(**Q1**) Consider the $\mathbb{R}^1$ map with fixed point $x^* = \alpha$ represented by the Taylor series

$$x_{n+1} = F(x_n) = \alpha + \beta_1 (x - \alpha) + \beta_2 (x - \alpha)^2 + \beta_3 (x - \alpha)^3 + \beta_4 (x - \alpha)^4 + \cdots$$

where $\beta_{1,2,3,4,\ldots}$ are constants. Let $G(x) = F(x + \alpha) - \alpha$. Show that G has a fixed point at the origin and

$$D_s\{F\}(\alpha) = D_s\{G\}(0).$$

(**Q2**) Consider the system $x_{n+1} = F_\mu(x_n)$, with $F_\mu(x_n) = \mu + x_n^2$ where $x_n, \mu \in \mathbb{R}$.

(a) Find the fixed points of the system in terms of μ.

(b) Find the value of x, and the corresponding value of the parameter μ, at which there is a saddle-node bifurcation.

(c) Find the value of x, and the corresponding value of the parameter μ, at which there is a flip bifurcation. Is it super- or subcritical?

(**Q3**) Let $I = [a, b]$ be a closed interval and $F : I \to I$ be a continuous function. Show that F has a fixed point in I. (Hint: Intermediate Value Theorem).

(**Q4**) Let $I = [a, b]$ be a closed interval and F be a continuous function such that $F(I) \supset I$. Show that F has a fixed point in I. (Hint: Intermediate Value Theorem).

(**Q5**) Show that if the mapping $x_{n+1} = F(x_n)$ with $F(x)$ continuous has a period-2 orbit, then it also has a fixed point. (Hint: Intermediate Value Theorem).

(**Q6**) Let $F : I \to I$ be a continuous map of $I = [0, 1]$. Show that if F has a prime period-3 orbit, then F has a fixed point and a prime period-2 point. This completes the proof of the simple Sharkovskii theorem.

(**Q7**) Show that the mapping $x_{n+1} = F(x_n)$

(a) has no prime period-k orbits for $k \geq 2$ if $F'(x) > 0$;

(b) has a unique fixed point and no prime period-k orbits for $k \geq 3$ if $F'(x) < 0$.

(Hint: consider the ordering of the x_j in a periodic orbit $(x_0, x_1, \cdots, x_{k-1})$; for $F'(x) < 0$, consider the sign of the derivative of $F^k(x)$.)

(**Q8**) Consider the $\mathbb{R}^1$ family of mappings

$$x_{n+1} = G_\mu(x_n) = \mu x_n \left(1 - x_n^4\right) \qquad (\mu > 0).$$

(a) Find the fixed point of this mapping with $x > 0$. For which range of values of μ does it exist?

(b) Find the value of μ for which the fixed point with $x > 0$ undergoes a flip bifurcation and discuss its nature.

(c) The mapping undergoes a sequence of period–doubling bifurcations as μ increases. Describe briefly this phenomenon.

(d) Describe the nature of all period–doubling bifurcations of this mapping.

(**Q9**) Consider the $\mathbb{R}^1$ mapping

$$x_{n+1} = F_\mu(x_n) \quad \text{with} \quad F_\mu(x) = \mu x - x^3 \;\; \text{and} \;\; \mu > 0.$$

(a) Find the fixed points of the mapping F_μ.

(b) Discuss the existence and stability of the fixed points in terms of μ, and thereby show that the mapping undergoes bifurcations for $\mu = 1$ and $\mu = 2$.

(c) Describe the bifurcation which arises at $\mu = 1$. Sketch the fixed points of F_μ on a (μ, x) bifurcation diagram for $0 < \mu < 3$. Indicate stability on your sketch.

(d) Determine whether the flip bifurcations at $\mu = 2$ are supercritical or subcritical by computing the Schwarzian derivative of F_μ. What are the implications of this result for period doubling?

(e) Consider the perturbed mapping

$$F_{\mu,\delta}(x) = \mu x - x^3 + \delta$$

such that $F_{\mu,0}(x) = F_\mu(x)$. For a fixed, small value of $\delta > 0$, sketch on a (μ, x) diagram the position of the fixed points of $F_{\mu,\delta}$.
(Hint: To sketch the position of the fixed points $x(\mu)$, it is convenient to consider the graph of the inverse relationship $\mu(x)$ and use reflection about the line $x = \mu$ to deduce the curves $x(\mu)$; there is then no need to solve the cubic equation for the fixed points explicitly).

(f) Show that the mapping $F_{\mu,\delta}$ undergoes a bifurcation for $\mu = 1 + 3(\delta/2)^{2/3}$. What is the nature of this bifurcation?

(**Q10**) Prove the following theorem:

Theorem.[Saddle-Node Bifurcation Theorem] Let $f_\mu(x)$ be a function that is C^3 in both variables. Assume that there is a μ_c, x_c such that

(a) $x_c = f_{\mu_c}(x_c)$;

(b) $a = f''_{\mu_c}(x_c) \neq 0$;

(c) $b = \left.\dfrac{\partial f_\mu}{\partial \mu}\right|_{x=x_c,\mu=\mu_c} \neq 0$;

(d) $f'_{\mu_c}(x_c) = 1$.

Then there exists a C^2 function $\mu = \mu(x)$ such that

(i) $\mu(x_c) = \mu_c$;

(ii) $f_{\mu(x)}(x) = x$ for all x near x_c; and

(iii) $\mu(x) = \mu_c - \dfrac{a}{2b}(x - x_c)^2 + O(|x - x_c|^3)$.

Conclude that f_μ undergoes a saddle-node bifurcation at $\mu = \mu_c$ and f_μ has fixed points $x_\pm(\mu) = x_c \pm \sqrt{\dfrac{-2b(\mu - \mu_c)}{a}} + O(|\mu - \mu_c|)$. [Hint: use the implicit function theorem.]

Compare the statement of the SNB Theorem and **Q2**.

(**Q11**) Prove the following theorem:

Theorem.[Period-Doubling/Flip Bifurcation Theorem] Let $f_\mu(x)$ be a function that is C^4 in both variables. Assume that there is a μ_c such that

(a) $0 = f_\mu(0)$ for all μ near μ_c;

(b) $f'_{\mu_c}(0) = -1$;

(c) $a = f'''_{\mu_c}(0) \neq 0$; and

(d) $b = \left.\dfrac{\partial \left(f^2_\mu\right)'}{\partial \mu}\right|_{x=0,\mu=\mu_c} \neq 0;$

(e) $f'_{\mu_c}(x_c) = 1$.

Then there exists a C^4 function $\mu = \mu(x)$ defined near $x = 0$ such that

(i) $\mu(0) = \mu_c$;

(ii) $f_{\mu(x)}(x) \neq x$, $f^2_{\mu(x)}(x) = x$ for all $x \neq 0$ near 0; and

(iii) $\mu(x) = \mu_c - \dfrac{a}{2b}(x - x_c)^2 + O(|x - x_c|^3)$.

Conclude that f_μ undergoes a saddle-node bifurcation at $\mu = \mu_c$ and f_μ has fixed points $x_\pm(\mu) = \pm\sqrt{\dfrac{-2b(\mu - \mu_c)}{a}} + O(|\mu - \mu_c|)$.

Hint: use the implicit function theorem for the function

$$H(x, \mu) = \begin{cases} \dfrac{f^2_\mu(x) - x}{x} & \text{if } x \neq 0, \\ (f^2_\mu)'(0) & \text{if } x = 0. \end{cases}$$

Compare the statement of the above Theorem, **Q9**e and our work in class.

At Examples Class 3 on Tuesday 16 November the solution to Questions 8 & 9 will be discussed.