(Q1) Consider the set Σ of sequences of two symbols $0, 1$, i.e., $\Sigma = \{0,1\}^{\mathbb{N}}$, with a distance between sequences $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma$ and $\mathbf{t} = (t_0, t_1, t_2, \cdots) \in \Sigma$ defined by

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

(a) Show that d defines a metric on Σ.

(b) Show that $d(\mathbf{s}, \mathbf{t}) \leq 2$.

(Q2) With the same definitions as in Question 1, let $\mathbf{s}, \mathbf{t}$ and $\mathbf{r}$ be the periodic sequences

$$\mathbf{s} = (1,1,2,1,1,2,1,1,2,\cdots),\ \ \mathbf{t} = (1,2,1,2,1,2,1,2,\cdots),\ \ \mathbf{r} = (2,1,2,1,2,1,2,1,\cdots).$$

Calculate $d(\mathbf{s}, \mathbf{t})$, $d(\mathbf{t}, \mathbf{r})$ and $d(\mathbf{r}, \mathbf{s})$.

(Q3) Let $\Sigma' \subset \Sigma = \{0,1\}^{\mathbb{N}}$ be the set of all sequences $\mathbf{s}$ of two symbols $0, 1$ with $s_{j+1} = 0$ if $s_j = 1$ (i.e., the sequences in Σ' do not have two consecutive 1's).

(a) Confirm that the shift map σ preserves Σ'.

(b) Show that periodic points are dense in Σ'.

(c) Show that there is a dense orbit in Σ'.

(d) How many fixed points of σ are there in Σ'? How many period-2 and period-3 orbits?

(Q4) Consider the one-sided shift map σ acting on sequences of N symbols, i.e., acting on $\Sigma = \{1, 2, \cdots, N\}^{\mathbb{N}}$.

(a) How many fixed points of σ^k are there?

(b) How many period-2 and period-4 orbits of σ are there in Σ? How many prime period-2 and -4 orbits are there?

(Q5) Consider a one–dimensional mapping $F(x_n)$ with m prime periodic orbit

$$\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{m-1}).$$

Show that the Liapunov exponent of an orbit attracted to this periodic orbit is given by

$$\lambda = \frac{1}{m} \ln \left| \prod_{i=0}^{m-1} F'(x_i) \right|.$$

Thereby, show that $\lambda < 0$.

(Q6) Find the Liapunov exponent of the logistic map $F_\mu(x) = \mu x(1-x)$ for $x \in [0,1]$ where:

(a) $1 < \mu < 3$
(Hint: You may assume that: (a) there exists at most one attracting period orbit for the logistic map; and (b) the basin of attraction for this attracting periodic orbit comprises the entire closed interval $[0,1]$ minus any repelling fixed points).

(b) $3 < \mu < 1 + \sqrt{6}$.

(Hint: use the result of Question 5).

(Q7) Let $f : [0,1] \to [0,1]$ be defined as follows

$$f(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/4, \\ -(x - \frac{1}{4})(\frac{7}{8} - x) & \text{if } 1/4 < x < 7/8, \\ 2(x - 7/8) & \text{if } 7/8 \leq x \leq 1. \end{cases}$$

Let $I_0 = [0, \frac{1}{4}]$ and $I_1 = [\frac{7}{8}, 1]$. The aim of this exercise is to show that there is an invariant set $\Lambda \subset [0,1]$ and a homeomorphism $h : \Lambda \to \Sigma'$ (see Q3) such that $h \circ f|\Lambda = \sigma \circ h$.

(a) Show that $I_0 \cup I_1 \subset f(I_0)$ and $I_0 \subset f(I_1)$.

(b) Show that if $\omega \in \Sigma'$, then the set $I_\omega = \{x \in [0,1] \ :\ f^n(x) \in I_{\omega_n} \text{ for all } n\}$ is non-empty, and contains a single point.

(c) Let $\Lambda = \cap_{n \geq 0} f^{-n}([0,1])$. Show that if $x \in \Lambda$ iff $f^n(x) \in [0,1]$ for all $n \geq 0$.

(d) Show that if $x \in \Lambda$, then $x \in I_0 \cup I_1$. Conclude that $f^n(x) \in \Lambda$ for all $n \geq 0$. Hence show that the itinerary map $h(x) = \omega$ is well-defined.

(e) Prove that h is continuous, 1-1 and onto.

(f) How many periodic orbits of period 2, 3 and 6 does f have?

(Q8) Let $f(x) = 4x(1-x)$ and let $\Sigma = \{0,1\}^{\mathbb{N}}$. Prove that there is a continuous surjection h such that

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \downarrow h & & \downarrow h \\ I & \xrightarrow{f} & I \end{array}$$

commutes (σ is the shift map). Describe the set of points where h fails to be injective, i.e. the set of $\omega \in \Sigma$ where $h^{-1}(h(\omega))$ contains more than one point. [Hint: find intervals J_0, J_1 with disjoint interiors such that $f(J_i) = I$ and $I = J_0 \cup J_1$. Try to define an itinerary map...]

At Examples Class 4 on Friday 3rd December the solution to Questions 3 and 7 will be discussed.