(**Q1**) Consider the logistic map

$$x_{n+1} = F(x_n) = \mu x_n(1 - x_n), \qquad \text{where } \mu > 0 \quad \text{and} \quad x \in [0, 1].$$

(a) Find the fixed points of this map. For which values of μ do they exist?

(b) Find a period-2 orbit of the map (i.e., find x_0 and $x_1 \neq x_0$ such that $x_1 = F(x_0)$ and $F(x_1) = x_0$). For which values of μ does it exist?

(**Q2**) Show that the discrete system

$$x_{n+1} = \frac{1}{4} - \frac{1}{2a} - a^2 x_n^2$$

is equivalent to the logistic map with $\mu = a$. (Hint: the variable transformation relating the two systems is affine; i.e., $y_n = \alpha x_n + \beta$.)

(**Q3**) Show that the map

$$\theta_{n+1} \equiv 2\theta_n \mod 1$$

can be transformed into the logistic map with $\mu = 4$ and $0 \leq x_n \leq 1$ by the change of variable $x_n = \sin^2(\pi\theta_n)$. Find x_0 such that $x_8 = x_0$ but $x_1, \ldots, x_7 \neq x_0$.

(**Q4**) Find the Poincaré map of the autonomous system

$$\ddot{x} + 2\dot{x} + 5x = 0$$

for $\Sigma = \{(x, y) = (x, 0), x > 0\}$.

(**Q5**) Find the Poincaré map of the periodically forced system $\ddot{x} + x = \cos 2t$.

(**Q6**) If $\{x_n\}$ satisfies the recurrence relation

$$x_{n+1} = \lambda x_n e^{-x_n}$$

where $0 < \lambda < 1$ and $x_0 > 0$, show that $x_n \to 0$ as $n \to \infty$. Find the fixed point different from 0 which exists for $\lambda > 1$.

(**Q7**) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Compute e^{tA}. What does this imply for the orbits of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$?

(**Q8**) Consider the one-dimensional map $x_{n+1} = F(x_n)$, where $F(x) = x - hx^3$.

(a) Compute F^2, the second iterate of the map.

(b) Deduce that $x = \sqrt{2/h}$ belongs to a period-2 orbit.

(c) What is the other point of this periodic orbit?

(**Q9**) Consider the system $x_{n+1} = -\frac{2}{3}x_n + y_n, \quad y_{n+1} = \frac{1}{3}(-4x_n + 5y_n)$, with $x_0 = \alpha$, $y_0 = \beta$. Show that $(x_n, y_n) \to (0, 0)$ as $n \to \infty$, for any choice of α and β. Show also that the convergence is faster if $\alpha = \beta$ than if $\alpha \neq \beta$.

(**Q10**) For a positive real number x let $[x]$ be the floor of x–the largest integer that is less than or equal to x – and let $\{x\} = x - [x]$ be the fractional part of x. The **Gauss map** is defined as $g(x) = \{\frac{1}{x}\}$ if $x \neq 0$ and $g(0) = 0$. Define a discrete DS

$$x_{n+1} = g(x_n).$$

(a) Show that the range of g is $[0, 1)$.

(b) Show that g has a fixed point at $x = 0$.

(c) Show that for each positive integer k there is a fixed point x_k^* of g such that $x_k^* = \frac{1}{k} - \frac{1}{k^3} + \cdots$ for $k \geq 2$. [Hint: to see the plausibility, draw the graphs of $y = g(x)$ and $y = x$.]

(d) Let a_i, $i = 1, \ldots$ be non-negative integers which satisfy the property that $a_i = 0$ implies $a_j = 0$ for all $j \geq i$. Let $\alpha = [a_1, \ldots]$ denote the continued fraction

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}}. \tag{1}$$

It is a fact that every irrational real number $\alpha \in [0, 1)$ has a unique continued fraction expansion of this form. Assuming this fact, prove that if $x_1 = \alpha$ and $\forall n \geq 1$

$$x_{n+1} = g(x_n) \tag{2}$$

then $\forall n \geq 1$

$$a_n = [1/x_n]. \tag{3}$$

(e) Conversely, argue that if we take (2–3) to be the definition of the a_i, then $\alpha = x_1$ is equal to the right-hand side of (1). In other words, from the Gauss map, we can derive the continued fraction expansion of α.

(f) Assume that $\alpha = p/q$ is a rational number in $[0, 1]$ and write $x_n = p_n/q_n$ where p_n, q_n are coprime (by convention $0 = 0/1$). Show that while $x_n \neq 0$ the denominator q_n is monotonically decreasing: $q_{n+1} < q_n$. Show that there is some N such that $x_n = 0$ for $n \geq N$.

(g) Say that a continued fraction $[a_1, \ldots]$ has a tail of zeros if there is an N s.t. $a_n = 0$ for $n \geq N$. Prove that α is rational iff its continued fraction $[a_1, \ldots]$ has a tail of zeros iff α is eventually fixed by g.

(h) Let $\alpha \in [0, 1)$ be a prime-period-2 periodic point of the Gauss map. Show that α is the root of a quadratic polynomial with rational coefficients. What are the coefficients in terms of the continued fraction expansion of α?

(i) Say that a point $x = x_1$ is eventually periodic if there is some n such that x_n is a periodic point. For example, you showed above that all rationals are eventually periodic. Fact (Lagrange): every eventually periodic irrational number is the root of a quadratic polynomial with rational coefficients.

(h) [Maple] Compute the continued fraction expansion of $\alpha = e - 2$. The answer was known to Euler, whose 300th birthday was April 15, 2007.

At Examples Class 1 on Tuesday 5 October the solutions to Questions 1 and 6 will be discussed.