times \mathbf{R}^3 \end{array}$$

commutes. ☐

5.4. **Non-riemannian examples.** Theorem 5.2 applies to certain finslerian and sub-riemanian metrics on $\mathbf{H}$. Let $\hat{h}_i$ be a positive-definite quadratic form on $\mathfrak{g}$ and let $F \in C^\omega(\mathbf{R}^3 - \{0\})$ be strictly convex and positively homogeneous of degree 1. The function $H = L^*F$ defines an analytic Finsler metric on $T^*\mathbf{H}$. Theorem 5.2 implies that the hamiltonian flow of $\bar{H}$ on $T^*(\Delta\backslash\mathbf{H})$ is integrable. One might take F to be equal to $(a_1^\rho + a_2^\rho + a_3^\rho)^{1/\rho}$ for all $a \in \mathbf{R}^3$, some $\rho \in 2\mathbf{Z}$.

On the other hand, a left-invariant sub-riemannian metric on $\mathbf{G}$ is induced by a quadratic form on $\mathfrak{g}$ that is congruent mod $\mathbf{G}$ to $\hat{g}(x) = Aa^2 + Bb^2 + Cc^2$ where $A, B, C \geq 0$ and $AB + AC + BC > 0$ (c.f. equation 1). Since $\hat{g} + \epsilon C$ is proper for ϵ sufficiently small, Theorem 5.2 implies that the geodesic flow of the sub-riemannian metric $H = L^*s$ determined by $\hat{g}$ is integrable on $T^*(\Delta\backslash\mathbf{H})$.

6. Geometric surfaces and 3-manifolds

Theorem 1.2. The momentum map of $\mathbf{K}$'s action on $T^*(\Delta\backslash\mathbf{H})$ is $\Psi_{\mathbf{K}}(\Delta G, X) = c_1 + c_2 + c_3$ (c.f. Equation 3). It is well-known that $\Psi_{\mathbf{K}}^{-1}(0)/\mathbf{K}$ is symplectomorphic to $T^*(\Delta\backslash\mathbf{H}/\mathbf{K})$. It is easily seen that $\mathcal{B} \cap \{c_1 + c_2 + c_3 = 0\}$ is open and dense in $\{c_1 + c_2 + c_3 = 0\} \subset \mathfrak{h}$. This fact implies Theorem 1.2. ☐

Corollary 1.2. Let $(M, \mathbf{k})$ be a compact geometric $\widetilde{SL}(2;\mathbf{R})$-manifold. Since, $\widetilde{SL}(2;\mathbf{R})$ is naturally isometric to the connected component of $\widetilde{\mathrm{O}}_+(2,1)$ containing the identity, $\tilde{\mathbf{G}}_o$, $(M, \mathbf{k})$ is isometric to $(\Lambda\backslash\tilde{\mathbf{G}}_o, \mathbf{g})$ where Λ is a discrete cocompact subgroup of the isometry group of $(\tilde{\mathbf{G}}_o, \mathbf{g})$ that acts freely on $\tilde{\mathbf{G}}_o$ and $\mathbf{g}$ is a left-invariant metric on $\tilde{\mathbf{G}}_o$ that is simultaneously invariant under the right action of $\mathbf{K}_o$. Theorem 4.15 in [31] along with the fact that the universal covering group of the isometry group of $(\tilde{\mathbf{G}}, \mathbf{g})$ is $\tilde{\mathbf{G}} \times \tilde{\mathbf{K}}$ imply that there is a covering $\hat{M} = \tilde{\mathbf{G}}/Q$ of M, where $Q < \tilde{\mathbf{K}}$ is an infinite group and the covering map $\hat{M} \to M$ is the map $\tilde{\mathbf{G}}/Q \to \Gamma\backslash\tilde{\mathbf{G}}/Q$ for some discrete cocompact subgroup $\Gamma < \tilde{\mathbf{G}}$. The proof of Corollary 1.2 is therefore essentially the same proof as that of Theorems 1.1 and 1.2. □

7. Concluding Comments

The examples in this paper leave several interesting questions. It still remains an open question if there are examples of real-analytic geodesic flows on compact surfaces of genus more than one that are integrable. Evidently, the best information concerning the dynamics of a real-analytic geodesic flow on a compact surface of genus more than one is the Bolotin-Negrini theorem [4], which says that there is a horseshoe in a neighbourhood of each minimizing periodic orbit. Little is known about the dynamics on the remainder of the phase space. Might it still be possible that there are integrable geodesic flows on these surfaces?

A second, more general question, is: given a manifold Σ, what is the smallest dimensional manifold Σ' in the homotopy class of Σ which admits an integrable geodesic flow?

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