(**Q1**) Consider the $\mathbb{R}^1$ map with fixed point $x^* = \alpha$ represented by the Taylor series

$$x_{n+1} = F(x_n) = \alpha + \beta_1 (x - \alpha) + \beta_2 (x - \alpha)^2 + \beta_3 (x - \alpha)^3 + \beta_4 (x - \alpha)^4 + \cdots$$

where $\beta_{1,2,3,4,\ldots}$ are constants. Let $G(x) = F(x + \alpha) - \alpha$. Show that $G$ has a fixed point at the origin and

$$D_s\{F\}(\alpha) = D_s\{G\}(0).$$

**Solution.**

We have that $G(0) = F(\alpha) - \alpha = 0$ since $F$ fixes $\alpha$. The chain implies that $G^{(k)}(x) = F^{(k)}(x + \alpha)$ for all $k \geq 1$. This implies that $D_s\{G\}(x) = D_s\{F\}(x + \alpha)$ for all $x$, which proves the claim.

(**Q2**) Consider the system $x_{n+1} = F_\mu(x_n)$, with $F_\mu(x_n) = \mu + x_n^2$ where $x_n, \mu \in \mathbb{R}$.

(a) Find the fixed points of the system in terms of $\mu$.

(b) Find the value of $x$, and the corresponding value of the parameter $\mu$, at which there is a saddle-node bifurcation.

(c) Find the value of $x$, and the corresponding value of the parameter $\mu$, at which there is a flip bifurcation. Is it super- or subcritical?

**Solution.** See the solution to Q1 in assignment 3.

(**Q3**) Let $I = [a, b]$ be a closed interval and $F : I \to I$ be a continuous function. Show that $F$ has a fixed point in $I$. (Hint: Intermediate Value Theorem).

**Solution.**

Let $f(x) = F(x) - x$. Since $F$ maps $I$ into itself, $F(a) \geq a$ and $F(b) \leq b$. Thus $f(a) \geq 0$ and $f(b) \leq 0$. Since $f$ is continuous, the intermediate value theorem says that there is an $\eta \in [a, b]$ such that $f(\eta) = 0$. Thus $F(\eta) = \eta$.

(**Q4**) Let $I = [a, b]$ be a closed interval and $F$ be a continuous function such that $F(I) \supset I$. Show that $F$ has a fixed point in $I$. (Hint: Intermediate Value Theorem).

**Solution.**

Since $F(I) \supset I$, we have that either $F(a) < a$ and $F(b) > b$ or $F(a) > b$ and $F(b) < a$. In the both cases, the intermediate value theorem can be applied as in Q3 to show the claim.

(**Q5**) Show that if the mapping $x_{n+1} = F(x_n)$ with $F(x)$ continuous has a period-2 orbit, then it also has a fixed point. (Hint: Intermediate Value Theorem).

**Solution.** Let $\{a, b\}$ be the period-2 orbit with $a < b$. Then $F(a) = b$ and $F(b) = a$. The intermediate theorem implies that $F(I) \supset I$ where $I = [a, b]$. [Check: let $a < y < b$. Then $F(a) > y$ and $F(b) < y$. Therefore there is an $x \in (a, b)$ s.t. $F(x) = y$.] Now apply Q4.

(**Q6**) Let $F : I \to I$ be a continuous map of $I = [0, 1]$. Show that if $F$ has a prime period-3 orbit, then $F$ has a fixed point and a prime period-2 point. This completes the proof of the simple Sharkovskii theorem.

**Solution.** Assume that the orbit is $\{a, b, c\}$ with $a < b < c$ and $F(a) = b, F(b) = c, F(c) = a$. $F$ maps the intervals $J_0 = [a, b]$ and $J_1 = [b, c]$ by $J_1 \subset F(J_0)$ and $J_0 \cup J_1 \subset F(J_1)$. Therefore, $J_1 \subset F(J_1)$ so $F$ has a fixed point in $J_1$. On the other hand, we showed that there are intervals $K_1 \subset J_0$ and $K_0 \subset J_1$ such that

$$F(K_0) = K_1, \quad F(K_1) = K_2 = J_1, \quad K_0 \subset F(K_2).$$

Therefore $K_0 \subset F^2(K_0)$ so $F^2$ has a fixed point $x$ in $K_0$. Since $x \in J_1, F(x) \in J_0$ and $x = F^2(x) \in J_1$, we see that if $x$ were a fixed point then $x \in J_0 \cap J_1 = \{b\}$. Since $b$ is not fixed, this proves that $x$ is a prime period-2 point.

(**Q7**) Show that the mapping $x_{n+1} = F(x_n)$

(a) has no prime period-$k$ orbits for $k \geq 2$ if $F'(x) > 0$;

(b) has a unique fixed point and no prime period-$k$ orbits for $k \geq 3$ if $F'(x) < 0$. (Hint: consider the ordering of the $x_j$ in a periodic orbit $(x_0, x_1, \cdots, x_{k-1})$; for $F'(x) < 0$, consider the sign of the derivative of $F^k(x)$.)

**Solution.** N.B.: The question must say *prime* period. It is false without this specification.

(a) Assume that $\{x_j\}$ is a *prime* period-$k \geq 2$ orbit, where $x_0$ is the smallest point on the orbit and $x_j = F^j(x_0)$. We see that $x_0 < x_1$ or $x_1 < x_0$. In the former case, since $F$ is increasing $x_j = F^j(x_0) < F^j(x_1) = x_{j+1}$ for all $j$, and so $x_0 < x_1 < \ldots < x_{k-1} < x_k$ But $x_k = x_0$. Absurd. The argument when $x_1 < x_0$ is similar.

(b) $F$ has a fixed point: because $F(x) - x \to -\infty$ as $x \to \infty$ (since $F$ is decreasing so $F(x) \leq F(0)$ for $x \geq 0$) and $F(x) - x \to +\infty$ as $x \to -\infty$ (since $F$ is decreasing $F(x) \geq F(0)$ for $x \leq 0$). The intermediate value theorem implies that $F(x) - x$ must vanish.

Assume that $F$ has a *prime* period-$k$ orbit with $k \geq 3$. Let $\{x_j\}$ be a prime period-$k \geq 3$ orbit, where $x_0$ is the smallest point on the orbit and $x_j = F^j(x_0)$. Observe that $F^2$ is increasing. Since the orbit has prime period $k \geq 3$, either $x_0 > x_2$ or $x_0 < x_2$. In the first case, since $F^2$ is increasing $x_2 = F^2(x_0) < F^2(x_2) = x_4$ and more generally, $x_0 < x_2 < x_4 < \cdots < x_{2n} < \cdots$. This contradicts the fact that the set $\{x_{2j}\}$ is finite (since the periodic orbit is finite). The case $x_0 > x_2$ is similar. This proves the claim.

(**Q8**) Consider the $\mathbb{R}^1$ family of mappings

$$x_{n+1} = G_\mu(x_n) = \mu x_n \left(1 - x_n^4\right) \qquad (\mu > 0).$$

(a) Find the fixed point of this mapping with $x > 0$. For which range of values of $\mu$ does it exist?

(b) Find the value of $\mu$ for which the fixed point with $x > 0$ undergoes a flip bifurcation and discuss its nature.

(c) The mapping undergoes a sequence of period–doubling bifurcations as $\mu$ increases. Describe briefly this phenomenon.

(d) Describe the nature of all period–doubling bifurcations of this mapping.

**Solution.**

(a) We wish to find those $x$ solving $x = G_\mu(x)$, that is,

$$x = \mu x(1 - x^4) \qquad \text{so}$$
$$x = \pm i\left(1 - \frac{1}{\mu}\right)^{\frac{1}{4}}, \text{ or } \qquad x = \pm\left(1 - \frac{1}{\mu}\right)^{\frac{1}{4}} \text{ or } \qquad x = 0. \tag{0.1}$$

We see that $x = 0$ is real for all $\mu$ and $x_\pm = \pm\left(1 - \frac{1}{\mu}\right)^{\frac{1}{4}}$ is real for $\mu \geq 1$ and $x_+ > 0$ for $\mu > 1$.

(b) We compute that $G'_\mu = -5\mu x^4 + \mu$, so the derivative of $G_\mu$ at the fixed point $x_+$ is equal to

$$-4\mu + 5. \tag{0.2}$$

In order to have a flip bifurcation, we need $G'_\mu(x_+) = -1$, so $\mu = 3/2$ and $x_+ = \frac{1}{3^{\frac{1}{4}}}$.

To determine the nature of the flip bifurcation, we compute the Schwartzian derivative,

$$\begin{aligned} D\{G_\mu\}(x) &= \frac{G'''_\mu(x)}{G'_\mu(x)} - \frac{3\left(G''_\mu(x)\right)^2}{2\left(G'_\mu(x)\right)^2} \\ &= -\frac{300x^6 + 60x^2}{25x^8 - 10x^4 + 1} \\ &= -\frac{300x^6 + 60x^2}{(5x^4 - 1)^2}. \end{aligned} \tag{0.3}$$

Since the denominator is positive everywhere except at $x = \pm 1/\sqrt[4]{5}$, we conclude that $D\{G_\mu\}(x_+) < 0$. Therefore, the bifurcation is a supercritical flip bifurcation.

(c–d) Since $D\{G_\mu\}(x) < 0$ for $x \neq \pm 1/\sqrt[4]{5}$, we see that $D\{G^n_\mu\}(x) < 0$ for all $x \neq \pm 1/\sqrt[4]{5}$ and all $n$. This implies that all period-doubling bifurcations are supercritical, so a stable period $2^n$ periodic orbit becomes unstable and gives birth to a nearby period $2^{n+1}$ stable orbit. See figure 0.1.

(**Q9**) Consider the $\mathbb{R}^1$ mapping

$$x_{n+1} = F_\mu(x_n) \quad \text{with} \quad F_\mu(x) = \mu x - x^3 \quad \text{and} \quad \mu > 0.$$

(a) Find the fixed points of the mapping $F_\mu$.

(b) Discuss the existence and stability of the fixed points in terms of $\mu$, and thereby show that the mapping undergoes bifurcations for $\mu = 1$ and $\mu = 2$.

(c) Describe the bifurcation which arises at $\mu = 1$. Sketch the fixed points of $F_\mu$ on a $(\mu, x)$ bifurcation diagram for $0 < \mu < 3$. Indicate stability on your sketch.

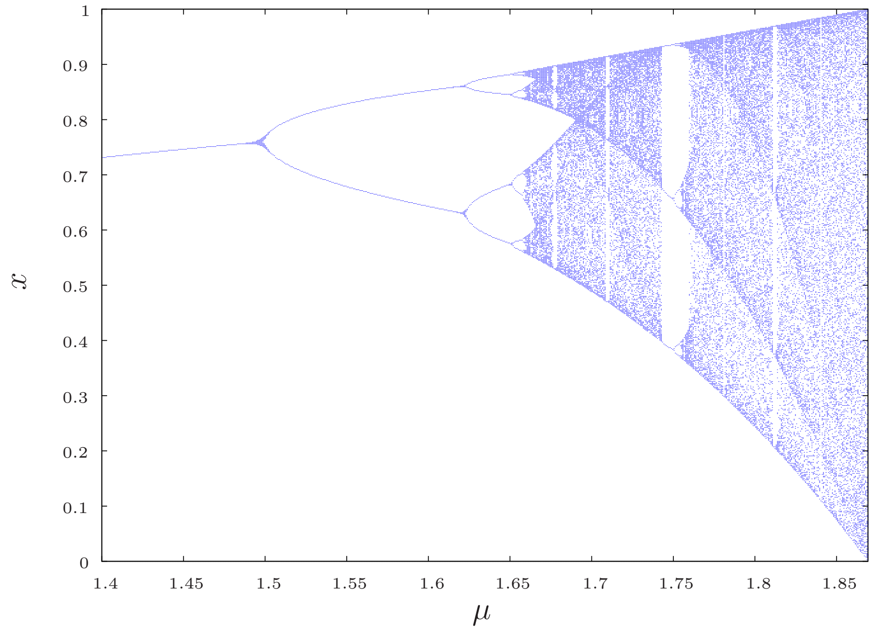



Figure 0.1: Period-doubling for the map $G_\mu$. The right-hand limit is $\mu = 5^{5/4}/4$, the parameter value beyond which $G_\mu$ no longer maps $[0,1]$ into $[0,1]$.

(d) Determine whether the flip bifurcations at $\mu = 2$ are supercritical or subcritical by computing the Schwarzian derivative of $F_\mu$. What are the implications of this result for period doubling?

(e) Consider the perturbed mapping

$$F_{\mu,\delta}(x) = \mu x - x^3 + \delta$$

such that $F_{\mu,0}(x) = F_\mu(x)$. For a fixed, small value of $\delta > 0$, sketch on a $(\mu, x)$ diagram the position of the fixed points of $F_{\mu,\delta}$.
(Hint: To sketch the position of the fixed points $x(\mu)$, it is convenient to consider the graph of the inverse relationship $\mu(x)$ and use reflection about the line $x = \mu$ to deduce the curves $x(\mu)$; there is then no need to solve the cubic equation for the fixed points explicitly).

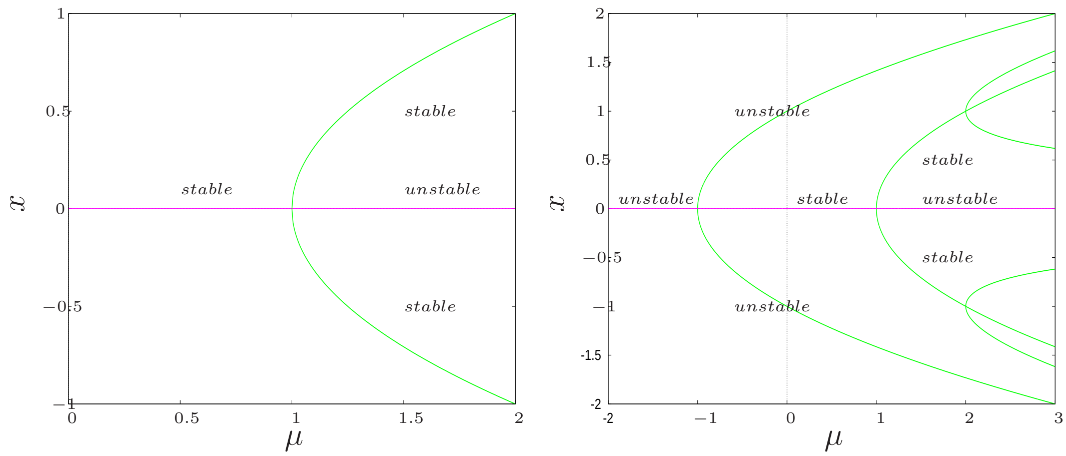
(f) Show that the mapping $F_{\mu,\delta}$ undergoes a bifurcation for $\mu = 1 + 3(\delta/2)^{2/3}$. What is the nature of this bifurcation?

**Solution.**

(a) We solve $x = F_\mu(x)$ to find that $x = 0$ or $1 = \mu - x^2$ so $x = \pm\sqrt{\mu - 1}$, for $\mu \geq 1$.

(b) Since $F'_\mu(0) = \mu$, at $\mu = 1$, the fixed point $x = 0$ goes from stable to unstable and bifurcates into a pair of stable fixed points at $x_\pm = \pm\sqrt{\mu - 1}$. At $\mu = 2$, $F'_\mu(x_\pm) = -1$, so these fixed points undergo a flip bifurcation at $\mu = 2$.

(c) See figure 0.2.




(a) $\mu \in (1, 3)$ (b) $\mu \in (-2, 4)$

Figure 0.2: Bifurcation diagram for the map $F_\mu$. The bifurcation at $\mu = 1$ is called a *pitchfork bifurcation*. Several pitchfork bifurcations are shown in (b).

(d) We compute the Schwartzian derivative of $F_\mu$ is

$$D\{F_\mu\}(x) = -\frac{6\,(6x^2 + \mu)}{(3x^2 - \mu)^2} \tag{0.4}$$

which is $< 0$ for all $\mu > 0$ and all $x$. It follows that the flip bifurcation at $\mu = 2$ is supercritial, as are all subsequent period-doubling bifurcations (see 0.2).

(e) See figure 0.3.

(f) The fixed points of $F_{\mu,\delta}$ satisfy

$$x = F_{\mu,\delta}(x) = \mu x - x^3 + \delta \qquad \Longrightarrow \mu = \frac{x^3 + x - \delta}{x}. \tag{0.5}$$

This describes $\mu$ as a function of the fixed point $x$. One computes that

$$F'_{\mu,\delta}(x) = \mu - 3x^2 \qquad \text{which equals 1 when } \mu = 1 + 3x^2. \tag{0.6}$$

When we equate (0.5) and (0.6), we find that the only real solution is $x = -\left(\frac{\delta}{2}\right)^{\frac{1}{3}}$ whence $\mu = 3\left(\frac{\delta}{2}\right)^{\frac{2}{3}} + 1$, as required.
We note that this is a 'blue-sky' or saddle-node bifurcation that takes place. Note though, that this is 'global', too, because the other fixed point has changed stability (its derivative passes through $-1$, in analogy with the pitchfork bifurcation above).

(**Q10**) Prove the following theorem:

**Theorem.**[Saddle-Node Bifurcation Theorem] Let $f_\mu(x)$ be a function that is $C^3$ in both variables. Assume that there is a $\mu_c, x_c$ such that

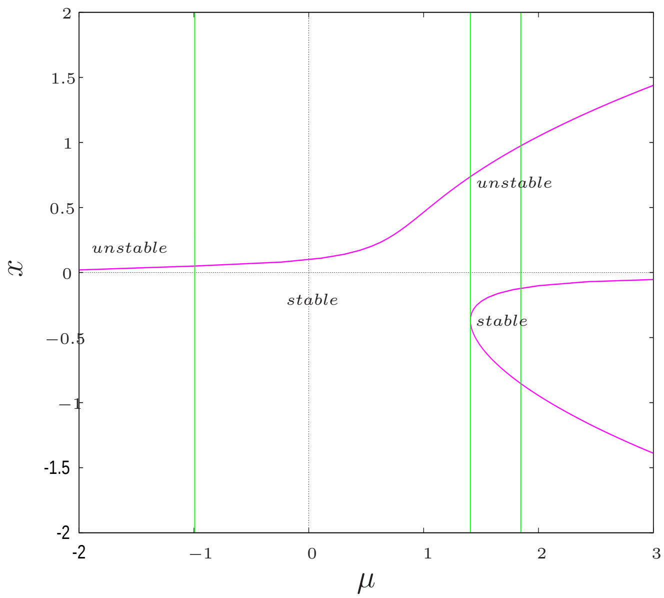



Figure 0.3: Bifurcation diagram for the map $F_{\mu,\delta}$ with $\delta = 1/10$. At approximately $\mu = 1.4$, the fixed point $x \simeq 0.74$ becomes unstable and a pair of stable fixed points appear (lower right). This graph looks like the pitchfork above, where the two tines have been slid off the central one.

(a) $x_c = f_{\mu_c}(x_c)$;

(b) $a = f''_{\mu_c}(x_c) \neq 0$;

(c) $b = \left.\dfrac{\partial f_\mu}{\partial \mu}\right|_{x=x_c,\mu=\mu_c} \neq 0$;

(d) $f'_{\mu_c}(x_c) = 1$.

Then there exists a $C^2$ function $\mu = \mu(x)$ such that

(i) $\mu(x_c) = \mu_c$;

(ii) $f_{\mu(x)}(x) = x$ for all $x$ near $x_c$; and

(iii) $\mu(x) = \mu_c - \dfrac{a}{2b}(x - x_c)^2 + O(|x - x_c|^3)$.

Conclude that $f_\mu$ undergoes a saddle-node bifurcation at $\mu = \mu_c$ and $f_\mu$ has fixed points $x_\pm(\mu) = x_c \pm \sqrt{\dfrac{-2b(\mu - \mu_c)}{a}} + O(|\mu - \mu_c|)$. [Hint: use the implicit function theorem.]

Compare the statement of the SNB Theorem and **Q2**.

**Solution.** We will use the hint supplied. Let us define

$$g(x, \mu) = f_\mu(x) - x, \tag{0.7}$$

which is $C^3$ in both variables. The hypotheses imply

$$g(x_c,\mu_c)=0 \qquad \left.\frac{\partial g}{\partial x}\right|_{x_c,\mu_c}=0, \qquad \left.\frac{\partial g}{\partial \mu}\right|_{x_c,\mu_c}=a\neq 0, \tag{0.8}$$

The final fact allows us to use the implicit function theorem. This implies that there is a $C^3$ function $\mu=\mu(x)$ that is defined in an interval $J$ containing $x_c$, with $\mu(x_c)=\mu_c$, such that

$$g(x,\mu(x))=0 \qquad \forall x\in J. \tag{0.9}$$

Applying implicit differentation to (0.9), we get

$$0=\frac{\partial}{\partial x}g(x,\mu(x))=f'_\mu(x)-1+\frac{\partial g}{\partial \mu}\cdot\mu'(x) \qquad \text{so}$$
$$\mu'(x_c)=0 \tag{0.10}$$

and

$$0=\frac{\partial^2}{\partial x^2}g(x,\mu(x))=f''_\mu(x)+\frac{\partial^2 g}{\partial x\partial \mu}\cdot\mu'(x)+\frac{\partial g}{\partial x}\cdot\mu''(x) \qquad \text{so}$$
$$\mu''(x_c)=-\frac{a}{b}. \tag{0.11}$$

Taylor's theorem implies that $\mu(x)=\mu(x_c)+\mu'(x_c)(x-x_c)+\frac{1}{2}\mu''(x_c)(x-x_c)^2+O(|x-x_c|^3)$. Thus

$$\mu(x)=\mu_c-\frac{a}{2b}(x-x_c)^2+O(|x-x_c|^3) \qquad \text{whence} \tag{0.12}$$
$$x=x_c\pm\sqrt{\frac{-2b(\mu-\mu_c)}{a}}+O(|\mu-\mu_c|). \tag{0.13}$$

(**Q11**) Prove the following theorem:

**Theorem.**[Period-Doubling/Flip Bifurcation Theorem] Let $f_\mu(x)$ be a function that is $C^4$ in both variables. Assume that there is a $\mu_c$ such that

(a) $0=f_\mu(0)$ for all $\mu$ near $\mu_c$;

(b) $f'_{\mu_c}(0)=-1$;

(c) $a=f'''_{\mu_c}(0)\neq 0$; and

(d) $b=\left.\dfrac{\partial\left(f_\mu^2\right)'}{\partial\mu}\right|_{x=0,\mu=\mu_c}\neq 0$;

(e) $f'_{\mu_c}(x_c)=1$.

Then there exists a $C^4$ function $\mu=\mu(x)$ defined near $x=0$ such that

(i) $\mu(0)=\mu_c$;

(ii) $f_{\mu(x)}(x) \neq x$, $f^2_{\mu(x)}(x) = x$ for all $x \neq 0$ near 0; and

(iii) $\mu(x) = \mu_c - \frac{a}{2b}(x - x_c)^2 + O(|x - x_c|^3)$.

Conclude that $f_\mu$ undergoes a saddle-node bifurcation at $\mu = \mu_c$ and $f_\mu$ has fixed points $x_\pm(\mu) = \pm\sqrt{\frac{-2b(\mu - \mu_c)}{a}} + O(|\mu - \mu_c|)$.

Hint: use the implicit function theorem for the function

$$H(x, \mu) = \begin{cases} \dfrac{f^2_\mu(x) - x}{x} & \text{if } x \neq 0, \\ (f^2_\mu)'(0) & \text{if } x = 0. \end{cases}$$

Compare the statement of the above Theorem, **Q9**e and our work in class.