(**Q1**) Consider the 2×2 real-valued matrix A, with trace τ and determinant Δ. In this question, you should make use of the Cayley–Hamilton theorem: matrix A satisfies its own characteristic equation; i.e., $A^2 - \tau A + \Delta I = 0$ where I is the identity matrix.

(a) Assuming that $\tau = 0$, show that

$$A^{2n} = (-1)^n \Delta^n I, \quad A^{2n+1} = (-1)^n \Delta^n A.$$

(b) Assuming that $\Delta = 0$, show that

$$A^n = \tau^{n-1} A.$$

Deduce simple expressions for $\exp(A)$ in each case.

Solution.

(a) $\tau = 0$: The Cayley-Hamilton theorem then says that $A^2 = -\Delta I$, whence $A^{2n} = (-\Delta)^n I$ and $A^{2n+1} = (-\Delta)^n A$. Then

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} \times A^n = \sum_{n=0} \frac{(-1)^n \Delta^n}{(2n)!} \times I + \sum_{n=0}^{\infty} \frac{(-1)^n \Delta^n}{(2n+1)!} \times A = \cos(\Delta) I + \sin(\Delta) A.$$

(b) $\Delta = 0$: In this case, $A^1 = \tau^0 A$ and $A^2 = \tau^1 A$ which proves the claim above for $n = 1, 2$. Assume that $A^n = \tau^{n-1} A$ for an n. Then $A^{n+1} = A(A^n) = \tau^{n-1} A^2 = \tau^n A$, which proves the claim by induction. Then

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} \times A^n = I + \sum_{n=1}^{\infty} \frac{\tau^{n-1}}{n!} A = I + \tau^{-1} A \left(-1 + \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \right) = I - \tau^{-1} A + \frac{e^\tau}{\tau} A.$$

(**Q2**) Let

$$x_{n+1} = g(x_n) = x_n + \epsilon \sin(2\pi x_n) \qquad \text{where} \quad x \in [0,1], |\epsilon| < \frac{1}{2\pi}.$$

(a) Show that g maps the unit interval to itself.

(b) Show that g has exactly three fixed points in the unit interval. If $\epsilon > 0$, then two are sources, and one is a sink.

(c) Determine the (un)stable manifold of each fixed point.

(d) Argue/Prove that the Hartman-Grobman conjugacy is defined on an entire (un)stable manifold.

Solution.

(a) To show that $g([0,1]) \subset [0,1]$, it suffices to show that the maximum (resp. minimum) of $g|[0,1]$ is ≤ 1 (resp. ≥ 1). Since $g'(x) = 1 + 2\pi\epsilon \cos(2\pi x)$, the hypothesis that $|\epsilon| < \frac{1}{\epsilon}$ implies that $g'(x) > 0$ for all x. Thus $0 = g(0) \leq g(x) \leq g(1) = 1$ for all $x \in [0,1]$.
This proves that g maps the unit interval to itself (and we also showed g is onto).

(b) A fixed point solves $x = g(x)$ iff $0 = \epsilon \sin(2\pi x)$ iff $2x$ is an integer. Thus $x = 0, \frac{1}{2}, 1$ are the fixed points of g on $[0,1]$. Since $g'(0) = 1 + 2\pi\epsilon = g'(1)$ and $g'(\frac{1}{2}) = 1 - 2\pi\epsilon$, if $\frac{1}{2\pi} > \epsilon > 0$, then $g'(0) = g'(1) > 1$ and $|g'(0)| < 1$. This proves that $0, 1$ are sources and $1/2$ is a sink.

(c) On the interval $(0, 1/2)$, $g(x) > x$, so an orbit with $x_0 \in (0, 1/2)$ is a monotone increasing sequence x_n. Since g maps the interval $[0,1]$ to itself, the sequence must converge to a point x_*. This point is a fixed of g since $x_* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = g(x_*)$. Moreover, x_* cannot be a source, so $x_* = 1/2$. This proves that the stable manifold of $1/2$ includes $(0, 1/2]$. A symmetric argument shows that it also includes $[1/2, 1)$.
Finally, the unstable manifold of 0 must be $[0, 1/2)$ and that of 1 must be $(1/2, 1]$.

(d) Take the fixed point $1/2$. The H-G theorem says that there is an interval surrounding $1/2$ on which g is conjugate to $x \mapsto \lambda(x - 1/2) + 1/2$. Take a point $x_0 \in (0,1)$, the stable manifold of $1/2$. Since $x_n \to 1/2$, there is an N s.t. $x_N = g^N(x_0)$ lands in the interval. Define $h(x_0) = g^{-N} \cdot h(x_N)$, where h is the H-G conjugacy. It suffices to check that this extension of h is independent of the choice of N.

Remark. The 'right' way to view our DS is to take $x \mod 1$. Then $0 \equiv 1$ and the phase portrait is easily sketched.

(**Q3**) Let

$$\mathbf{x}_{n+1} = \begin{bmatrix} -1 & 1/2 \\ 1 & 0 \end{bmatrix} \mathbf{x}_n = \mathbf{A}\,\mathbf{x}_n$$

be a DS in the plane. Show that $\mathbf{0}$ is a hyperbolic fixed point. Determine the stable and unstable subspaces and sketch the phase portrait.

Solution. Since the DS is linear, its fixed points solve $(\mathbf{A} - I)\mathbf{x} = 0$. Since $\det(\mathbf{A} - I) = 1 - 1/2 \neq 0$, the only fixed point is $\mathbf{x} = \mathbf{0}$.

The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + \lambda - \frac{1}{2}$ which has roots $\lambda_\pm = \frac{-1\pm\sqrt{3}}{2}$. As $0 < \lambda_+ < 1$ and $|\lambda_-| > 1$, the origin is a saddle. Since $\lambda_- < 0$, the orbits along its eigenspace will oscillate from one side of the origin to the other.

E^+: this is the eigenspace of λ_+, which is the kernel of

$$\mathbf{A} - \lambda_+ I = \begin{bmatrix} -1 - \lambda_+ & 1/2 \\ 1 & -\lambda_+ \end{bmatrix}.$$

The vector $v_+ = \begin{bmatrix} \lambda_+ \\ 1 \end{bmatrix}$ spans E^+.

E^-: this is the eigenspace of λ_-, which is the kernel of

$$\mathbf{A} - \lambda_- I = \begin{bmatrix} -1 - \lambda_- & 1/2 \\ 1 & -\lambda_- \end{bmatrix}.$$

The vector $v_- = \begin{bmatrix} \lambda_- \\ 1 \end{bmatrix}$ spans E^-.

(**Q4**) Find the fixed points of the nonlinear map

$$x_{n+1} = x_n^2 - 5x_n + y_n, \quad y_{n+1} = \frac{1}{2} y_n + x_n^2,$$

and discuss their stability. Compute the third-order Maclaurin series of the stable manifold of $(0,0)$ [You will need to diagonalize the linear part of the system first].

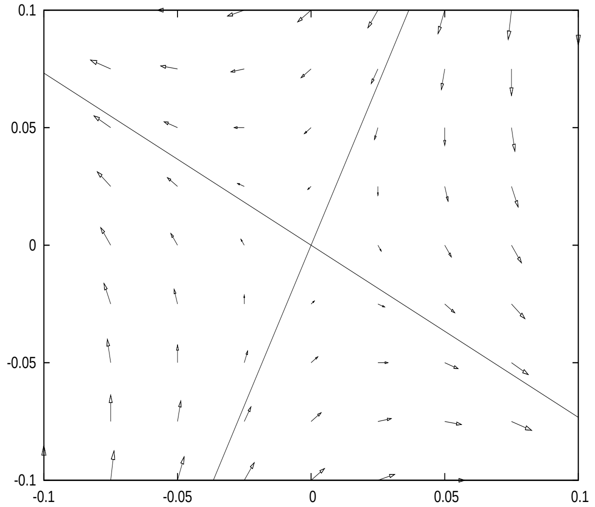

Figure .0.1: The phase portrait

Solution. The fixed point satisfy $x = -5x + y + x^2, y = y/2 + x^2$. Thus $y = 2x^2$ and so $0 = -6x + 3x^2$. Therefore, $x = 0, y = 0$ or $x = 2, y = 4$. We compute that

$$\mathbf{DF} = \begin{bmatrix} -5 + 2x & 1 \\ 2x & \frac{1}{2} \end{bmatrix}$$

so

$$\mathbf{DF}_{(0,0)} = \begin{bmatrix} -5 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad \mathbf{DF}_{(2,4)} = \begin{bmatrix} -1 & 1 \\ 4 & \frac{1}{2} \end{bmatrix}.$$

The eigenvalues of the first are $-5, 1/2$ so the origin is a saddle, while the second has eigenvalues $\frac{1}{4} \pm \frac{1}{4}\sqrt{65}$, so it is a source.

To compute W^+_{loc} we first must diagonalize $\mathbf{DF}_{(0,0)}$. The unstable eigenspace E^- is spanned by $v_- = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus stable eigenspace E^+ is spanned by $v_+ = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. The equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x} = u^+ v_+ + u^- v_- = \begin{bmatrix} 2u^+ + u^- \\ 11u^+ \end{bmatrix}$$

yields

$$u^+ = y/11, \qquad u^- = x - 2y/11.$$

We then get that

$$\begin{aligned} u^+_{n+1} &= y_{n+1}/11, \\ &= \tfrac{1}{2} y_n/11 + x_n^2/11 \\ &= \tfrac{1}{2} u_n^+ + (2u_n^+ + u_n^-)^2/11, \\ &= \tfrac{1}{2} u_n^+ + \tfrac{4}{11}(u_n^+)^2 \\ &\quad + \tfrac{4}{11} u_n^+ u_n^- + \tfrac{1}{11}(u_n^-)^2, \end{aligned} \qquad \begin{aligned} u^-_{n+1} &= x_{n+1} - 2y_{n+1}/11, \\ &= -5x_n + y_n - y_n/11 + x_n^2 - 2x_n^2/11, \\ &= -5u_n^- + 9(2u_n^+ + u_n^-)^2/11, \\ &= -5u_n^- + \tfrac{36}{11}(u_n^+)^2 \\ &\quad + \tfrac{36}{11} u_n^+ u_n^- + \tfrac{9}{11}(u_n^-)^2 \end{aligned}$$

We assume that W^+_{loc} is the graph $u^- = g(u^+)$. Since our coordinate system is chosen to be along $E^{\pm}$, $g(0) = 0$ and $g'(0) = 0$. Therefore, we can write

$$g(u) = a_2 u^2 + a_3 u^3 + O(u^4).$$

Assuming that $(u_n^+, u_n^-) \in W^+_{loc}$ we can compute u^-_{n+1} in two ways.

(a) $u^-_{n+1} = g(u^+_{n+1})$ since $(u^+_{n+1}, u^-_{n+1}) \in W^+_{loc}$ due to the invariance of W^+_{loc}. Thus ($\cdots$ are the terms of degree 4 or more)

$$\begin{aligned} u^-_{n+1} &= a_2(u^+_{n+1})^2 + a_3(u^+_{n+1})^3 + \cdots, \\ &= a_2(u_n^+)^2/4 + (\frac{4}{11}a_2 + \frac{1}{8}a_3)(u_n^+)^3 + \cdots, \end{aligned}$$

where we have used the fact that on W^+_{loc}, $u^+_{n+1} = \frac{1}{2}u_n^+ + \frac{4}{11}(u_n^+)^2 + \frac{4}{11}a_2(u_n^+)^3 + \cdots$ which means that $(u^+_{n+1})^2 = (u_n^+)^2/4 + \frac{4}{11}(u_n^+)^3 + \cdots$ and $(u^+_{n+1})^3 = (u_n^+)^3/8$.

(b) On the other hand u^-_{n+1} is determined by the dynamical system, while $u_n^- = g(u_n^+)$ so

$$\begin{aligned} u^-_{n+1} &= -5u_n^- + \frac{36}{11}(u_n^+)^2 + \frac{36}{11}u_n^+ u_n^- + \frac{9}{11}(u_n^-)^2, \\ &= (-5a_2 + \frac{36}{11})(u_n^+)^2 + (-5a_3 + \frac{36}{11}a_2)(u_n)^3 + \cdots \end{aligned}$$

Since these two expression must be equal, we get that

$$a_2/4 = -5a_2 + \frac{36}{11}, \qquad \frac{4}{11}a_2 + \frac{1}{8}a_3 = -5a_3 + \frac{36}{11}a_2$$

These equations have the solutions $a_2 = 48/77$ and $a_3 = 256a_2/451 = 12288/34727$. Thus

$$g(u) = \frac{48}{77}u^2 + \frac{12288}{34727}u^3 + O(u^4).$$

(**Q5**) Let

$$x_{n+1} = x_n/10 + x_n y_n, \quad y_{n+1} = 2y_n + x_n^2. \qquad (DS)$$

Compute the third-order Maclaurin series of a transformation $\mathbf{u} = \mathbf{Q}(\mathbf{x})$ which transforms this (DS) into the linear DS

$$u_{n+1} = u_n/10, \quad v_{n+1} = 2v_n. \qquad (LDS)$$

That is, find $\mathbf{Q}$ such that $\mathbf{QF} = \mathbf{AQ}$ where $\mathbf{F}$ is the map defined by the RHS of (DS) and $\mathbf{A}$ is the matrix defined by the RHS of (LDS).

Solution. Let $\mathbf{u} = (u, v)$ and let $\mathbf{F}(\mathbf{x}) = (x/10 + xy, 2y + x^2)$ where $\mathbf{x} = (x, y)$. We will compute $\mathbf{Q}$ inductively, beginning with its degree 2 terms. Before we do this, note that the linear part of our map $\mathbf{F}$ is already diagonal. To kill the quadratic terms xy, it will suffice to choose an xy term in $\mathbf{Q}$, etc. Thus, we will choose

$$\mathbf{Q}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} axy \\ bx^2 \end{bmatrix} + \cdots,$$

where $\cdots$ indicates terms of degree 3 or more. We compute that $\mathbf{u}_{n+1} = \mathbf{Q}(\mathbf{F}(\mathbf{x_n}))$ is equal to

$$\begin{aligned}\mathbf{u}_{n+1} &= \mathbf{Q}(\mathbf{x_{n+1}}),\\ &= \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} + \begin{bmatrix} a x_{n+1} y_{n+1} \\ b x_{n+1}^2 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} x_n/10 + x_n y_n \\ 2y_n + x_n^2 \end{bmatrix} + \begin{bmatrix} a x_n y_n/5 \\ b x_n^2/100 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} x_n/10 \\ 2y_n \end{bmatrix} + \begin{bmatrix} (a/5+1) x_n y_n \\ (b/100+1) x_n^2 \end{bmatrix} + \cdots \end{aligned}$$

On the other hand, we want $\mathbf{u}_{n+1} = \mathbf{A}\mathbf{u}_n = \mathbf{A}\mathbf{Q}(\mathbf{x_n})$ where $\mathbf{A}$ is the linear part of $\mathbf{F}$. Thus

$$\begin{aligned}\mathbf{u}_{n+1} &= \mathbf{A}\mathbf{Q}(\mathbf{x_n}),\\ &= \begin{bmatrix} x_n/10 \\ 2y_n \end{bmatrix} + \begin{bmatrix} a x_n y_n/10 \\ 2b x_n^2 \end{bmatrix} + \cdots \end{aligned}$$

This yields the equations

$$a/5 + 1 = a/10, \qquad b/100 + 1 = 2b,$$

so $a = -10$ and $b = 100/199$.

To compute the cubic terms in $\mathbf{Q}$ we make the observation that the only cubic terms to be killed are those that were created when we kill the quadratic terms. These will be of the form xy^2 and x^3 (created from xy) and x^2y (created from x^2). Thus, we will put

$$\mathbf{Q}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} axy \\ bx^2 \end{bmatrix} + \begin{bmatrix} a_0 x^3 + a_2 x y^2 \\ b_2 x^2 y \end{bmatrix} + \cdots$$

where $\cdots$ indicates terms of degree 4 or more. We compute that $\mathbf{u}_{n+1} = \mathbf{Q}(\mathbf{F}(\mathbf{x_n}))$ is equal to

$$\begin{aligned}\mathbf{u}_{n+1} &= \mathbf{Q}(\mathbf{x_{n+1}}),\\ &= \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} + \begin{bmatrix} a x_{n+1} y_{n+1} \\ b x_{n+1}^2 \end{bmatrix} + \begin{bmatrix} a_0 x_{n+1}^3 + a_2 x_{n+1} y_{n+1}^2 \\ b_2 x_{n+1}^2 y_{n+1} \end{bmatrix} + \cdots \end{aligned}$$

The first term (vector) contributes no cubics, while the second term contributes the cubic terms $a x_n^3/10 + 2a x_n y_n^2$ and $b x_n^2 y_n/5$. The third term (vector) contributes $a_0 x_n^3/10^3 + a_2 x_n y_n^2 2^2/10$ and $b_2 x_n^2 y_n 2/10^2$. The cubic part of $\mathbf{u}_{n+1}$ is therefore

$$(*) = \begin{bmatrix} (a_0/1000 + a/10) x_n^3 + (2a_2/5 + 2a) x_n y_n^2 \\ (b_2/50 + b/5) x_n^2 y_n \end{bmatrix}$$

On the other hand, we want $\mathbf{u}_{n+1} = \mathbf{A}\mathbf{u}_n = \mathbf{A}\mathbf{Q}(\mathbf{x_n})$. The cubic part of this expression for $\mathbf{u}_{n+1}$ is just $\mathbf{A}$ times the cubic part of $\mathbf{Q}$:

$$(**) = \begin{bmatrix} a_0 x_n^3/10 + a_2 x_n y_n^2/10 \\ 2b_2 x_n^2 y_n \end{bmatrix} + \cdots$$

Since the expressions (*) and (**) should be equal, we get

$$a_0/1000 + a/10 = a_0/10, \quad 2a_2/5 + 2a = a_2/10, \quad b_2/50 + b/5 = 2b_2$$

The coefficients are found to be

$$a_0 = -1000/99, a_2 = 200/3, b_2 = 1000/19701.$$

Thus

$$\mathbf{Q}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} x - 10xy - \frac{1000}{99}x^3 + \frac{200}{3}xy^2 \\ y + \frac{100}{199}x^2 + \frac{1000}{19701}x^2 y \end{bmatrix} + \mathbf{O}(|\mathbf{x}|^4).$$

(**Q6**) Stable and unstable manifolds can be defined for saddles of continuous dynamical systems in the same way as they are defined for discrete dynamical systems. In particular, the stable manifold can be written as an expansion

$$u^- = a_2(u^+)^2 + a_3(u^+)^3 + a_4(u^+)^4 + a_5(u^+)^5 + \cdots.$$

The constant coefficients a_i can then be obtained by equating two expressions for $\dot{u}^-$. Consider the system

$$\left.\begin{aligned} \dot{x} &= -x + y^2 \\ \dot{y} &= y - x^2 \end{aligned}\right\}.$$

Find the coefficients a_2, a_3, a_4 and a_5 for the expansion of the stable manifold through $(0, 0)$.

Solution. Since the linear part of the vector field is already diagonal, we see that the stable subspace is $y = 0$ and the unstable subspace is $x = 0$. Thus, we use $u^+ = x$ as the coordinate on E^+ and $u^- = y$ on E^-.

We assume that W^+_{loc} is the graph $y = g(x)$. And since the linearized system is already diagonal, $g(0) = 0 = g'(0)$. Thus, we suppose that $g(x) = a_2 x^2 + \cdots + a_5 x^5 + O(x^6)$.

On the one hand, if $(x, y = g(x)) \in W^+_{loc}$, then we compute that

$$\begin{aligned}\dot{y} &= g'(x)\dot{x} \\ &= (2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4)(-x + g(x)^2) + O(x^6) \\ &= -(2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + 5a_5 x^5) + (2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4) a_2^2 x^4 + O(x^6) \\ &= -2a_2 x^2 - 3a_3 x^3 - 4a_4 x^4 + (-5a_5 + 2a_2^3) x^5 + O(x^6).\end{aligned}$$

On the other hand, the derivative of $y = g(x)$ is given by

$$\begin{aligned}\dot{y} &= y - x^2 = g(x) - x^2 \\ &= (a_2 - 1)x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + O(x^6).\end{aligned}$$

Equating these two expressions for $\dot{y}$ yields the equations

$$-2a_2 = a_2 - 1, \quad -3a_3 = a_3, \quad -4a_4 = a_4, \quad -5a_5 + 2a_2^3 = a_5$$

so

$$a_2 = 1/3, \quad a_5 = 1/81, \quad a_3 = a_4 = 0.$$

Thus

$$g(x) = x^2/3 + x^5/81 + O(x^6).$$

(**Q7**) Consider the equilibrium $\mathbf{0}$ of the system

$$\left.\begin{aligned}\dot{x} &= -x \\ \dot{y} &= 2y - x^2\end{aligned}\right\}.$$

Give a series expansion of the stable manifold in the new variables, as far as 3rd order terms.

Solution. This is virtually identical to Q5. See the Maple solution sheet.

(**Q8**) Consider a $\mathbb{R}^2$ system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with an equilibrium point $\mathbf{x} = \mathbf{0}$ at which the linearised system has an imaginary pair of eigenvalues $\lambda = i\sigma$, $\overline{\lambda} = -i\sigma$, where $\sigma > 0$. In terms of a complex variable z, the dynamics are assumed to be given by

$$\dot{z} = \lambda z + az^2 + bz\overline{z} + c\overline{z}^2 + mz^2\overline{z} + \cdots \tag{0.1}$$

Following the treatment of discrete systems in §3.3 of the course notes, show that

(a) the quadratic terms in (0.1) can be eliminated by introducing the new variable $w = z + \alpha z^2 + \beta z\overline{z} + \gamma\overline{z}^2$ with suitable α, β, γ;

(b) the further variable transformation $\zeta = w + dw^3 + ew^2\overline{w} + gw\overline{w}^2 + h\overline{w}^3$ with suitable d, e, g, h allows the elimination of all cubic terms apart from the term proportional to $\zeta^2\overline{\zeta}$, leading to

$$\dot{\zeta} = \lambda\zeta + q\zeta^2\overline{\zeta} + O(|\zeta|^4), \quad \text{with} \quad q = m + \frac{iab}{\sigma} - \frac{i|b|^2}{\sigma} - \frac{2i|c|^2}{3\sigma}.$$

Conclude that the origin is stable if $\operatorname{Re} q < 0$ and unstable if $\operatorname{Re} q > 0$.

(**Q9**) Let

$$\dot{x} = -y + x^2, \quad \dot{y} = x + y^2 x.$$

Is $(0,0)$ a stable or unstable equilibrium?

Solution. You find that $\lambda = \pm i = \pm\sqrt{-1}$ and $z = x + iy$. The differential equations are transformed into

$$\dot{z} = iz + \frac{1}{4}z^2 + \frac{1}{2}z\overline{z} + \frac{1}{4}\overline{z}^2 - \frac{i}{8}z^2\overline{z} + \cdots, \tag{0.2}$$

which gives us $\bar{\lambda}q = -\frac{7}{24}$. So the origin is stable.

(**Q10**) Determine the stability of the origin for the dynamical system

$$\begin{aligned}x_{n+1} &= x_n - y_n + x_n^2 + y_n^3, \\ y_{n+1} &= x_n.\end{aligned}$$

Solution. If we write our system as $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, we get that

$$d\mathbf{f}_{\mathbf{0}} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

which has eigenvalues $\lambda = \dfrac{1 \pm \sqrt{-3}}{2}$, where are cube roots of -1. An eigenvector $\mathbf{t}$ of $d\mathbf{f}_{\mathbf{0}}^T$ (transpose) is $[-\lambda, 1]^T$, which gives

$$z = \langle \mathbf{t}, \mathbf{x} \rangle = -\lambda x + y.$$

We can solve for x and y and get

$$x = \frac{1}{\sqrt{-3}}(z - \bar{z}), \qquad y = uz + \bar{u}\bar{z}$$

where we take λ to be the root with a positive imaginary part and $u = \frac{1}{2}\left(1 - \frac{1}{\sqrt{-3}}\right)$.

We can substitute in to get

$$\begin{aligned}z_{n+1} &= -\lambda x_{n+1} + y_{n+1} \\ &= -\lambda(x_n - y_n + x_n^2 + y_n^3) + x_n \\ &= (1-\lambda)x_n + \lambda y_n + \frac{\lambda}{3}(z_n^2 - 2z_n\overline{z}_n + \overline{z}_n^2) + \\ &\quad \lambda(u^3 z_n^3 + 3u^2\bar{u}z_n^2\overline{z}_n + 3u\bar{u}^2 z_n\overline{z}_n^2 + \bar{u}^3\overline{z}_n^3) \\ &= \lambda z_n + az_n^2 + bz_n\overline{z}_n + c\overline{z}_n^2 + ez_n^2\overline{z}_n + \cdots,\end{aligned} \tag{0.3}$$

where

$$a = c = \frac{\lambda}{3} \qquad b = -\frac{2\lambda}{3} \qquad e = 3\lambda u^2\bar{u} = -\frac{1}{\sqrt{-3}}.$$

We know that there is a transformation $\zeta = h(z)$ such that the DS in (0.3) is transformed into $\zeta_{n+1} = \lambda\zeta_n + \mathfrak{m}\zeta_n^2\bar{\zeta}_n + O(|\zeta_n|^4)$ where

$$\mathfrak{m} = e + \frac{(2\lambda - 1)ab}{\lambda(1-\lambda)} + \frac{\lambda|b|^2}{\lambda - 1} + \frac{2\lambda|c|^2}{\lambda^3 - 1}, \tag{0.4}$$

and the stability is determined by $\hbar = \operatorname{Re}(\bar{\lambda}\mathfrak{m})$. We compute that

$$\begin{aligned}\bar{\lambda}\mathfrak{m} &= -\bar{\lambda}\frac{1}{\sqrt{-3}} + \bar{\lambda}\frac{(2\lambda-1)\lambda^2}{\lambda(1-\lambda)} \times \frac{-2}{9} + \bar{\lambda}\frac{\lambda}{\lambda - 1} \times \frac{2}{3} + 2\bar{\lambda}\frac{\lambda}{\lambda^3 - 1} \times \frac{1}{3} \\ &\vdots \\ &= -\frac{1}{2} + \sqrt{-1} \cdot \mathbf{R}.\end{aligned}$$

Since $\operatorname{Re}(\bar{\lambda}\mathfrak{m}) < 0$, so the equilibrium is stable.

At Examples Class 2 on Friday 29th October the solutions to Questions 5,6 and 10 will be discussed.