(**Q1**) Consider the 2×2 real-valued matrix A, with trace τ and determinant Δ. In this question, you should make use of the Cayley–Hamilton theorem: matrix A satisfies its own characteristic equation; i.e., $A^2 - \tau A + \Delta I = 0$ where I is the identity matrix.

(a) Assuming that $\tau = 0$, show that

$$A^{2n} = (-1)^n \Delta^n I, \quad A^{2n+1} = (-1)^n \Delta^n A.$$

(b) Assuming that $\Delta = 0$, show that

$$A^n = \tau^{n-1} A.$$

Deduce simple expressions for $\exp(A)$ in each case.

(**Q2**) Let

$$x_{n+1} = g(x_n) = x_n + \epsilon \sin(2\pi x_n) \qquad \text{where} \quad x \in [0,1], |\epsilon| < \frac{1}{2\pi}.$$

(a) Show that g maps the unit interval to itself.

(b) Show that g has exactly three fixed points in the unit interval. If $\epsilon > 0$, then two are sources, and one is a sink.

(c) Determine the (un)stable manifold of each fixed point.

(d) Argue/Prove that the Hartman-Grobman conjugacy is defined on an entire (un)stable manifold.

(**Q3**) Let

$$\mathbf{x}_{n+1} = \begin{bmatrix} -1 & 1/2 \\ 1 & 0 \end{bmatrix} \mathbf{x}_n = \mathbf{A}\, \mathbf{x}_n$$

be a DS in the plane. Show that $\mathbf{0}$ is a hyperbolic fixed point. Determine the stable and unstable subspaces and sketch the phase portrait.

(**Q4**) Find the fixed points of the nonlinear map

$$x_{n+1} = x_n^2 - 5x_n + y_n, \quad y_{n+1} = \frac{1}{2} y_n + x_n^2,$$

and discuss their stability. Compute the third-order Maclaurin series of the stable manifold of $(0,0)$ [You will need to diagonalize the linear part of the system first].

(**Q5**) Let

$$x_{n+1} = x_n/10 + x_n y_n, \quad y_{n+1} = 2y_n + x_n^2. \qquad (DS)$$

Compute the third-order Maclaurin series of a transformation $\mathbf{u} = \mathbf{Q}(\mathbf{x})$ which transforms this (DS) into the linear DS

$$u_{n+1} = u_n/10, \quad v_{n+1} = 2v_n. \qquad (LDS)$$

That is, find $\mathbf{Q}$ such that $\mathbf{QF} = \mathbf{AQ}$ where $\mathbf{F}$ is the map defined by the RHS of (DS) and $\mathbf{A}$ is the matrix defined by the RHS of (LDS).

(**Q6**) Stable and unstable manifolds can be defined for saddles of continuous dynamical systems in the same way as they are defined for discrete dynamical systems. In particular, the stable manifold can be written as an expansion

$$u^- = a_2(u^+)^2 + a_3(u^+)^3 + a_4(u^+)^4 + a_5(u^+)^5 + \cdots.$$

The constant coefficients a_i can then be obtained by equating two expressions for $\dot{u}^-$. Consider the system

$$\left.\begin{aligned} \dot{x} &= -x + y^2 \\ \dot{y} &= y - x^2 \end{aligned}\right\}.$$

Find the coefficients a_2, a_3, a_4 and a_5 for the expansion of the stable manifold through $(0,0)$.

(**Q7**) Consider the equilibrium $\mathbf{0}$ of the system

$$\left.\begin{aligned} \dot{x} &= -x \\ \dot{y} &= 2y - x^2 \end{aligned}\right\}.$$

Give a series expansion of the stable manifold in the new variables, as far as 3rd order terms.

(**Q8**) Consider a $\mathbb{R}^2$ system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with an equilibrium point $\mathbf{x} = \mathbf{0}$ at which the linearised system has an imaginary pair of eigenvalues $\lambda = i\sigma$, $\overline{\lambda} = -i\sigma$, where $\sigma > 0$. In terms of a complex variable z, the dynamics are assumed to be given by

$$\dot{z} = \lambda z + az^2 + bz\overline{z} + c\overline{z}^2 + mz^2\overline{z} + \cdots \qquad (0.1)$$

Following the treatment of discrete systems in §3.3 of the course notes, show that

(a) the quadratic terms in (0.1) can be eliminated by introducing the new variable $w = z + \alpha z^2 + \beta z\overline{z} + \gamma \overline{z}^2$ with suitable α, β, γ;

(b) the further variable transformation $\zeta = w + dw^3 + ew^2\overline{w} + gw\overline{w}^2 + h\overline{w}^3$ with suitable d, e, g, h allows the elimination of all cubic terms apart from the term proportional to $\zeta^2\overline{\zeta}$, leading to

$$\dot{\zeta} = \lambda\zeta + q\zeta^2\overline{\zeta} + O(|\zeta|^4), \quad \text{with} \quad q = m + \frac{iab}{\sigma} - \frac{i|b|^2}{\sigma} - \frac{2i|c|^2}{3\sigma}.$$

Conclude that the origin is stable if $\mathrm{Re}\, q < 0$ and unstable if $\mathrm{Re}\, q > 0$.

(**Q9**) Let

$$\dot{x} = -y + x^2, \quad \dot{y} = x + y^2 x.$$

Is $(0,0)$ a stable or unstable equilibrium?

(**Q10**) Determine the stability of the origin for the dynamical system

$$\begin{aligned} x_{n+1} &= x_n - y_n + x_n^2 + y_n^3, \\ y_{n+1} &= x_n. \end{aligned}$$

At Examples Class 2 on Friday 29th October the solutions to Questions 5,6 and 10 will be discussed.