

(Q1) Consider the logistic map

$$x_{n+1} = F(x_n) = \mu x_n(1 - x_n), \quad \text{where } \mu > 0 \quad \text{and} \quad x \in [0, 1].$$

- (a) Find the fixed points of this map. For which values of μ do they exist?
 (b) Find a period-2 orbit of the map (i.e., find x_0 and $x_1 \neq x_0$ such that $x_1 = F(x_0)$ and $F(x_1) = x_0$). For which values of μ does it exist?

(Q2) Show that the discrete system

$$x_{n+1} = \frac{1}{4} - \frac{1}{2a} - a^2 x_n^2$$

is equivalent to the logistic map with $\mu = a$. (Hint: the variable transformation relating the two systems is affine; i.e., $y_n = \alpha x_n + \beta$.)

(Q3) Show that the map

$$\theta_{n+1} \equiv 2\theta_n \pmod{1}$$

can be transformed into the logistic map with $\mu = 4$ and $0 \leq x_n \leq 1$ by the change of variable $x_n = \sin^2(\pi\theta_n)$. Find x_0 such that $x_8 = x_0$ but $x_1, \dots, x_7 \neq x_0$.

(Q4) Find the Poincaré map of the autonomous system

$$\ddot{x} + 2\dot{x} + 5x = 0$$

for $\Sigma = \{(x, y) = (x, 0), x > 0\}$.

(Q5) Find the Poincaré map of the periodically forced system $\ddot{x} + x = \cos 2t$.

(Q6) If $\{x_n\}$ satisfies the recurrence relation

$$x_{n+1} = \lambda x_n e^{-x_n}$$

where $0 < \lambda < 1$ and $x_0 > 0$, show that $x_n \rightarrow 0$ as $n \rightarrow \infty$. Find the fixed point different from 0 which exists for $\lambda > 1$.

(Q7) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Compute e^{tA} . What does this imply for the orbits of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$?

(Q8) Consider the one-dimensional map $x_{n+1} = F(x_n)$, where $F(x) = x - hx^3$.

- (a) Compute F^2 , the second iterate of the map.
 (b) Deduce that $x = \sqrt{2/h}$ belongs to a period-2 orbit.
 (c) What is the other point of this periodic orbit?

(Q9) Consider the system $x_{n+1} = -\frac{2}{3}x_n + y_n$, $y_{n+1} = \frac{1}{3}(-4x_n + 5y_n)$, with $x_0 = \alpha$, $y_0 = \beta$. Show that $(x_n, y_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$, for any choice of α and β . Show also that the convergence is faster if $\alpha = \beta$ than if $\alpha \neq \beta$.

(Q10) For a positive real number x let $[x]$ be the floor of x —the largest integer that is less than or equal to x —and let $\{x\} = x - [x]$ be the fractional part of x . The **Gauss map** is defined as $g(x) = \{\frac{1}{x}\}$ if $x \neq 0$ and $g(0) = 0$. Define a discrete DS

$$x_{n+1} = g(x_n).$$

- (a) Show that the range of g is $[0, 1)$.
- (b) Show that g has a fixed point at $x = 0$.
- (c) Show that for each positive integer k there is a fixed point x_k^* of g such that $x_k^* = \frac{1}{k} - \frac{1}{k^3} + \dots$ for $k \geq 2$. [Hint: to see the plausibility, draw the graphs of $y = g(x)$ and $y = x$.]
- (d) Let $a_i, i = 1, \dots$ be non-negative integers which satisfy the property that $a_i = 0$ implies $a_j = 0$ for all $j \geq i$. Let $\alpha = [a_1, \dots]$ denote the continued fraction

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}. \quad (1)$$

It is a fact that every irrational real number $\alpha \in [0, 1)$ has a unique continued fraction expansion of this form. Assuming this fact, prove that if $x_1 = \alpha$ and $\forall n \geq 1$

$$x_{n+1} = g(x_n) \quad (2)$$

then $\forall n \geq 1$

$$a_n = [1/x_n]. \quad (3)$$

- (e) Conversely, argue that if we take (2–3) to be the definition of the a_i , then $\alpha = x_1$ is equal to the right-hand side of (1). In other words, from the Gauss map, we can derive the continued fraction expansion of α .
- (f) Assume that $\alpha = p/q$ is a rational number in $[0, 1]$ and write $x_n = p_n/q_n$ where p_n, q_n are coprime (by convention $0 = 0/1$). Show that while $x_n \neq 0$ the denominator q_n is monotonically decreasing: $q_{n+1} < q_n$. Show that there is some N such that $x_n = 0$ for $n \geq N$.
- (g) Say that a continued fraction $[a_1, \dots]$ has a tail of zeros if there is an N s.t. $a_n = 0$ for $n \geq N$. Prove that α is rational iff its continued fraction $[a_1, \dots]$ has a tail of zeros iff α is eventually fixed by g .
- (h) Let $\alpha \in [0, 1)$ be a prime-period-2 periodic point of the Gauss map. Show that α is the root of a quadratic polynomial with rational coefficients. What are the coefficients in terms of the continued fraction expansion of α ?
- (i) Say that a point $x = x_1$ is eventually periodic if there is some n such that x_n is a periodic point. For example, you showed above that all rationals are eventually periodic. Fact (Lagrange): every eventually periodic irrational number is the root of a quadratic polynomial with rational coefficients.
- (h) [Maple] Compute the continued fraction expansion of $\alpha = e - 2$. The answer was known to Euler, whose 300th birthday was April 15, 2007.