(**Q1**) Consider the set Σ of sequences of two symbols $0, 1$, i.e., $\Sigma = \{0,1\}^{\mathbb{N}}$, with a distance between sequences $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma$ and $\mathbf{t} = (t_0, t_1, t_2, \cdots) \in \Sigma$ defined by

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

(a) Show that d defines a metric on Σ.

(b) Show that $d(\mathbf{s}, \mathbf{t}) \leq 2$.

Solution.

(a) Let $\eta, \omega, \xi \in \Sigma$. We want to prove that

i. $d(\eta, \omega) \geq 0$ and $d(\eta, \omega) = 0$ iff $\eta = \omega$;

ii. $d(\eta, \omega) = d(\omega, \eta)$;

iii. $d(\eta, \xi) \leq d(\eta, \omega) + d(\omega, \xi)$.

The first two properties are obvious. The third property, the triangle inequality, follows from the triangle inequality for real numbers: since $|\eta_i - \xi_i| \leq |\eta_i - \omega_i| + |\omega_i - \xi_i|$ for all i, $\sum_{i=0}^{\infty} \frac{|\eta_i - \xi_i|}{2^i} \leq \sum_{i=0}^{\infty} \frac{|\eta_i - \omega_i|}{2^i} + \sum_{i=0}^{\infty} \frac{|\omega_i - \xi_i|}{2^i}$. This proves that $d(\eta, \xi) \leq d(\eta, \omega) + d(\omega, \xi)$.

(b) For all $\eta, \xi \in \Sigma$, $|\eta_i - \xi_i| \leq 1$ for all i. Thus $d(\eta, \xi) \leq \sum_{i=0}^{\infty} 2^{-i} = 2$.

(**Q2**) With the same definitions as in Question 1, let $\mathbf{s}, \mathbf{t}$ and $\mathbf{r}$ be the periodic sequences

$$\mathbf{s} = (1,1,2,1,1,2,1,1,2,\cdots), \;\; \mathbf{t} = (1,2,1,2,1,2,1,2,\cdots), \;\; \mathbf{r} = (2,1,2,1,2,1,2,1,\cdots).$$

Calculate $d(\mathbf{s}, \mathbf{t})$, $d(\mathbf{t}, \mathbf{r})$ and $d(\mathbf{r}, \mathbf{s})$.

Solution. We see that $\mathbf{s} - \mathbf{t} = (0, -1, 1, -1, 0, 0, 0, -1, 1, -1, \cdots)$ which is periodic of period 6. It follows that

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} 2^{-6i}(0/1 + 1/2 + 1/4 + 1/8 + 0/16 + 0/32) = \frac{7}{8} \times \frac{1}{1 - 2^{-6}} = \frac{56}{63}.$$

Since $\mathbf{t} - \mathbf{r} = (-1, 1, -1, 1, \cdots)$ we see that $d(\mathbf{t}, \mathbf{r}) = 2$.

Since $\mathbf{s} - \mathbf{r} = (-1, 0, 0, 0, -1, 1, \cdots)$ is periodic of period 6, we see that

$$d(\mathbf{s}, \mathbf{r}) = \sum_{i=0}^{\infty} 2^{-6i}(1/1 + 0/2 + 0/4 + 0/8 + 1/16 + 1/32) = \frac{35}{32} \times \frac{1}{1 - 2^{-6}} = \frac{70}{63}.$$

(**Q3**) Let $\Sigma' \subset \Sigma = \{0,1\}^{\mathbb{N}}$ be the set of all sequences $\mathbf{s}$ of two symbols $0, 1$ with $s_{j+1} = 0$ if $s_j = 1$ (i.e., the sequences in Σ' do not have two consecutive 1's).

(a) Confirm that the shift map σ preserves Σ'.

(b) Show that periodic points are dense in Σ'.

(c) Show that there is a dense orbit in Σ'.

(d) How many fixed points of σ are there in Σ'? How many period-2 and period-3 orbits?

Solution.

(a) Let $\eta \in \Sigma'$ so that if $\eta_j = 1$, then $\eta_{j+1} = 0$ for all $j \geq 0$. Therefore, if $\eta_{j+1} = 1$, then $\eta_{j+2} = 0$ for all $j \geq 0$. Since $\sigma(\eta)_j = \eta_{j+1}$, this shows that $\sigma(\eta) \in \Sigma'$.

(b) Let $\eta \in \Sigma'$ and $\epsilon > 0$ be given. We want to find a periodic point $\omega \in \Sigma'$ of σ such that $d(\eta, \omega) < \epsilon$.

Let $N > 0$ be such that $2^{-N} < \epsilon$. Let $\omega_i = \eta_i$ for $i = 0, \ldots, N$. We would like to then define $\omega_{i+N+1} = \omega_i$ for all i. However, we may have a problem if $\eta_N = 1$ and η_0, which would imply $\omega_N = 1 = \omega_{N+1}$, and so ω would not be in Σ'. If this occurs, then increase N by 1. Since $\eta \in \Sigma'$, this choice guarantees that $\omega \in \Sigma'$. It is clear that $d(\eta, \omega) < \epsilon$ and ω is a periodic point of σ.

(c) Let us do the following: say that a "word" of length k is a sequence of zeroes and ones of length k. A word is "admissible" if it satisfies the condition that any 1 must be followed by a 0 (except if the 1 is the final letter of the word). We can concatenate admissible words w_1 and w_2 as follows: if w_1 ends with a 1 and w_2 begins with a 1, then we put a 0 between the words: $w_1 0 w_2$; in all other cases, $w_1 w_2$ is an admissible word. We will denote the concatenation operation by $w_1 \cdot w_2$.

Since the set of admissible words of length k is finite, the set of all admissible words is countable. Let $w_1, \ldots, w_n, \ldots$ be an enumeration of all admissible words. Let $\omega = w_1 \cdot w_2 \cdots w_n \cdots$ be the concatenation of all admissible words. Since each 1 that appears in ω is followed by a 0, $\omega \in \Sigma'$.

Claim: The orbit of ω is dense in Σ.

Check: Let $\eta \in \Sigma'$ and let $\epsilon > 0$ be given. Let $N > \log_2 \epsilon^{-1} + 1$. From the definition of the metric d, we know that if $\eta'_i = \eta_i$ for $i = 0, \ldots, N$, then $d(\eta', \eta), \epsilon$.

Let $w = \eta_0, \eta_1, \ldots, \eta_N$. This is an admissible word of length $N+1$. Therefore, w appears in ω, or in other words $\omega = w_0 \cdot w \cdots$ where w_0 is an admissible word of length K for some K (and there is no 0 padded between w_0 and w). Therefore $\sigma^K(\omega) = w \cdots$. Thus

$$d(\sigma^K(\omega), \eta) < \epsilon.$$

This prove that the orbit of ω is dense.

(d) Observe that to find the periodic points of $\sigma|\Sigma'$, we can find the periodic points of σ that lie in Σ'. Observe also that each period-k periodic point of $\sigma|\Sigma'$ corresponds to a unique admissible word w of length k such that ww is also admissible (no 0 padding).

$k = 1$: (fixed point) Only 0 and 1 are admissible length-1 words and 00 is admissible but 11 is not. Thus $\sigma|\Sigma'$ has only one fixed point: $\omega = (0, 0, 0, \ldots)$.

$k = 2$: Only 00, 10, 01 are admissible length-2 words and each produces a periodic point of $\sigma|\Sigma'$.

$k = 3$: Only 000, 100, 010, 001 and 101 are admissible length-3 words. The first four produce period-3 points for $\sigma|\Sigma'$ but the fifth does not produce a point in Σ since 101101 is not admissible.

In total, there are: 1 fixed point, 3 period-2 points and 4 period-4 points. There are 2 prime period-2 points and 3 prime period-3 points.

(**Q4**) Consider the one-sided shift map σ acting on sequences of N symbols, i.e., acting on $\Sigma = \{1, 2, \cdots, N\}^{\mathbb{N}}$.

(a) How many fixed points of σ^k are there?

(b) How many period-2 and period-4 orbits of σ are there in Σ? How many prime period-2 and -4 orbits are there?

Solution.

(a) Each fixed point of σ^k corresponds uniquely to a word of length k in N symbols, so there are N^k fixed points.

(b) N^2 and N^4. As there are N^1 fixed points, there are $N^2 - N^1$ prime period-2 points. Since a period-4 point which is not prime period-4 must also be a period-2 point, there are $N^4 - N^2$ prime period-4 points.

(**Q5**) Consider a one–dimensional mapping $F(x_n)$ with m prime periodic orbit

$$\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{m-1}).$$

Show that the Liapunov exponent of an orbit attracted to this periodic orbit is given by

$$\lambda = \frac{1}{m} \ln \left| \prod_{i=0}^{m-1} F'(x_i) \right|.$$

Thereby, show that $\lambda < 0$.

Solution. Let y_0 have the orbit y_i which converges to this periodic orbit. Possibly after we have relabeled elements in the periodic orbit, we can assume that for each i: $y_{i+km} \to x_i$ as $k \to \infty$. Then, assuming that F' is continuous at each point of the periodic orbit, $F'(y_{i+km}) \to F'(x_i)$.

We know that

$$\begin{aligned}
\lambda(y_0) &= \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N} \ln |F'(y_j)| \\
&= \lim_{N\to\infty} \frac{1}{mN} \sum_{k=0}^{N} \sum_{j=0}^{m-1} \ln |F'(y_{j+km})| \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \frac{1}{m} \sum_{j=0}^{m-1} \ln |F'(y_{j+km})| \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \frac{1}{m} \sum_{j=0}^{m-1} \ln |F'(x_j)| \times \frac{\ln |F'(y_{j+km})|}{\ln |F'(x_j)|}
\end{aligned}$$

Let $\epsilon > 0$ be given. Since $y_{i+km} \to x_j$ as $k \to \infty$, there is a K such that for all $k \geq K$, $1 - \epsilon < \frac{\ln|F'(y_{j+km})|}{\ln|F'(x_j)|} < 1 + \epsilon$ for all j. Now, since K is fixed relative to N,

$$\begin{aligned}\lambda(y_0) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{K} \frac{1}{m} \sum_{j=0}^{m-1} \ln|F'(y_{j+km})| + \frac{1}{N} \sum_{k=K+1}^{N} \frac{1}{m} \sum_{j=0}^{m-1} \ln|F'(y_{j+km})| \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=K+1}^{N} \frac{1}{m} \sum_{j=0}^{m-1} \ln|F'(y_{j+km})|.\end{aligned}$$

Thus

$$(1-\epsilon) \times \frac{1}{m} \sum_{j=0}^{m-1} \ln|F'(x_j)| \leq \lambda(y_0) \leq (1+\epsilon) \times \frac{1}{m} \sum_{j=0}^{m-1} \ln|F'(x_j)|.$$

Since $\epsilon > 0$ was arbitrary, this proves the claim.

Since $\mathbf{x}$ is an attracting periodic orbit, $|F'(x_i)| < 1$ for all i. This proves that $\lambda(y_0) < 0$.

(**Q6**) Find the Liapunov exponent of the logistic map $F_\mu(x) = \mu x(1-x)$ for $x \in [0,1]$ where:

(a) $1 < \mu < 3$
(Hint: You may assume that: (a) there exists at most one attracting period orbit for the logistic map; and (b) the basin of attraction for this attracting periodic orbit comprises the entire closed interval $[0,1]$ minus any repelling fixed points).

(b) $3 < \mu < 1 + \sqrt{6}$.

(Hint: use the result of Question 5).

Solution.

(a) $1 < \mu < 3$
By Q1 of PS 1, there is a unique attracting fixed point $x = 1 - 1/\mu$ in this range. We have that $F'_\mu(x) = \mu - 2(-1+\mu) = -\mu + 2$. By the hint and Q5, the Lyapunov exponent of any $y_0 \neq 0$ is $\ln|2-\mu|$.

(b) $3 < \mu < 1 + \sqrt{6}$.
By Q1 of PS 1, there is a unique attracting period-2 orbit x_+, x_-. We have that $F'_\mu(x_+)F'_\mu(x_-) = (\mu - (\mu + 1 + \sqrt{a}) \times (\mu - (\mu + 1 - \sqrt{a}) = 1 - a$ where $a = (\mu-3)(\mu-1)$. By Q5, we have that the Lyapunov exponent of any $y_0 \neq 0, 1 - 1/\mu$ is $\ln|1-a|^{\frac{1}{2}}$.

(**Q7**) Let $f : [0,1] \to [0,1]$ be defined as follows

$$f(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/4, \\ -(x - \frac{1}{4})(\frac{7}{8} - x) & \text{if } 1/4 < x < 7/8, \\ 2(x - 7/8) & \text{if } 7/8 \leq x \leq 1. \end{cases}$$

Let $I_0 = [0, \frac{1}{4}]$ and $I_1 = [\frac{7}{8}, 1]$. The aim of this exercise is to show that there is an invariant set $\Lambda \subset [0,1]$ and a homeomorphism $h : \Lambda \to \Sigma'$ (see Q3) such that $h \circ f|\Lambda = \sigma \circ h$.

(a) Show that $I_0 \cup I_1 \subset f(I_0)$ and $I_0 \subset f(I_1)$.

(b) Show that if $\omega \in \Sigma'$, then the set $I_\omega = \{x \in [0,1] \ : \ f^n(x) \in I_{\omega_n} \text{ for all } n\}$ is non-empty, and contains a single point.

(c) Let $\Lambda = \cap_{n\geq 0} f^{-n}([0,1])$. Show that if $x \in \Lambda$ iff $f^n(x) \in [0,1]$ for all $n \geq 0$.

(d) Show that if $x \in \Lambda$, then $x \in I_0 \cup I_1$. Conclude that $f^n(x) \in \Lambda$ for all $n \geq 0$. Hence show that the itinerary map $h(x) = \omega$ is well-defined.

(e) Prove that h is continuous, 1-1 and onto.

(f) How many periodic orbits of period 2, 3 and 6 does f have?

Solution.

(a) Since $f : x \mapsto 4x$ on I_0, it $I = [0,1] \subset f(I_0)$. Since $f|I_1$ is affine, with $f(7/8) = 0$ and $f(1) = 1/4$, f maps I_1 onto I_0.
Let us note that $f|I_0$ and $f|I_1$ is a 1-1 map.

(b) Let $\omega \in \Sigma'$. Define $I_{\omega_0,\ldots,\omega_n} = \{x \in I \ : \ f^k(x) \in I_{\omega_k} \text{ for } k = 0,\ldots,n\}$.
Claim: For all $n \geq 0$, and all $\omega \in \Sigma'$, $I_{\omega_0,\ldots,\omega_n}$ is a non-empty interval.
Check: For $n = 0$, this is trivially true. Assume that it is true for $0,\ldots,n-1$. Now, by the induction hypothesis $I_{\omega_1,\ldots,\omega_n}$ is a non-empty interval that is contained in I_{ω_1}.
There are several possibilities to verify. If $\omega_0 = 0$, $\omega_1 = 0$, then part (a) shows that there is a unique interval in $K \subset I_{\omega_0}$ s.t. $f(K) = I_{\omega_1,\ldots,\omega_n}$. This interval K is the sought after interval $I_{\omega_0,\omega_1,\ldots,\omega_n}$.
The argument is similar for $\omega_0 = 0, \omega_1 = 1$ and $\omega_0 = 1, \omega_1 = 0$. However, the argument fails when $\omega_0 = 1 = \omega_1$ – which is fortunate, because that cannot occur when $\omega \in \Sigma'$!
Thus, we have proven the claim by induction.

Claim: For all $n \geq 0$, and all $\omega \in \Sigma'$, $I_{\omega_0,\ldots,\omega_n}$ has length $\leq 1/2^{n-1}$.
Check: The claim is true for $n = 0$. For $n \geq 1$, we observe that $f'|I_0 \cup I_1 \geq 2$. Therefore, if $x_0, y_0 \in I_{\omega_0,\ldots,\omega_n}$, then $x_k = f^k(x_0), y_k = f^k(y_0) \in I_{\omega_k}$ for each $k = 0,\ldots,n$. The mean-value theorem says that

$$|x_n - y_n| \geq 2^1|x_{n-1} - y_{n-1}| \geq 2^2|x_{n-2} - y_{n-2}| \geq \cdots \geq 2^{n+1}|x_0 - y_0|.$$

Since x_n, y_n both lie in I_0 or I_1, their distance apart is at most $1/4$. Thus

$$|x_0 - y_0| \leq 1/2^{n-1}.$$

This shows that any two points in $I_{\omega_0,\ldots,\omega_n}$ are at most $1/2^{n-1}$ apart. This proves the claim.
Clearly, $I_{\omega_0} \supset I_{\omega_0,\omega_1} \supset \cdots \supset I_{\omega_0,\ldots,\omega_n} \supset \cdots$, so we have a nested sequence of compact intervals so their intersection I_ω is non-empty.
Since the length of these intervals converges to 0, I_ω contains a single point.

(c) Let $x \in \Lambda$. Then $x \in f^{-n}(I)$ for all $n \geq 0$. Therefore $f^n(x) \in I$ for all $n \geq 0$. On the other hand, if $f^n(x) \in I$ for all $n \geq 0$, then $x \in f^{-n}(I)$ for all $n \geq 0$, so $x \in \Lambda$.

(d) Assume that $x \in \Lambda$. Then $f(x) \in \Lambda$ by (c). The formula for $f(x)$ shows that if $x \notin I_0 \cup I_1$, then $f(x) < 0$ so $f(x) \notin \Lambda$. Hence $x \in \Lambda$ implies that $x \in I_0 \cup I_1$. Since Λ is f-invariant, $f^n(x) \in I_0 \cup I_1$ for all $n \geq 0$. Since I_0, I_1 are disjoint, the itinerary map – $h(x) = \omega$ iff $f^n(x) \in I_{\omega_n}$ for all n – is well-defined.

(e) h is continuous: Let $\epsilon > 0$ be given, and let $x \in \Lambda$. Let $\omega = h(x)$ and let $N > \log_2 \epsilon^{-1}$.

From (b), if $|x - y| < 2^{-N-1}$, and $y \in \Lambda$, then $y \in I_{\omega_0,\ldots,\omega_N}$ so x and y share the same itinerary up to the N-th iterate. Thus

$$y \in \Lambda, |x - y| < 2^{-N-1} \implies d(h(x), h(y)) < \epsilon.$$

h is 1-1: If $h(x) = \omega = h(y)$, then from (b), the distance between x and y is at most 2^{-n+1} for all $n \geq 0$. Hence $x = y$.

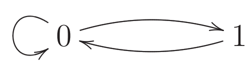
h is onto: From (b), I_ω is non-empty for all $\omega \in \Sigma'$. Thus, there is an x s.t. $h(x) = \omega$.

(f) Any periodic point of f lies in Λ and our previous work shows that $f|\Lambda$ is conjugate to $\sigma|\Sigma'$. Therefore, we can do all our calculations with the shift map.

We computed the answer for periods 2 and 3 in Q3. For period 6, we can write out all admissible length 6 words, and then pare this list down, as we did in the earlier examples. However, here is a better method.

Recall that a periodic point of period k for $\sigma|\Sigma'$ corresponds to a closed path on the graph G of length k.

G :



Let A be the adjacency matrix of G: that is $A_{ij} = 1$ iff there is an oriented edge in G running from vertex i to vertex j.

Claim: The number of closed paths of length k in the graph G is Trace(A^k).

Check: For $k = 1$ this is true, as a closed path of length 1 is just a loop from vertex i to vertex i.

Let us suppose that $(i_1, i_2, \ldots, i_{k+1})$ is a path of length k in the graph G, which means that we start at vertex i_1, proceed to i_2, etc. It follows that $A_{i_s,i_{s+1}} = 1$ for all $s = 1, \ldots, k$. On the other hand, if $A_{i_s,i_{s+1}} = 1$ for all $s = 1, \ldots, k$, then there is a path $(i_1, i_2, \ldots, i_{k+1})$ is a path of length k in the graph G.

Observe that

$$A^k_{a,b} = \sum_{i_2, i_3 \cdots, i_k} A_{a,i_2} A_{i_2,i_3} \cdots A_{i_k,b}.$$

Since $A_{a,i_2} A_{i_2,i_3} \cdots A_{i_k,b}$ is non-zero (whence 1) iff there is a path of length k in G from a to b, it follows that

$$A^k_{a,b} = \{\text{length } k \text{ paths in G, from } a \text{ to } b\}.$$

The number of closed paths in G of length k is then $\sum_a A^k_{a,a}$ which is the trace of A^k. This proves the claim.

In our case

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

since one cannot be at 1 and stay at 1. Then

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, A^6 = \begin{bmatrix} 13 & * \\ * & 5 \end{bmatrix},$$

so there are $13 + 5 = 18$ period-6 points.
Therefore there are $18 - (4-1) - (3-1) - 1 = 12$ prime period-6 points.
Remark: The eigenvalues of A are $\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$. So

$$\text{Trace}(A^k) = \lambda_+^k + \lambda_-^k \sim 1.6^k.$$

The number of periodic points therefore grows exponentially.

(**Q8**) Let $f(x) = 4x(1-x)$ and let $\Sigma = \{0,1\}^{\mathbb{N}}$. Prove that there is a continuous surjection h such that

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \Big\downarrow h & & h \Big\downarrow \\ I & \xrightarrow{f} & I \end{array}$$

commutes (σ is the shift map). Describe the set of points where h fails to be injective, i.e. the set of $\omega \in \Sigma$ where $h^{-1}(h(\omega))$ contains more than one point. [Hint: find intervals J_0, J_1 with disjoint interiors such that $f(J_i) = I$ and $I = J_0 \cup J_1$. Try to define an itinerary map...]

Solution. Following the hint, let $J_0 = [0,p]$ and $J_1 = [p,1]$ where $p = \frac{1}{2}$. It is clear these intervals satisfy the properties suggested in the hint. For a point $x \in I$ whose orbit does not contain p, the itinerary of x is unambiguously defined. If the orbit of x contains p at say the k-th step, then $f^k(x) = p$, $f^{k+1}(x) = f(p) = 1$, $f^{k+2}(x) = f(1) = 0$ and then $f^{k+2+j}(0) = 0$ for all $j \geq 0$. Thus, the itinerary of x is unambiguous except at the k-th step, where $f^k(x) = p$ lies in both J_0 and J_1. In this case, the possible itineraries are:

$$\omega_0 \cdots \omega_{k-1} 0100 \cdots, \qquad \text{or} \qquad \omega_0 \cdots \omega_{k-1} 1100 \cdots,$$

where in both cases $\omega_0 \cdots \omega_{k-1}$ is the same sequence determined by $f^i(x) \in J_{\omega_i}$ for $i = 0, \ldots, k-1$.

We therefore see that every point $x \in I$ can be assigned at most 2 itineraries. Moreover, since $f(J_i) = J_0 \cup J_1$, the IVT argument implies that for any itinerary $\omega \in \Sigma$, there is an $x \in I$ which has an itinerary ω.

At Examples Class 4 on Friday 3rd December the solution to Questions 3 and 7 will be discussed.